On the Difficulty of Combining Actual and Potential Criteria for an Increase in Social Welfare

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Summary: This paper examines two problems associated with the use of potential Pareto criteria in welfare economics. The first problem is the well-known intransitivity of the compensation criteria *à la* Kaldor-Hicks-Scitovsky. The second problem is the possible incompatibility between the Chipman-Moore-Samuelson criterion and the Pareto principle. The main result of this paper is that, in order to avoid either of these problems, it is necessary and sufficient that the domain to which these criteria are used is such that the Chipman-Moore-Samuelson criterion encompasses completely the Pareto criterion. When interpreted in a standard economic environment, this result is shown to be equivalent to Gorman's requirement of non-crossing between utility possibility frontiers.

Keywords and Phrases: Pareto principle, potential Pareto criteria, binary relations, transitivity, utility possibility frontiers

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1 Introduction

The Pareto principle asserts that a sufficient condition for ranking an economic alternative above another is when no one in society strictly prefers the latter and at least one person strictly prefers the former.[1] Economists like to think that, with this principle, they have at their disposal a (relatively) non-controversial criterion to assess whether or not economic transformations are worth doing.[2] A major problem with this criterion is that the domain of cases to which it can be applied conclusively is rather narrow.

To overcome this difficulty in part, without invoking other ethical principles, it has been suggested that this domain be extended from *actual* cases to *potential* or hypothetical ones. In substance, this is the motivation behind the famous Kaldor [17] - Hicks [16] - Scitovsky [22] compensation (KHSC) criterion which, even today, serves as a justification for using many tools of applied welfare economics such as consumer surpluses (see for instance Boadway [2], Boadway and Bruce [3], Dierker and Leninghaus [12] and Ruiz-Castillo [20]). Recall that the KHSC criterion ranks alternative x above alternative y if there exists, in the set of alternatives potentially achievable from x, an alternative z (not necessarily distinct from x) which Pareto dominates y. By including the Pareto criterion as a subrelation, the KHSC criterion clearly satisfies the Pareto principle. The usual argument in favor of the KHSC criterion is that it extends (albeit incompletely) the power of the Pareto principle at a low cost in terms of ethical defensibility.

The merit of this argument can be (and has been) debated. The relevance of hypothetical situations for assessing the social desirability of actual ones is certainly not a well established principle (see e.g. Little [18] and Blackorby and Donaldson [**?**]). But the KHSC suffers from a more immediate (and well-known) defect. It can lead to contradictory (intransitive) policy recommendations. Less well-known is the exact set of conditions that one needs to impose on the domain of alternatives in order to avoid these inconsistencies. Given the widespread use of this criterion in cost-benefit analysis, such a lack of knowledge is a bit

[1] This statement is that of what is sometimes called in the literature the strong Pareto principle (see Weymark [24]). The weak version requires that everyone in society strictly prefers one state over another as a sufficient condition for ranking the former above the latter. The difference between the two versions has no consequence for the problem at hand.

[2] Despite its large acceptance, the Pareto principle may conflict with other widely accepted values such as a minimal respect of individual rights (as illutrated in the well-known Sen's [23] Paradox) or a basic requirement of Bayesian rationality when the collective decision involve uncertainty (see for example Mongin [19]).

surprising. Some time ago, Gorman [14] conjectured that the occurrence of these intransitivities was related to the crossing of utility possibility frontiers. Such a conjecture is difficult to interpret since Gorman did not make precise whether he meant that the non-crossing of utility frontiers is a necessary condition for eliminating cyclical uses of the KHSC criterion or that it is a sufficient condition for this elimination or both. Neither was he clear about how rich the domain of alternatives need to be in order for the conjecture to hold. However, if we interpret the conjecture as saying that the two requirements of non-crossing of utility possibility frontiers and of transitive use of the KHSC criterion are *equivalent*, and provided of course that the conjecture is correct, it is clear that the assumption that the KHSC criterion can be used consistently is stringent. For, economic domains that generate utility possibility frontiers that never cross are rather exceptional.

The truth and the exact meaning of Gorman's conjecture notwithstanding, the relative ease with which inconsistent applications of the KHSC criterion can be obtained has motivated several economists (see for example Samuelson [21] and Chipman and Moore ([6], [7], [8], [10]) to develop another approach for extending the Pareto criterion to potential states; this is the Chipman-Moore-Samuelson (CMS) criterion.[3] According to this criterion, alternative x is at least as good as alternative y if *any* alternative potentially feasible from y is Pareto dominated by *some* alternative potentially feasible from x. It has the advantage of providing a transitive (although typically incomplete) ranking of *sets* of alternatives (Chipman and Moore ([6]; theorem 1)). For this reason, the CMS criterion has been perceived by some (e.g. Chipman and Moore ([6]; [8]; [10]), Chipman [5] and Ruiz-Castillo [20]) to be the natural analogue (for comparing sets of potentially feasible alternatives) to the Pareto criterion (for comparing actual alternatives).

The problem with the CMS criterion is that, contrary to the KHSC criterion, it does *not* satisfy the Pareto principle. It may indeed fail to recommend an actual Pareto improvement if it is the case that some alternative, potentially feasible *before* the implementation of the policy, is Pareto dominated by no alternative potentially feasible *after* the adoption of the policy. In short the CMS criterion is concerned only with *potential* welfare. It says nothing about *actual* welfare as evaluated by the Pareto principle. Given the wide acceptance of the Pareto criterion and, more fundamentally, the fact that the whole purpose of constructing the CMS criterion (or the KHSC one) is to extend the usefulness of the Pareto criterion, this state of affairs is clearly unsatisfactory.

In this paper the two problems - i.e. the possibility of intransitive uses of

[3] Also known as the Samuelson criterion. For a thorough historical survey of this criterion in relation to the KHSC one, see Chipman and Moore [9].

the KHSC criterion and the possibility of using the CMS criterion in a way that violates the Pareto principle - are put in parallel and examined more closely. The issue of compatibility between the CMS and the Pareto criteria is addressed by studying the properties of the set of all binary relations that encompass the CMS criterion as a subrelation while satisfying the Pareto principle. The KHSC is an example of such a binary relation. For the purpose of using such binary relations as foundations for policy recommendations, it is clearly important to require *at least one* of them to be transitive. Yet the main results of this paper are that, when the alternatives are interpreted as n-dimensional utility vectors drawn from compact and strictly comprehensive utility possibility sets, the requirement of transitivity for at least one such binary relation is equivalent to imposing the Pareto criterion to be a subrelation of the CMS one. In other words requiring a transitive binary relation compatible with the CMS criterion to satisfy the Pareto principle amounts to making the latter completely redundant. It is also shown that the redundancy of the Pareto criterion with respect to the CMS one is equivalent to the transitivity of the KHSC criterion. Finally, it is established that when the KHSC criterion is transitive, it does not offer any more guidance than the CMS one to compare alternative social states.

