Ranking societies with different numbers of individuals: A characterization of the Average Utilitarian criterion*

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Abstract

The paper provides an axiomatic characterization of a family of criteria for comparing societies with a (possibly) *variable* number of individuals. A society is described as a finite ordered list of *vectors*, each such vector being interpreted as providing the values of the normatively relevant attributes of the individual to which it corresponds. Every criterion in the family can be thought of as comparing societies on the basis of their *average utility*, for some utility function defined over the considered attributes. A comparison of our characterization of generalized average utilitarianism with that of Blackorby, Bossert, and Donaldson (1999) obtained in a welfarist setting is also provided.

Keywords: social rankings, variable population, generalized average utilitarianism, axiomatic

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*We are pleased to dedicate this work to Serge-Christophe Kolm. While we are not sure that he will find attractive the proposal for comparing societies with variable population that is examined in this paper, we are certain that our discussions with him on many related issues and our reading of his work has been incredibly influential in shaping our view on the subject. We are also indebted, with the usual disclaiming qualification, to David Donaldson, Maurice Salles and an anonymous referee for their helpful comments.

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1 Introduction

At an abstract level, one can view the various theories of justice that have been discussed in economics and philosophy in the last 50 years or so, including of course that of Serge-Christophe Kolm (2005), as attempts to providing criteria for comparing alternative *societies* on the basis of their "ethical goodness". The compared societies can be truly distinct societies, such as India and China. They can also be the same society examined at two different points of time (say India today and India twenty years ago) or, more counterfactually, before and after a tax reform or demographic shock. As they are envisaged in much theories that we are aware, societies are described as lists of *attributes bundles*, as many bundles as there are individuals in the society. However, the theories and approaches differ markedly with respect to the choice of the individual attributes that are deemed normatively relevant to describe a society. In classical social choice theory, developed along the lines of Arrow (1950) famous impossibility theorem, the only considered attribute is the individual's ordinal preference ordering over social alternatives so that a society can be considered as a list of preference orderings. The scope of classical Arrovian theory was, however, restricted to normative comparisons involving societies with the same number of individuals. In the welfarist approach, defended forcefully by Blackorby, Bossert, and Donaldson (2005), among many others, the relevant attribute is the individual's utility so that a society can be summarized by the list of utility numbers achieved by the individuals. The welfarist tradition has explored with considerable detail the problems raised by comparing societies with different numbers of people (see e.g. Parfit (1984), Blackorby, Bossert, and Donaldson (2005) and Broome (2004) and the literature surveyed herein for examples of welfarist approaches to variable population ethics). As another example of a normative approach for comparing societies, one finds Sen (1985) multidimensional functioning and capabilities theory in which the individual attributes are taken to be various functionings (nutrition, education, health, etc.). As far as we are aware, there have not been many attempts to apply Sen's theory to societies involving different numbers of individuals.

In this chapter, we discuss a general method for comparing all logically conceivable finite lists of attributes bundles, without entering into the (philosophically important) matter of identifying what these attributes are. Taking the relevant individual attributes as given, we formulate several principles that can be used for comparing alternative lists of attributes bundles and we show that these principles characterize a rather specific method for comparing societies. For the criterion that we characterize can be thought of as comparing societies on the basis of their *average utility*, for some utility function defined over the set of attributes bundles.

If we take the view that utility is the only relevant individual attribute so that a society can be seen as a list of utility numbers, then this chapter may be seen as providing an alternative characterization of the family of social orderings called *average generalized utilitarianism* by Blackorby, Bossert, and Donaldson (2005) (p. 171-172, 198). With average general utilitarianism, individual utilities are first transformed by some function before being averaged.

Yet, the main interest of the characterization provided in this chapter is that it applies as well to *non-welfarist* multidimensional contexts where utility is not the only relevant attribute for appraising the situation of an individual. In any such context where the situation of an individual is summarized by a vector of attributes, the properties that we impose on the ranking of all societies happen to force one to aggregate these attributes into a "utility" function and to compare different societies on the basis of their average utility. Of course, when interpreted in this perspective, the "utility function" that aggregates individual's attributes need not be that which corresponds to individual's subjective well-being. It should, rather, be thought of the value assigned to the individual attribute bundle by the social planner or, more generally, by the theoretician of justice. While the main theorem of this chapter does not impose anything on the utility function beside monotonicity, it is quite easy to require it to satisfy additional properties if further assumptions are imposed on the social ordering. Important such properties that can be thought of here are attributes-inequality aversion ones which, if imposed on the social ordering, imply specific additional conditions on the utility function.

The main characterization result that is obtained in this chapter is derived in a straightforward fashion from the characterization of the family of uniform expected utility rankings obtained in Gravel, Marchant, and Sen (2007) in the somewhat different context of individual decision under complete uncertainty. In decisions under complete uncertainty, we are interested in ranking finite sets of consequences. In normative economics, we are interested in ranking finite lists of attribute bundles interpreted as societies. As societies are more structured objects than sets (in particular, it makes sense to duplicate an individual in a society but it does not make sense in general to duplicate an element in a set), the axiomatic structure that characterizes average utilitarianism turns out to be simpler than the one which characterizes uniform expected utility in the complete uncertainty setting considered in Gravel, Marchant, and Sen (2007).

The remaining part of this chapter is organized as follows. In the next section, we introduce the formal model, the main axioms and the criterion used to compare societies. Section 3 provides the main result of the paper and section 4 indicates how some additional restrictions on the utility function can be obtained from imposing additional properties on the social ordering. Section 5 concludes.

2 Notation and basic definitions

2.1 Notation

The sets of integers, non-negative integers, strictly positive integers, real numbers, non-negative real numbers and strictly positive real numbers are denoted by $\mathbb{N}$, $\mathbb{N}_+$, $\mathbb{N}_{++}$, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_{++}$ respectively. The cardinality of any set A is denoted by $\#A$ and the k-fold Cartesian product of a set A with itself is denoted by A^k. The inner product of an $n \times m$ matrix a by an $m \times r$ matrix b is denoted by $a.b$. The k-dimensional unit vector is denoted by 1^k. Given a vector v in $\mathbb{R}^k$ and a positive real number ε, we denote by $N_\varepsilon(v)$ an *ε-neighborhood around* v defined by $N_\varepsilon(v) = \{x \in \mathbb{R}^k :\mid x_h - v_h \mid < \varepsilon$ for all $h = 1, ..., k\}$. If Φ is a function from a subset A of $\mathbb{R}^k$ to $\mathbb{R}$ and a is a vector in A, we denote, for every strictly positive real number Δ such that $(a_1, ..., a_j + \Delta, ..., a_k) \in A$, by $\Phi_j^\Delta(a)$ its (discrete right hand side) jth derivative defined by:

$$\Phi_j^\Delta(\mathbf{a}) = \frac{\Phi(a_1, ..., a_j + \Delta, ..., a_k) - \Phi(\mathbf{a})}{\Delta} \tag{1}$$

A *permutation matrix* π is a square matrix whose entries are either 0 or 1 and sum also to 1 in every line and every column. Our notation for vectors inequalities is $\geqq$, $\geq$ and $>$. A *binary relation* $\succsim$ on a set Ω is a subset of $\Omega \times \Omega$. Following the convention in economics, we write $x \succsim y$ instead of $(x, y) \in \succsim$. Given a binary relation $\succsim$, we define its *symmetric factor* $\sim$ by $x \sim y \Longleftrightarrow x \succsim y$ and $y \succsim x$ and its *asymmetric* factor $\succ$ by $x \succ y \Longleftrightarrow x \succsim y$ and not $(y \succsim x)$. A binary relation $\succsim$ on Ω is *reflexive* if the statement $x \succsim x$ holds for every x in Ω, is *transitive* if $x \succsim z$ always follows $x \succsim y$ and $y \succsim z$ for any x, y, $z \in \Omega$ and is *complete* if $x \succsim y$ or $y \succsim x$ holds for every distinct x and y in Ω. A symmetric, reflexive and transitive binary relation is called an *equivalence relation* and a reflexive, transitive and complete binary relation is called an *ordering.* Given an equivalence relation $\sim$ on Ω, and some $\omega \in \Omega$, we denote by $E_\sim(\omega)$ the equivalence class of ω under $\sim$ defined by $E_\sim(\omega) = \{\omega' \in \Omega \mid \omega' \sim \omega\}$. It can be seen immediately that if $\sim$ is an equivalence relation, one has $E_\sim(\omega) \neq \emptyset$ for every ω, $E_\sim(\omega) = E_\sim(\omega')$ or $E_\sim(\omega) \cap E_\sim(\omega') = \emptyset$ for every elements ω and ω' in Ω and $\bigcup_{\omega \in \Omega} E_\sim(\omega) = \Omega$ so that the equivalence class of all elements of Ω under $\sim$ form a partition of Ω. Such a partition is called the *quotient of* Ω *under* $\sim$.