Although a stringent requirement on its own, this redundancy of the Pareto criterion with respect to the CMS one can be interpreted in the light of Gorman's [14] conjecture. More specifically, it is shown below that the transitivity of at least one binary relation compatible with the Pareto and the CMS criteria (and in particular of the transitivity of the KHSC criterion) is formally equivalent to the assumption that no utility possibility frontiers intersect. Gorman's conjecture is thus confirmed for the transitivity of the KHSC criterion. Furthermore, the non-crossing of utility possibility frontiers is also necessary and sufficient for the CMS criterion to be applied in accordance with the Pareto principle without inconsistencies.

The stringency of these conditions appears quite damaging for the practice of resorting to hypothetical states, either by using the KHSC or the CMS criterion, to evaluate policies. This is clear for the case of the KHSC criterion. If intransitive, this criterion can not serve as a (sensible) basis for making policy recommendation. If transitive, the results of this paper show that the KHSC criterion is indistinguishable from the CMS one so that this latter can be used instead. To that respect, the result of this paper goes along with the conclusions obtained by Bossert [4]. Bossert [4] shows that the *acyclicity* of the KHSC criterion (a rationality property weaker than transitivity) is equivalent to the existence of a complete and transitive social ranking of utility allocations. If such a social ranking exists, it can be used to make policy evaluation and there is no need to resorting to hypothetical alternatives to evaluate actual ones. In

Bossert [4], each actual state is assumed to be selected from a set of potential states by a continuous choice function. Such an assumption, which rules out the possibility for two distinct actual states to belong to the same set of potential states, is dispensed with in this paper. But as indicated, such a move does not salvage the KHSC criterion from its difficulties.

But the results of this paper are also damaging for the CMS criterion itself. In effect, an implication of the results herein is that, unless utility possibility frontiers never cross, it is possible to find an actual Pareto improvement that no transitive binary relation compatible with the CMS criterion (including of course the CMS criterion itself) will recommend. Hence the only remaining defense of the CMS criterion would be to consider it as an ethical principle in its own right that is to be used independently from the Pareto principle. As the controversies around the original KHSC criteria illustrates, there has been a widespread skepticism with respect to the relevance of *supplementing* actual welfare considerations by hypothetical ones. It is therefore doubtful that the idea of *substituting* potential welfare considerations to actual ones would command wider ethical support.

The organization of the rest of the paper is as follows. The next section sets out the notation and discusses the relationships between the Pareto criterion, the KHSC criterion, the CMS criterion and the set of all binary relations that encompass the CMS criterion while respecting the Pareto principle. The third section discusses, states and proves the main results on the equivalence between the transitivity of any binary relation compatible with both the Pareto criterion and the CMS one, the requirement for the Pareto criterion to be a subrelation of the CMS one and Gorman's requirement of non-crossing of utility-possibility frontiers. The fourth section concludes.

2 The general setting

2.1 Notation

The possibly non-standard notation general notation of this paper is as follows. The logical operators are , $\neg$ ("not"), $\wedge$ ("and") and $\vee$ ("or"). The set of integers, non-negative integers, and strictly positive integers are denoted, respectively, by I, I_+ and I_{++} while the set of real numbers, non-negative real numbers and strictly positive real numbers are denoted by $\Re$, $\Re_+$ and $\Re_{++}$ respectively. By a binary relation B on a set Ω, it is meant a subset of $\Omega \times \Omega$. Following the practice in economics, I write $x \ B \ y$ instead of $(x, y) \in \ B$. For a binary relation B, its *asymmetric* factor B_a is defined by $x\ B_a\ y \Longleftrightarrow (x\ B\ y) \wedge \neg(y\ B\ x)$, its *symmetric* factor B_s by $x\ B_s\ y \Longleftrightarrow (x\ B\ y) \wedge (y\ B\ x)$, its *non-comparable* factor B_n by $x\ B_n\ y \Longleftrightarrow \neg(x\ B\ y) \wedge \neg(y\ B\ x)$ and its *inverse* factor B_i by $x\ B_i\ y \Longleftrightarrow \neg(y\ B\ x)$. A binary relation B on Ω is *reflexive* if $x\ B\ x$ for all $x \in \Omega$, is *asymmetric* if it is equal to its asymmetric factor, is *complete* if its non-comparable factor is empty, is *transitive* if $x\ B\ z$ follows from $x\ B\ y$ and $y\ B\ z$ for any x, y and z, and is *acyclic* if for every finite sequence $\{x_i\}_{i=1}^{K}$ of elements of Ω, $(x_j\ B_a\ x_{j+1}\ \forall\ j = 1, ...K-1) \Rightarrow \neg(x_K\ B_a\ x_1)$. As can be easily shown, a transitive binary relation is acyclic. Given two binary relations B and B' on Ω, we say that B is a *subrelation* of B' (or alternatively that B' is *compatible* with B) when $B \subseteq B'$. Given a binary relation B on a set Ω and a subset Y of Ω, the (possibly empty) set $M_B(Y)$ of B-*maximal elements* of Y is defined by $M_B(Y) = \{y \in Y \mid y'\ B_a\ y \text{ for no } y' \text{in } Y\}$. A *quasi-ordering* is a reflexive and transitive binary relation and an *ordering* is a complete quasi-ordering. The notation used for the usual quasi-orderings of vectors in $\Re^m$ (for $m > 1$) is $\geq, >$ and $>>$.

2.2 Definitions and statement of the problem

Let Ω be an arbitrary set of economic states and let q be a quasi-ordering on Ω with asymmetric, symmetric and non-comparable factors q_a, q_s and q_n respectively. One can interpret q as the ranking of economic states induced by some version of the Pareto criterion. Corresponding to any state $y \in \Omega$ is a set $Y \subseteq \Omega$ of states potentially achievable from y through appropriate *compensation* payments. As in Samuelson [21] any such a set is referred to as a *situation*. A pair (y, Y) made of an economic state and its corresponding situation is referred to as a *position* of the economy. Denoting by $P(\Omega)$ the set of all non-empty subsets of Ω, let $S \subseteq \Omega \times P(\Omega)$ be the set of all conceivable positions of the economy. The following assumption is imposed throughout on S.

Axiom 1 $\forall (y, Y) \in S,\ y \in M_q(Y)$.

That is, for every position, an actual state is assumed to be Pareto-optimal in its situation. To the best of my knowledge, every use of the KHSC or the CMS criterion in the literature stands on such an assumption. It is not difficult to see that any weakening of this condition increases the ease with which one can obtain intransitive applications of the KHSC criterion or any binary relation compatible with both the Pareto and the CMS criteria.

Given q, one can define its *extension* $E(q)$ from Ω to $P(\Omega)$ as follows.

Definition 1 $\forall\ Y,\ Y^{'} \in P\left(\Omega\right), Y\ E(q)\ Y^{'} \Longleftrightarrow \forall\ y' \in Y', \exists\ y \in Y$ *such that* $y\ q\ y^{'}$.

It is immediate to see that $E(q)$ is reflexive and transitive. In a similar manner, one can define the *asymmetric extension* $E(q_a)$ of q by substituting q_a for q in definition 1. As can be easily seen, $E(q_a)$ is asymmetric and transitive but is *not* the asymmetric factor of $E(q)$. However $E(q_a) \subset E(q)_a$ on a domain of positions satisfying axiom 1. The difference between $E(q_a)$ and $E(q)_a$ is illustrated in figure 1 (with q taken to be the usual quasi-ordering $\geq$ of vectors in $\Re^2$) where $X\ E(q)_a\ Y$ but not $X\ E(q_a)\ Y$.