2.2 Basic definitions

We assume that there are k (with $k \in \mathbb{N}_{++}$) individual attributes that are perfectly measurable. We view a society s as a finite ordered list of vectors in $\mathbb{R}^k$, every such vector being interpreted as the set of values taken by the attribute for the individual to which it corresponds. Let $n(s)$ denote the number of individuals living in society s. Then we can depict any society s as an $n(s) \times k$ matrix:

$$s = \begin{bmatrix} s_{11} & \cdots & s_{1k} \\ \cdots & \cdots & \cdots \\ s_{n(s)1} & \cdots & s_{n(s)k} \end{bmatrix}$$

where s_{ij}, for $i = 1, ..., n(s)$ and $j = 1, ..., k$, is interpreted as the amount of attribute j received by individual i in society s. Denote by s_i the vector of attributes received by i in s. A society s with $n(s)$ individuals is therefore an element of $\mathbb{R}^{n(s)k}$ and the set of all logically conceivable such societies is $\bigcup_{l \in \mathbb{N}_{++}} \mathbb{R}^{lk}$. If s is a society in $\mathbb{R}^{mk}$ and s' is a society in $\mathbb{R}^{nk}$, we denote by (s, s') the society in $\mathbb{R}^{(m+n)k}$ where the m first individuals get, in the same order, the bundles obtained by the corresponding individuals in s and the n last individuals get the bundles obtained, in the same order, by the n individuals in s'. To alleviate notation, we write, for any attribute bundle x in $\mathbb{R}^k$, the one-individual society (x) as x.

We notice that depicting societies as ordered lists of attributes bundles make sense only if we adopt an *anonymous* postulate that "the name of the individuals does not matter". We adopt this postulate throughout, even though we are fully aware that it rides on an implicit assumption that the attributes bundle received by an individual constitutes the *only* information on the individual's situation that is deemed normatively relevant. It is therefore important for the interpretation of our framework that one adopts a wide list of attributes, which could include many consumptions goods, health, education levels and, possibly, Rawlsian primary goods such as the "social basis of self respect". Possibly, one should also view these bundles of goods and primary goods as being distinguished by the time at which they are made available if one wants to adopt a life time perspective. A clear discussion on the explicit modeling of this anonymity postulate within a richer - albeit welfarist - formal framework is provided in Blackorby, Bossert, and Donaldson (2005) (ch. 3.9).

We should also mention that our framework leads one to consider all conceivable societies, including societies made of a single individual. While these kinds of "societies" may be considered to lie outside the realm of theories of justice, there are several instances where normative comparisons of small communities are made. For instance, it is not uncommon in applied welfare economics to compare the well-being of an individual with that of a family. Of course we are doing much more than that here since we also accept to normatively compare a family of three persons with, say, the whole People Republic of China. Notice also that we agree to rank all single-individual societies and thus, implicitly, all bundles of attributes from a social point of view.

If we accept to compare in this fashion all societies, we shall make the comparisons on the basis of a social ordering $\succsim$, with asymmetric and symmetric factors $\succ$ and $\sim$ respectively. We interpret the statement $s \succsim s'$ as meaning "the distribution of the k attributes in society s is at at least as just as the corresponding distribution in society s'". A similar interpretation is given to the statements $s \succ s'$ ("strictly more just") and $s \sim s'$ ("equally just").

In this chapter, we identify the properties (axioms) of the ordering $\succsim$ that are necessary and sufficient for the existence of a (utility) function $u : \mathbb{R}^k \to \mathbb{R}$ such that, for every societies s and s' in $\bigcup_{l \in \mathbb{N}_{++}} \mathbb{R}^{lk}$, one has:

$$s \succsim s' \iff \sum_{i=1}^{n(s)} \frac{u(s_i)}{n(s)} \geq \sum_{i=1}^{n(s')} \frac{u(s'_i)}{n(s')} \tag{2}$$

An ordering satisfying this property could therefore be thought of as resulting from the comparisons of the *average utility* level achieved by individuals in the two compared societies for some utility function. Notice that the utility function that appears in this formula is the same for all individuals, so that all individuals are treated symmetrically. Of course if we put ourselves initially in a welfarist paradigm in which individual attributes are interpreted to be utility levels (in which case $k = 1$), then societies are defined as lists of utility levels and the ordering defined by formula (2) corresponds to what is called "generalized average utilitarianism" by Blackorby, Bossert, and Donaldson (1999). Notice that formula (2) defines in fact a *family* of social criteria, with as many members as there are logically conceivable utility functions. We shall discuss below how one could restrict this family by imposing additional axioms on the social ranking. We refer to any ranking that satisfies (2) for some function u as to an *Average Utilitarian (AU)* ranking.

Let us now introduce the five axioms which, as it will turn out, characterize the AU family of social rankings.