Either q or $E(q)$ can be used to rank (in general incompletely) positions in S. The standard Pareto ranking of positions is based on a comparison of their first component - their actual states - by means of the quasi-ordering q. Formally, the Pareto criterion defines the following binary relation P on S.

Definition 2 $\forall\ (y, Y), (y', Y') \in S, (y, Y)\ P\ (y', Y') \Longleftrightarrow y\ q\ y'$

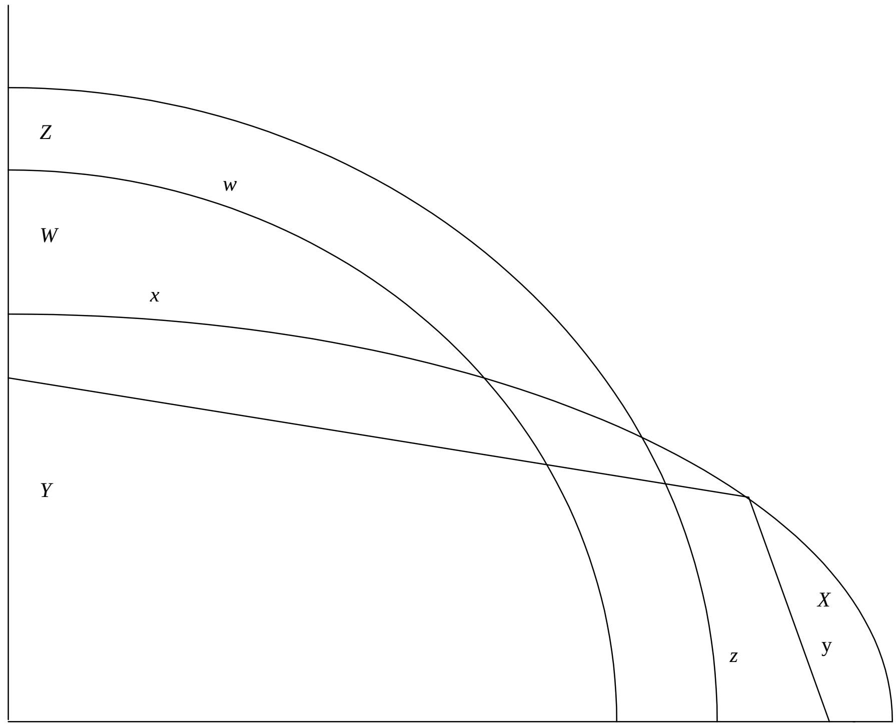
The criterion proposed originally by Samuelson [21] and formalized by Chipman and Moore [6] discussed in the introduction is obtained by comparing the second component of any two positions - their situation - by means of the quasi-ordering $E(q)$. For latter use, we define formally the CMS criterion as follows.

Definition 3 $\forall\ (y, Y), (y', Y') \in S, (y, Y)\ CMS\ (y', Y') \Longleftrightarrow Y\ E(q)\ Y'$.

The Pareto ranking does not use any information on the situations of the positions. It is a ranking based purely on *actual* considerations. On the other hand the CMS ranking does not use any information on the actual states of the positions. Hence, it may not be able to compare two positions, such as (w, W) and (x, X) in Figure 1, where one actual state strictly dominates the other in terms of q. It is a ranking based entirely on *potential* considerations. It seems safe to say that many ethical systems, whatever is their opinion on the introduction of potential considerations in normative appraisal, would agree that actual considerations should bear a priority over potential ones. This first requirement, referred in the introduction as the Pareto principle, is captured by the following notion of *asymmetric compatibility* with q.

Definition 4 *(Pareto principle) A binary relation* $\succeq$ *(with asymmetric and symmetric factors* $\succ$ *and* $\sim$ *respectively) on* S *is asymmetrically compatible with* q *if and only if for all* $(y, Y),\ (y', Y') \in S,\ (y\ q_a\ y') \Rightarrow (y, Y) \succ (y', Y')$.

In itself, CMS is clearly not asymmetrically compatible with q. Yet, as discussed earlier, many authors have argued in favor of extending the field of usefulness of q by comparing alternative positions on the basis of the ranking



of their situations in terms of $E(q)$. There are at least two ways by which such an extension could be performed. The first would be to require, as stated in the following definition, the ranking of social states $\succeq$ to be also asymmetrically compatible with $E(q)$.

Definition 5 *A binary relation $\succeq$ (with asymmetric and symmetric factors $\succ$ and $\sim$ respectively) on S is asymmetrically compatible with $E(q)$ if and only if for all $(y, Y),\ (y', Y') \in S$, $Y\ E(q)_a\ Y' \Rightarrow (y, Y) \succ (y', Y')$.*

The other would be to require $\succeq$ to be asymmetrically compatible with $E(q_a)$ (but not necessarily with $E(q)$). Since $E(q_a)$ is a subrelation of $E(q)_a$, the requirement for the social ranking to be asymmetrically compatible with $E(q)$ is stronger than that of asymmetric compatibility with $E(q_a)$.

A possible problem with either of these requirements of asymmetric compatibility with $E(q)$ is that they may conflict with the Pareto principle when the ranking of two positions (y, Y) and (y', Y') such that $y\ q_a\ y'$ and $Y'\ E(q)_a\ Y$ (or $Y'\ E(q_a)\ Y$) is involved. As it turns out however, on a domain of positions satisfying axiom 1, this type of conflict can never happen. More specifically, the following result (whose straightforward proof is omitted) holds.

Lemma 1 *Let S be a domain of positions satisfying axiom 1. Then for every positions (y, Y) and $(y', Y') \in S$, $y\ q_a\ y' \Rightarrow \neg(Y'\ E(q)_a\ Y) \Rightarrow \neg(Y'\ E(q_a)\ Y)$.*

Given lemma 1, a ranking of social states that is asymmetrically compatible with both q and $E(q)$ (or $E(q_a)$) would appear therefore to be a natural translation of the idea of giving priority to actual considerations while resorting to potential considerations when actual ones are not conclusive. The KHSC criterion discussed in the introduction is clearly the result of such an idea. The KHSC criterion induces the following binary relation K on S.

Definition 6 $\forall\ (y,Y),(y^{'},Y)\in S,\ (y,Y)\ K\ (y',Y^{'}) \Longleftrightarrow Y\ E(q_a)\ \{y'\}$.

This definition of the KHSC criterion is that proposed originally by Kaldor [17]. The so-called Hicks [16] compensation criterion is nothing but the inverse factor of K while the so-called Scitovsky [22] criterion corresponds to its asymmetric factor. As can be easily verified, K is asymmetrically compatible with q and with $E(q_a)$ but is *not* asymmetrically compatible with $E(q)$. Let β be the set of all binary relations on S asymmetrically compatible with q and $E(q)$ and let β' denote the set of all binary relations on S that are asymmetrically compatible with q and $E(q_a)$. As mentioned, $\beta \subseteq \beta'$ and $K \in \beta'$.

It is well-known that K (or its asymmetric factor K_a) need not, in general, be transitive. An example of intransitivity is depicted in Figure 1 with (x,X) K (y,Y) and (w,W) K (x,X) but not (w,W) K (y,Y). For K to rank S transitively, some further restrictions must be imposed on S. What are these restrictions ? And more generally, what condition must S satisfy in order to guarantee the existence of at least one transitive binary relation in β (or in β')? In the present paper, attention is restricted to the set β. We shall therefore look for a restriction on the domain of positions of the economy that is necessary and sufficient for the existence of at least one transitive member of β. As shall also be seen in the next section (theorems 2 and 3), such a condition turns out also to be necessary and sufficient for K to be transitive. But the problem of finding (slightly) weaker condition that would secure the existence of transitive members of β' distinct from K and not contained in β shall not be considered.

In order to address the problem of finding a transitive element in β, the following binary relation R is first defined.

Definition 7 $\forall\ (y,Y),\ (y',Y')\in S,\ (y,Y)\ R\ (y',Y') \Longleftrightarrow y\ q_a\ y^{'} \vee Y\ E(q)_a\ Y^{'}$.

Under axiom 1 and lemma 1, R is thus the *intersection* of all binary relation belonging to β. It is immediate to see that, if S satisfies axiom 1 (and therefore if lemma 1 holds), R is asymmetric. The interest in studying the binary relation R is that its *acyclicity* is *necessary* for the existence of at least one transitive binary relation in β. This is stated formally in the following lemma.

Lemma 2 *Let S be a set of positions satisfying axiom 1 and let $\succeq \in \beta$. Then $\succeq$ is transitive only if R is acyclic.*

Proof: Let $\succeq$ be a transitive binary relation in β, and let, for some $K \in I_{++}$, $(y_i, Y_i)_{i=1}^{K}$ be a sequence of positions in S such that $(y_i, Y_i)\ R_a\ (y_{i+1}, Y_{i+1})$ (or $(y_i, Y_i)\ R\ (y_{i+1}, Y_{i+1})$ since R is asymmetric) for all $i = 1, ..., K-1$. That is, using definition 7, $(y_i, Y_i)_{i=1}^{K}$ is a sequence of positions satisfying $y_i\ q_a\ y_{i+1} \vee Y_i\ E(q)_a\ Y_{i+1}$ for all $i = 1, ..., K-1$. Since $\succeq$ is asymmetrically compatible with both q and $E(q)$, it follows that $(y_i, Y_i) \succ (y_{i+1}, Y_{i+1})$ for all $i = 1, ..., K-1$ and, since $\succeq$ is transitive, that $(y_1, Y_1) \succ (y_K, Y_K)$. Since $\succeq$ is asymmetrically compatible with both q and $E(q)$, this state of affairs rules out the possibility that $y_K\ q_a\ y_1$ holds as it rules out the possibility that $Y_K\ E(q)_a\ Y_1$ holds. Hence we have $\neg(y_K, Y_K)\ R_a\ (y_1, Y_1)$, which establishes that R is indeed acyclic. QED.

Since R itself is a binary relation asymmetrically compatible with both q and $E(q)$, an immediate consequence of this result is that there exists an acyclic binary relation in β if and only if R is acyclic.

In general, R need not be acyclic as the example of (w, W), (x, X), (y, Y), (z, Z) on figure 1 illustrates. As shown in the next section, a necessary and sufficient condition for the acyclicity of R is for the asymmetric factor of the Pareto criterion to be redundant with respect to the CMS one. Formally, this *asymmetric redundancy* condition is stated as follows.

Definition 8 *The binary relation q is asymmetrically redundant with respect to $E(q)$ on S if for every positions (x, X), $(y, Y) \in S$, $x\ q_a\ y \Rightarrow X\ E(q)_a\ Y$*

In words, the actual criterion is asymmetrically redundant with respect to the potential criterion if the domain of positions considered is such that every strict (asymmetric) actual improvement leads to a strict potential improvement in the CMS sense.

3 Results

3.1 Notation and interpretation

In this section, I consider a society of m individuals indexed by a set $M \subseteq I_{++}$, and I assume that $\Omega = \Re^m_+$, a typical element u of which being interpreted as a *utility allocation* among these m individuals. Furthermore, a situation $U \subseteq \Re^m_+$ is to be interpreted as a *utility possibility set.* In this setting, the Pareto criterion q is interpreted as the usual quasi-ordering $\geq$ of vectors (with asymmetric factor $>$). Interpreting $\geq$ as q amounts to using the strong Pareto principle in Weymark's [24] terminology (increasing utility of one individual without decreasing that of others suffices to improve social welfare). Given $U \subseteq \Re^m_+$, U^- denotes the *closure* of U in $\Re^m_+$, U^0 denotes the *interior* of U in $\Re^m_+$ and $F(U)$ denotes the frontier of U in $\Re^m_+$ (that is $F(U) = \{u \in U \mid u \in \left[\Re^m_+ \backslash U\right]^-\}$. Given any 2 points $u, v \in \Re^m_+$, $[u, v]$ denotes the closed segment connecting u to v (that is, $[u, v] = \left\{u^{'} \in \Re^m_+ \mid \exists\, t \in [0, 1] \text{ such that } u^{'} = (1 - t)\, u + tv\right\}$) and $]u, v[$ denotes the open segment connecting u to v (that is, $]u, v[=$ $\left\{u^{'} \in \Re^m_+ \mid \exists\, t \in]0, 1[\text{ such that } u^{'} = (1 - t)\, u + tv\right\}$). Given any $w \in \Re^m_+$, a *neighborhood* of radius ε around w is denoted $N_\varepsilon(w)$ and is defined by $N_\varepsilon(w) = \left\{w^{'} \in \Re^m_+ \mid \forall\, i = 1, ..., n, \ \mid w_i^{'} - w_i \mid < \varepsilon\right\}$. The $m - 1$ dimensional simplex is denoted S^{m-1} and is defined by $S^{m-1} = \{u \in \Re^m_+ \mid \sum_{i \in M} u_i = 1\}$. The Euclidean norm of a vector $u \in \Re^m_+$ is denoted by $\parallel u \parallel$ and the origin in $\Re^m_+$ is denoted by 0^m.