The first axiom is a "minimal increasingness" condition, which says that the k attributes that matter for normative appraisal are positively valued by the social planner. We state formally this axiom as follows.

Axiom 1 *(Minimal increasingness) For all attribute bundles x and $y \in \mathbb{R}^k$, $x \geqq y$ implies $x \succsim y$ and $x \geq y$ implies $x \succ y$.*

This axiom is a slight weakening of the axiom of the same name in Blackorby, Bossert and Donaldson. Of course, Blackorby, Bossert and Donaldson's axiom applies only to the case where there is only one attribute - welfare - while our formulation applies to any number of attributes. In our view, this minimal increasingness axiom is natural if the attributes can be interpreted as primary goods *à la* Rawls (1982) or, using the words of Sen (1987), as things that "people have reasons to value".

The second axiom is also very natural in the setting we are considering, especially in view of the discussion made above, since it requires the social ranking to be *anonymous*. That is, the names of the individuals who receive the attributes do not matter for normative evaluation so that all societies that distribute the same list of attributes among the same number of persons are normatively equivalent. We state formally this axiom, which has been called "same people anonymity" by Blackorby, Bossert and Donaldson, as follows.

Axiom 2 *(Anonymity) For every society* $s \in \bigcup_{l \in \mathbb{N}_{++}} \mathbb{R}^{lk}$ *and all* $n(s) \times n(s)$ *permutation matrix* π*, one has* $\pi.s \sim s$*.*

The third axiom that we impose on $\succsim$ is a continuity condition that is weaker than most conditions of the same kind, since it applies only to sets of attribute bundles which, if received by a single individual, are considered as weakly better than, or weakly worse than, a given society. It says that if a converging sequence of such bundles are considered weakly better (resp. weakly worse) than a given society, then the point of convergence of this sequence should also be considered weakly better (resp. weakly worse) than the considered society.

Axiom 3 *(Continuity) For every society* s*, the sets* $B(s) = \{x \in \mathbb{R}^k : x \succsim s\}$*,* $W(s) = \{x \in \mathbb{R}^k : s \succeq x\}$ *are both closed in* $\mathbb{R}^k$*.*

While weak, this axiom rules out rankings such as the Leximin one which compare ordered lists of bundles by first defining a continuous utility function on those bundles and by ranking vectors of the utilities associated to these bundles by the usual lexicographic ordering of $\bigcup_{l \in \mathbb{N}_{++}} \mathbb{R}^l$ (generalized to account for the different dimensions of the vectors thus compared). It is clear that such a ranking would violate continuity because it would be quite possible to observe a sequence of bundles that are ranked weakly above the worse bundle of a given society and that converge to this worse bundle. Yet, contrary to what is required by continuity, the society containing only one individual endowed with this bundle would be considered strictly worse than the original many individuals society by the leximin criterion.

The next axiom plays a crucial role in the characterization and captures the very idea of averaging numbers for comparing societies. We call it for this reason the *averaging axiom.* This axiom says that merging two distinct societies leads to a larger society that is normatively worse than the best of the two societies and better than the worse of the two societies. It says also, conversely, that if a society looses (gains) from bringing in members with specific attributes endowments, then this can only be because the distribution of attributes that is brought in is worse (better) than that already present in the original society. Using the language of population ethics, this axiom says that bringing in new members with specific endowments of attributes is worth doing if and only if the added distribution of endowment is better than the original one. This axiom is formally stated as follows.

Axiom 4 *(Averaging) For all societies* s *and* $s' \in \bigcup_{l \in \mathbb{N}_{++}} \mathbb{R}^{lk}$, $s \succsim s' \Leftrightarrow s \succsim (s, s') \Leftrightarrow (s, s') \succsim s'$.

When applied to an ordering, the averaging axiom implies some other properties that have been considered in the literature on population ethics. One of them is the axiom called "replication equivalence" by Blackorby, Bossert, and Donaldson (2005) (p. 197) and which states that, for societies where everyone gets the same attributes bundle, the *number* of society's members does not matter. This property clearly rules out social preferences of the type "small is beautiful" or, conversely, the biblical "be fertile and multiply". We state formally this property as follows.

Condition 1 *(Replication equivalence) For every attributes bundle* $x \in \mathbb{R}^k$ *and all societies* $s \in \bigcup_{l \in \mathbb{N}_{++}} \mathbb{R}^{lk}$ *such that* $s_i = x$ *for all* $i = 1, ..., n(s)$, *one has* $s \sim x$.

This condition is implied by averaging if $\succsim$ is reflexive. The proof of this claim is left to the reader.