3.2 Statement, proof and discussion of the main results

I first proceed by imposing some minimal properties on the domain $S \subseteq \Re^m_+ \times P(\Re^m_+)$ of positions that is considered. More specifically, I consider the following definition and axiom :

Definition 9 *A utility possibility set $U \subseteq \Re^m_+$ is* comprehensive *if and only if $(u \in U, u' \in \Re^m_+$ and $u' < u) \Longrightarrow u' \in U$ and is* strictly comprehensive *if and only if it is comprehensive and satisfies $(u \in U,\, u' \in \Re^m_+$ and $u' < u) \Longrightarrow (\exists\, \widehat{u} \in U$ such that $\widehat{u} >> u')$.*

Axiom 2 *(domain) S is a set of positions that satisfies (i) for all $(u, U) \in S$, U is compact and strictly comprehensive and $U \neq \{0^m\}$ (ii) for every $(u, U) \in S$, and for every $\widehat{u} \in U$, such that $\widehat{u} \neq 0^m$ there exists some strictly comprehensive and compact set $\widehat{U} \subseteq U$ such that $(\widehat{u}, \widehat{U}) \in S$ and (iii) for every (u, U), $(v, V) \in S$, for every $\widehat{u} \in U$, $(\overline{v} \in V$ and $\overline{v} > \widehat{u}) \Rightarrow (\overline{v}, \overline{U}) \in S$ for some $\overline{U} \subseteq \Re^m_+$ satisfying $U \subseteq \overline{U}$.*

Requiring utility possibility sets to be compact and strictly comprehensive is natural if these sets are interpreted as the image, under a m-dimensional vector of continuous and monotonically increasing utility functions, of a set of allocations of goods in a standard Arrow-Debreu economy satisfying a free-disposal assumption. Clause (ii) and (iii) of axiom 2 requires the domain of conceivable positions to be 'minimally rich'. More specifically, clause (ii) requires S to be such that, for any position (u, U) that it contains and for every utility allocation (other than the origin) belonging to U, it contains also a position where this utility allocation is the actual state and whose situation is a subset of U. In short, (ii) says that the domains is sufficiently rich to enable one to go 'everywhere inside' the utility possibility set of any position that it contains. Clause (iii) imposes a similar requirement on the 'outside' case. More precisely, clause (iii) says that, whenever a utility allocation $\overline{v}$ in the utility possibility set of a position Pareto dominates a utility allocation in the utility possibility set U of some other position, then there must be a position with $\overline{v}$ as the actual state and with a utility possibility set containing U as a subset.

The main results of the paper make some use of the following lemmas.

Lemma 3 *(Arrow [1]; lemma 1) Let* $U \subseteq \Re^m_+$ *be compact. Then, for every* $u \in U$*, there exists some* $\widehat{u} \in M_{\geq}(U)$ *such that* $\widehat{u} \geq u$*.*

Lemma 4 *Let* U *and* U' *be two comprehensive subsets of* $\Re^m_+$*. Then* $U \supseteq U' \Leftrightarrow U \ E(\geq) \ U'$*.*

Proof : The proof of the first part of the implication is obvious (see Chipman and Moore [6]; lemma 1). For the other part, assume that $U \ E(\geq) \ U'$ and let $u' \in U'$. From definition 1, there exists $u \in U$ such that $u \geq u'$, which, since U is comprehensive, implies in turn that $u' \in U$. QED

Lemma 5 *For any strictly comprehensive set* $U \subseteq \Re^m_+$*,* $M_{\geq}(U) = F(U)$*.*

Proof : I first show that $M_{\geq}(U) \subseteq F(U)$. Let $u \in M_{\geq}(U)$. Then, consider any $\widehat{u} \in \Re^m_+$ such that $\widehat{u} > u$. By definition of $M_{\geq}(U)$, $\widehat{u} \notin U$. Define the sequence $\left\{u^i\right\}_{i=1}^{\infty}$ by $u^i = u(1 - 1/i) + \widehat{u}/i$ for $i = 1, \ldots, \infty$. By construction, one has $u^i > u$ and thus $u^i \notin U$ for all $i = 1, \ldots, \infty$. But clearly, $\left\{u^i\right\}_{i=1}^{\infty}$ converges to u. Hence, $u \in [\Re^m_+ \backslash U]^-$.

To show that $F(U) \subseteq M_{\geq}(U)$, let $u' \in F(U)$ but, contrary to the claim, assume that $u' \notin M_{\geq}(U)$. Then, there is some $u \in U$ such that $u > u'$. Since U is strictly comprehensive, there exists some $\widehat{u} \in U$ such that $\widehat{u} >> u'$. Let $\delta = \min_{i \in M}(\widehat{u}_i - u'_i)$. Since $\widehat{u} >> u', \delta > 0$. Consider now any $v \in N_{\delta/2}(u')$. By definition of a neighborhood one has $\left|v_i - u'_i\right| < \delta/2 < \delta \leq (\widehat{u}_i - u'_i)$ for all $i \in M$. If $u'_i > v_i$, it is clear that $\widehat{u}_i > u'_i > v_i$. If $v_i > u'_i$, one has $v_i - u'_i < \widehat{u}_i - u'_i$ which clearly implies $v_i < \widehat{u}_i$. Thus $v << \widehat{u}$ and, since the set is comprehensive, $v \in U$. But this implies that $N_{\delta/2}(u^{'}) \subseteq U$, a contradiction of the original claim that $u^{'} \in F(U)$. QED

Lemma 6 *Let U be any non-empty comprehensive subset of $\Re^m_+$. Then U is connected.*

Proof: Since U is a non-empty comprehensive subset of$\Re^m_+$, $0^m \in U$. Let associate to any $u \in U$ the set $[0^m, u]$. $[0^m, u]$ is obviously connected. The rest of the proof consist in showing that $\underset{u \in U}{\cup} [0^m, u] = U$. It is straightforward to establish that $U \subseteq \underset{u \in U}{\cup} [0^m, u]$. In order to show that $\underset{u \in U}{\cup} [0^m, u] \subseteq U$, let $u' \in \underset{u \in U}{\cup} [0^m, u]$. Then, for some $u \in U$, $u' \in [0^m, u]$. By definition of $[0^m, u]$, $u' \leq u$. Since U is comprehensive, $u' \in U$. The fact that U is connected is an immediate consequence of the fact that U can be written as the union of a collection of connected sets all of which containing 0^m as a common element. QED

Lemma 7 *Let U be a closed subset of $\Re^m$. Then, $\forall\ u \in U$ and $\forall\ v \in \Re^m_+ \backslash U$, $F(U) \cap [u, v] \neq \emptyset$*

Proof : Suppose, by way of obtaining a contradiction, that for some $u \in U$ and for some $v \in \Re^m_+ \backslash U$, one has $F(U) \cap [u, v] = \emptyset$ or, equivalently:

$$[u, v] \subseteq \Re^m_+ \backslash F(U) \tag{1}$$

Since U is closed, it is easily seen that:

$$\Re^m_+ \backslash F(U) = [\Re^m_+ \backslash U] \cup U^0 \tag{2}$$

Hence, it follows from (1) and (2) that $[u, v]$ is contained in the union of two disjoint (relatively) open sets. Now recall that $u \in U$ so that $u \in U^0$ since $u \notin F(U)$; Hence $U^0 \cap [u, v] \neq \emptyset$. Recall also that $v \in \Re^m_+ \backslash U$ so that $[\Re^m_+ \backslash U] \cap [u, v]$. Since, $U^0 \cap [u, v]$ and $[\Re^m_+ \backslash U] \cap [u, v]$ are both open relatively to $[u, v]$, it follows that $[u, v]$ can be written as the union of two non-empty disjoint open sets. But this can not be since $[u, v]$ is connected. QED[4]

We can now proceed with the proof of the main results of the paper. It is first established that the asymmetric redundancy of the Pareto criterion with respect to the CMS one (as stated in definition 8) is necessary and sufficient for R to be acyclic (and therefore for the existence of at least one transitive binary relation asymmetrically compatible with both $\geq$ and $E(\geq)$.

[4] I thank an anonymous referee for the argument underlying the proof of this lemma.

Theorem 1 *Let S be a domain of positions satisfying axioms 1 and 2. Then R is acyclic if and only if $\geq$ is asymmetrically redundant with respect to $E(\geq)$.*

Proof : It is immediate to see that, due to the transitivity of $E(\geq)_a$, if $\geq$ is asymmetrically redundant with respect to $E(\geq)$ (that is if for every (u, U) and $(u', U') \in S$ $u > u' \Rightarrow U\ E(\geq)_a\ U'$) then R is transitive and, therefore, acyclic. For the other implication, assume by contraposition that, for two positions (u, U) and $(u', U') \in S$ one has $u > u'$ but not $U\ E(\geq)_a\ U'$. By definition of R, one has $(u, U)\ R_a\ (u', U')$. By lemma 1, $u > u'$ rules out the possibility for $U'\ E(\geq)\ U$ to hold so that the statement $\neg U\ E(\geq)_a\ U'$ implies $\neg U\ E(\geq)\ U'$. From definition 1, there exists a utility allocation $\widehat{u}' \in U'$ such that $\widehat{u} \geq \widehat{u}'$ for no $\widehat{u} \in U$. Since $\geq$ is reflexive $\widehat{u}' \notin U$. By lemma 3, there is no loss of generality in assuming that $\widehat{u}' \in M_{\geq}(U')$. Let $\widetilde{u} \in F(U) \cap [0, \widehat{u}']$. Such a $\widetilde{u}$ exists by lemma 7, belongs to $M_{\geq}(U)$ by lemma 5 and satisfies $\widetilde{u} < \widehat{u}'$ by construction. For this reason the segment $[\widetilde{u}, \widehat{u}']$ is not degenerate. Let therefore $\overline{u}$ be a utility allocation in the open segment $]\widetilde{u}, \widehat{u}'[$. Since $\overline{u} \in U'$ (due to the comprehensiveness of U'), there exists (by clause (*ii*) of axiom 2), a comprehensive and compact subset $\overline{U}$ of U' such that $(\overline{u}, \overline{U}) \in S$. By lemma 4, $U'\ E(\geq)\ \overline{U}$. Furthermore, since $\widehat{u}' > \overline{u}$ and $\overline{u} \in M_{\geq}(\overline{U})$, there can be no $\overline{v} \in \overline{U}$ such that $\overline{v} \geq \widehat{u}'$. Hence $\overline{U}$ $E(\geq)\ U'$ does not hold and one has $U'\ E(\geq)_a\ \overline{U}$. From definition 7, $(u', U')\ R_a$ $(\overline{u}, \overline{U})$. Now, since $\overline{u} > \widetilde{u}$, the segment $[\widetilde{u}, \overline{u}]$ is not degenerate and there is some $u'' \in\ ]\widetilde{u}, \overline{u}[\subseteq \overline{U}$ (since $\overline{U}$ is comprehensive). By clause (*iii*) of axiom 2, there exists a compact and comprehensive set $U'' \subseteq \Re^m_+$ containing U as a subset such that $(u'', U'') \in S$. Since $u'' \in\ ]\widetilde{u}, \overline{u}[$, $\overline{u} > u''$ which, by definition 7, implies $(\overline{u}, \overline{U})\ R_a\ (u'', U'')$. Note now that, since $U \subseteq U''$, one has by lemma 4 that U'' $E(\geq)\ U$. Furthermore, there is no $\widehat{u} \in U$ such that $\widehat{u} \geq u''$, for assuming the existence of such a $\widehat{u}$ would contradict, given the transitivity of $\geq$ and the fact that $u'' > \widetilde{u}$, the original statement that $\widetilde{u} \in M_{\geq}(U)$. Hence, $U\ E(\geq)\ U''$ does not hold and, as a consequence, one has $U''\ E(\geq)_a\ U$ and, from definition 7, $(u'', U'')\ R_a\ (u, U)$. But the ranking of the four positions (u, U), (u', U'), $(\overline{u}, \overline{U})$ and (u'', U'') gives us the required violation of acyclicity. QED

Theorem 1 thus states that a domain on which the asymmetric factor Pareto ranking is a subrelation of the asymmetric *CMS* one is the only circumstance, up to a violation of axiom 2, in which a transitive binary relation asymmetrically compatible with both $\geq$ and $E(\geq)$ can be obtained. This fact should be troublesome for the advocates of potential Pareto criteria. After all, the whole purpose of introducing potential considerations in cost-benefit analysis was to complete the Pareto criterion, not to make it superfluous.

As mentioned in the preceding section, the acyclicity of R is neither necessary nor sufficient for the transitivity of the KHSC criterion since the binary relation K does not strictly speaking belong to β. As it turns out however, on a domain satisfying axioms 1 and 2, the asymmetric redundancy of $\geq$ with respect to $E(\geq)$ is also necessary and sufficient for the transitivity of K. This is established in the following theorem.

Theorem 2 *Let S be a domain of positions satisfying axioms 1-2. Then K is transitive if and only if $\geq$ is asymmetrically redundant with respect to $E(\geq)$ on S.*

Proof : (necessity) Suppose that $\geq$ is not asymmetrically redundant with respect to $E(\geq)$ on S. That is, suppose that there exists two positions (u, U) and (u', U') in S such that $u > u' \wedge \neg(U\ E(\geq)_a\ U')$. From the very definition of K (definition 6) $(u, U)\ K\ (u', U')$ holds for any such pair of positions. Since, by lemma 1, $U'\ E(\geq)\ U$ does not hold, it follows from the statement $\neg(U\ E(\geq)_a\ U')$ that $U\ E(\geq)\ U'$ does not hold. From definition 1, there exists a utility allocation $\widehat{u}' \in U'$ such that $\widehat{u} \geq \widehat{u}'$ for no $\widehat{u} \in U$. Since $\geq$ is reflexive, $\widehat{u}' \notin U$. By lemma 3, there is no loss of generality in assuming that $\widehat{u}' \in M_{\geq}(U')$. Let $\widetilde{u} \in F(U) \cap [0, \widehat{u}']$. Such a $\widetilde{u}$ exists by lemma 7, belongs to $M_{\geq}(U)$ by lemma 5 and satisfies $\widetilde{u} < \widehat{u}'$ by construction. For this reason the segment $[\widetilde{u}, \widehat{u}']$ is not degenerate. Let therefore $\overline{u}$ be a utility allocation in the open segment $]\widetilde{u}, \widehat{u}'[$. Since $\overline{u} \in U'$ (due to the comprehensiveness of U'), there exists (by clause (ii) of axiom 2), a comprehensive and compact subset $\overline{U}$ of U' such that $(\overline{u}, \overline{U}) \in S$. By construction, $\widehat{u}' > \overline{u}$ so that $U'\ E(>)\ \{\overline{u}\}$ and, by definition 6, $(u', U')\ K\ (\overline{u}, \overline{U})$. Now, there is no $\widehat{u} \in U$ such that $\widehat{u} > \overline{u}$ for, assuming the existence of such a $\widehat{u}$ would contradict, given the transitivity of $\geq$ and the fact that $\overline{u} \in]\widetilde{u}, \widehat{u}'[$, the statement established previously that $\widetilde{u} \in M_{\geq}(U)$. Hence it is not the case that $U\ E(>)\ \{\overline{u}\}$ and thus, is also not the case that $(u, U)\ K\ (\overline{u}, \overline{U})$), which gives us the required intransitive chain.

(sufficiency). By contraposition, let (u, U), (u', U') and (u'', U'') be any three positions in S such that $(u, U)\ K\ (u', U')$ and $(u', U')\ K\ (u'', U'')$ are true but $(u, U)\ K\ (u'', U'')$ is false. From definitions 1 and 6, this implies that 1) there is some $\widehat{u} \in U$ such that $\widehat{u} > u'$, 2) there is some $\widehat{u}' \in U'$ such that $\widehat{u}' > u''$ and 3) there is no $\widetilde{u} \in U$ such that $\widetilde{u} > u''$. Clearly, given the transitivity of $\geq$ and the $\widehat{u}'$ referred to in statement 2), statement 3) implies that there is no $\widetilde{u} \in U$ such that $\widetilde{u} \geq \widehat{u}'$. From definition 1, we see that it is not the case that $U\ E(\geq)\ U'$ and thus, that $U\ E(\geq)_a\ U'$ is false as well. If $(\widehat{u}, U) \in S$ for the utility allocation $\widehat{u}$ referred to in statement 1, then we have obtained the required violation of the asymmetric redundancy of $\geq$ with respect to $E(\geq)$. If $(\widehat{u}, U) \notin S$, there exists from clause (ii) of axiom 2 some utility allocation $\overline{u}$ satisfying $u' < \overline{u} < \widehat{u}$ and a compact and comprehensive subset $\overline{U}$of U such that $(\overline{u}, \overline{U}) \in S$. Now $\overline{U}\ E(\geq)$ U' can not hold for assuming otherwise would contradict, given lemma 4, the previously established falsity of $U\ E(\geq)\ U'$. Hence it can not be the case that $\overline{U}\ E(\geq)_a\ U'$). Since we have $\overline{u} > u'$ for the positions (u', U') and $(\overline{u}, \overline{U}) \in S$, we have obtained the required violation of the asymmetric redundancy of $\geq$ with respect to $E(\geq)$. QED

Since the *KHSC* criterion is transitive if and only if $\geq$ is asymmetrically redundant with respect to $E(\geq)$, we can wonder what are its advantage as a criterion for comparing states of affairs. If intransitive, it does not provide any help in ranking economic alternatives. If transitive, Theorem 2 shows that the domain of alternatives is such that the Pareto criterion is entirely encompassed by the CMS criterion so that this latter criterion can be used instead. This uselessness of the KHSC criterion is further emphasized by the following theorem which establishes that when the KHSC criterion is transitive, its asymmetric factor is a subrelation of the asymmetric factor of CMS.

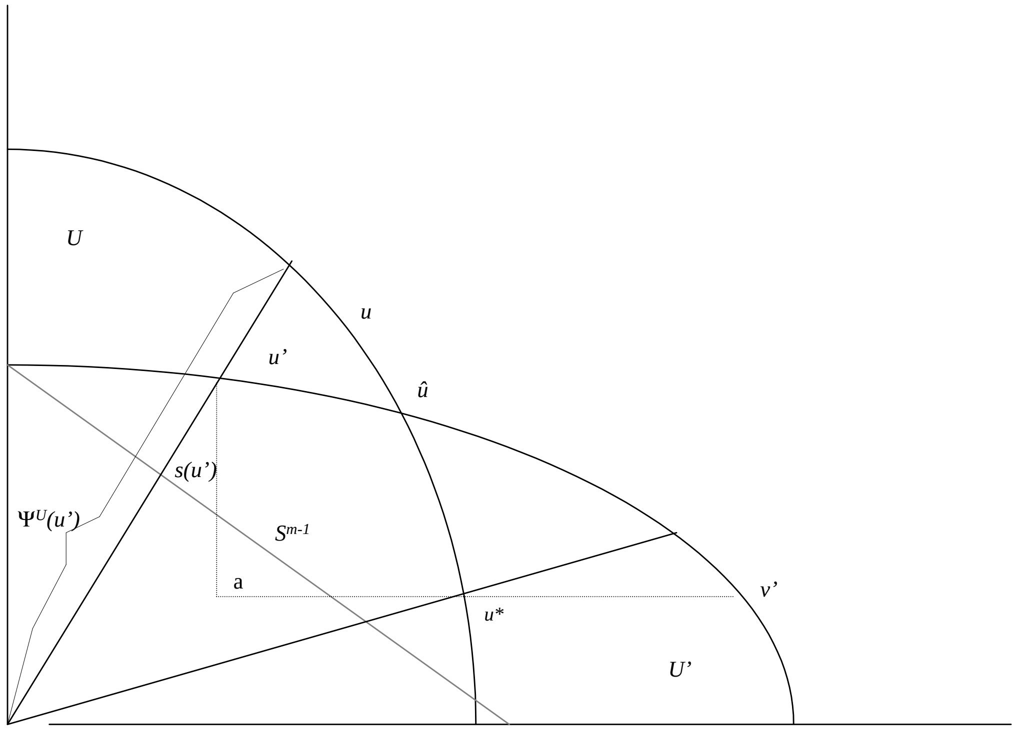
Theorem 3 *Let S be a domain of positions satisfying axioms 1 and 2. Then $\geq$ is asymmetrically redundant with respect to $E(\geq)$ only if $K \subseteq CMS_a$.*

Proof : By contraposition, assume that there are two positions (u, U), (u', U') $\in S$ such that $(u, U)\ K\ (u', U')$ but not $U\ E(\geq)_a\ U'$. From definition 6, there exists some $\widehat{u} \in U$ such that $\widehat{u} > u'$. By clause (ii) of axiom 2, there exists a position $(\overline{u}, \overline{U}) \in S$ with $\overline{u} \in]u', \widehat{u}[$ and $\overline{U} \subseteq U$. From lemma 4, the transitivity of $\geq$ and the fact that $U\ E(\geq)_a\ U'$ is false, $\overline{U}\ E(\geq)_a\ U'$ can not hold. Since $\overline{u} > u'$, this establishes the required violation of the asymmetric redundancy of $\geq$ with respect to $E(\geq)$. QED.

Hence, when K is transitive, it does not offer any more guidance than CMS to compare alternative positions.

A reinterpretation of the condition of asymmetric redundancy of $\geq$ with respect to $E(\geq)$ in terms of Gorman's [14]'s condition of non-crossing of utility possibility frontiers shall now be provided. Formally, the non-crossing condition is defined as follows.

Definition 10 *Two utility possibility sets U and U' have frontiers F and F' that* cross *at point $u \in \Re^m_+$ if $u \in F \cap F'$ and, for some $\varepsilon > 0$, $N_\varepsilon(u) \cap U \backslash U' \neq \emptyset$ and $N_\varepsilon(u) \cap U' \backslash U \neq \emptyset$.*



Theorem 4 *Let S be a domain of positions satisfying axioms 1 and 2. Then $\geq$ is asymmetrically redundant with respect to $E(\geq)$ if and only if there are no positions (u, U) and (u', U') in the domain such that $F(U)$ and $F(U')$ cross at some utility allocation v.*

Proof: (necessity). By contraposition, suppose that (u, U) and (u', U') are two positions such that $F(U)$ and $F(U')$ cross at v. Let $\widehat{u} \in U \backslash U'$ and $\widehat{u}' \in U' \backslash U$. The existence of such $\widehat{u}$ and $\widehat{u}'$ is secured by definition 10 and, without loss of generality by lemma 3, one can assume that $\widehat{u} \in M_{\geq}(U)$ and $\widehat{u}' \in M_{\geq}(U')$. Note that, by lemma 4, we have that $U'\ E(\geq)\ U$ does not hold (and therefore, that $U'\ E(\geq)_a\ U$ does not hold either). By lemma 7, let $\widetilde{u} \in F(U) \cap [0^m, \widehat{u}']$, let $\overline{u} \in]\widetilde{u}, \widehat{u}'[$ and, by clause (iii) of axiom 2, consider the position $(\overline{u}, \overline{U}) \in S$ satisfying $U \subseteq \overline{U}$. Clearly, $\widehat{u}' > \overline{u}$. On the other hand, since $U \subseteq \overline{U}$, one can not have $U'\ E(\geq)_a\ \overline{U}$. Hence $\geq$ is not asymmetrically redundant with respect to $E(\geq)$.

(sufficiency; see figure 2 for an illustration of the argument in $\Re^2_+$) By contraposition suppose that for two positions (u, U), $(u', U') \in S$, one has $u > u'$ but not $U\ E(\geq)_a\ U'$. From definition 1, there exists $v' \in U'$ such that $v \geq v'$ for no $v \in U$. In particular, this is true for u so that $u \geq v'$ does not hold and, by transitivity of $\geq$, $u' \geq v'$ does not hold either. From lemma 3, one can assume that $v' \in M_{\geq}(U')$ which, by lemma 5, amounts to assuming that $v' \in F(U')$. This latter fact, combined with the reflexivity and transitivity of $\geq$ rules out also the possibility for $v' \geq u'$ to hold. Since $\geq$ is reflexive, $v' \notin U$. Let $a \in \Re_+^m$ be the utility allocation defined, for every $i \in M$, by $a_i = \min(u_i, v_i')$. Since it is not the case that $u \geq v'$ or $v' \geq u$ hold, one has that $a < u$ and $a < v^{'}$. Since U is comprehensive, $a \in U$. From lemma 7, there exists $u^* \in [a, v'] \cap F(U)$. By construction, $u^* < v'$. Define now the (obviously continuous) function $s : \Re_+^m \rightarrow S^{m-1}$ by $s(u) = \frac{u}{\sum\limits_{i \in M} u_i}$ and, for any strictly comprehensive and compact subset set Z of $\Re_+^m$, the correspondence $\Phi^Z : S^{m-1} \rightarrow Z$ by $\Phi^Z(x) = \{z \in Z \mid \exists\, t \in \Re_+ \text{ such that } z = tx\}$. This correspondence is examined by Chipman and Moore ([6]; sect. 5; expr (3)). It can be seen easily that, if Z is compact and strictly comprehensive, it satisfies the property, derived from other axioms in Chipman and Moore ([6]; lemma 7), that $\forall\, z \in Z$, $\forall\, \varepsilon \in \Re_{++}$, $\exists\, \overline{z} \in N_\varepsilon(z) \cap Z$ such that $\overline{z} >> 0$. For this reason, the proof provided in ([6]; lemma 8) that the correspondance Φ^Z is continuous (see e.g. Debreu ([11];1.8) applies here. Hence, for a comprehensive and compact subset set Z of $\Re_+^m$ the correspondence $\Psi^Z : \Re_+^m \rightarrow Z$ defined by $\Psi^Z(u) = \Phi^Z(s(u))$ for every $u \in \Re_+^m$ is continuous as is the composition of a continuous correspondence with a continuous function (see Debreu ([11];1.8 *i*)) .Now for a strictly comprehensive and compact $U \subseteq \Re_+^m$, define the mapping $\mu^U : U \rightarrow \Re_+$ by

$$\mu^U(u) = \max_{u' \in \Psi^U(u)} \|u'\|$$

By virtue of the maximum theorem (See Debreu ([11];1.8 *k*)), $\mu_U(.)$ is a continuous function. Define now the function $f : U \cup U^{'} \rightarrow \Re$ by $f(u) = \mu^U(u) - \mu^{U^{'}}(u)$ for $u \in U \cup U^{'}$. It is a continuous function which, by lemma 6, is defined on a connected set $U \cup U^{'}$. Now it is easy to show that $f(u^*) = \|u^*\| - \mu^{U'}(u^*) < 0$ (since $u^* \leq v^{'}, v^{'} \in F(U')$ and $u^* \in U^{'}$). Analogously, $u \geq u^{'}$, together with the fact that $u \in F(U)$ and $u^{'} \in F(U')$, imply that $f(u') = \mu^U(u') - \|u'\| \geq 0$ (> 0 if $u' \neq u$). By the intermediate value theorem, there exists $\widehat{u} \in U \cap U^{'}$ such that $\Psi(\widehat{u}) = 0$. By construction, $\widehat{u}$ is a point where the two frontiers $F(U)$ and $F(U')$ cross. QED

It is worth emphasizing the stringency of the requirement that no utility possibility frontiers cross. Suppose that we interpret each utility possibility set as the image, under a given vector of continuous, monotonically increasing and quasi-concave utility functions, of the set of all allocations, among the n individuals, of some aggregate endowment of k goods (or of some aggregate production set). Within such a framework, it has been shown by Gorman ([13], [15]) that the non-crossing of utility possibility sets is equivalent to assuming that all individuals have quasi-homothetic preferences with parallel linear Engel's curve. There is very little evidence that real households behave as if they had preferences of this type.

4 Conclusion

In this paper, a set of necessary and sufficient conditions for either the CMS criterion to be used, consistently, in accordance with the Pareto principle or the KHSC criterion to be transitive is established. These conditions amount to requiring that, on the domain of alternatives under consideration, the Pareto quasi-ordering must be a subrelation of the CMS one. If this condition holds, the Pareto criterion is useless since it does not add any information to that already embodied in the CMS criterion. When interpreted on a standard economic domain, where the alternatives are utility vectors drawn from compact and strictly comprehensive utility possibility sets, this condition is equivalent to the requirement that no utility possibility frontiers intersect each other. In themselves or in connection with Gorman ([13], [15]) these results indicate that the practice of resorting to hypothetical alternatives to evaluate economic policies is blatantly flawed.

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