The next, and last, axiom that we consider is a strengthening of an axiom called "same number independence" by Blackorby, Bossert, and Donaldson (2005). Our axiom, which we call "same number enlargement consistency", asserts that normative judgements involving societies with the same number of people can be combined consistently. Specifically, it requires the ranking of any two societies with the same number of people to be robust to an enlargement of the two societies by the same number of individuals when the distribution of attributes among these new persons are ranked, at least weakly, in the same way than the ranking of the two initial societies. Formally, this axiom is stated as follows.

Axiom 5 *(Same number enlargement consistency) For all societies* s, s', s'' *and* $s''' \in \bigcup_{l \in \mathbb{N}_{++}} \mathbb{R}^{lk}$ *such that* $n(s) = n(s')$, $n(s'') = n(s''')$ *and* $s'' \succsim s'''$, $s \succsim s'$ *implies* $(s, s'') \succsim (s', s''')$ *and* $s \succ s'$ *implies* $(s, s'') \succ (s', s''')$.

When applied to an anonymous social ordering, this axiom implies a condition that could be called, following Blackorby, Bossert, and Donaldson (2005) (p. 159), "same number existence independence". This condition says that, in comparing societies with identical number of individuals, we should only focus on the distribution of attribute over the subset of individuals whose attributes bundle differ. Individuals that do not experience any change in their attribute bundles, and who are therefore "unconcerned" by the change, should not matter for the normative comparison of the two societies. The existence, or non existence, of these individuals should therefore not affect the ranking of the societies. We state this condition formally as follows.

Condition 2 *(Same number existence independence) For all societies s, s' and s" $\in \bigcup_{l \in \mathbb{N}_{++}} \mathbb{R}^{lk}$ such that $n(s) = n(s')$, $(s, s'') \succsim (s', s'')$ if and only if $s \succsim s'$.*

Let us show that an anonymous social ordering that satisfies the same number enlargement consistency axiom also satisfies the same number existence independence condition.

Claim 1 *Let $\succsim$ be an ordering on $\bigcup_{l \in \mathbb{N}_{++}} \mathbb{R}^{lk}$ satisfying same number enlargement consistency and anonymity. Then $\succsim$ satisfies same number existence independence.*

Proof. *Assume that $\succsim$ is an anonymous ordering that satisfies same number enlargement consistency and consider societies s, s' and $s'' \in \bigcup_{l \in \mathbb{N}_{++}} \mathbb{R}^{lk}$ such that $n(s) = n(s')$ and $s \succsim s'$. Since $\succsim$ is reflexive, $s'' \succsim s''$ and, by application of the first part of the same number enlargement consistency axiom, one has $(s, s'') \succsim (s', s'')$. In the other direction, assume that $s \succsim s'$ does not hold. Since $\succsim$ is complete, this means that $s' \succ s$. Since $s'' \succsim s''$ by reflexivity, we obtain, using the second part of the same number enlargement consistency axiom, that $(s'', s') \succ (s'', s)$ holds and, since $\succsim$ is anonymous, that $(s', s'') \succ (s, s'')$. Hence $(s, s'') \succsim (s', s'')$ does not hold and this completes the proof.* ■

It can be checked that any AU ranking satisfies anonymity, continuity, averaging and same number enlargement consistency. In the next section, we establish the converse proposition that any minimally increasing, continuous and anonymous ranking that satisfies averaging and same number enlargement must be a AU ranking.

3 Main Result

The result that we are going to establish rides heavily on the companion paper Gravel, Marchant, and Sen (2007) that characterizes a *uniform expected utility* criterion in a somewhat different - but formally close - setting of choice under complete uncertainty. In Gravel, Marchant, and Sen (2007), the objects that are compared are finite sets of consequences instead of vectors of attributes bundles.

The first step in establishing the result consists in showing that if a social ordering $\succsim$ satisfies the five axioms above, then it satisfies the three following conditions.

Condition 3 *(Existence of critical levels) For every society* $s \in \bigcup_{l \in \mathbb{N}_{++}} \mathbb{R}^{lk}$*, there exists an attribute bundle* $x \in \mathbb{R}^k$ *such that* $x \sim s$*.*

Condition 4 *(richness) For all societies* s *and* $s' \in \bigcup_{l \in \mathbb{N}_{++}} \mathbb{R}^{lk}$*, if there are bundles* x^* *and* x_* *in* $\mathbb{R}^k$ *such that* $(s, x^*) \succsim s' \succsim (s, x_*)$*, then there exists a bundle* $x \in \mathbb{R}^k$ *such that* $(s, x) \sim s'$*.*

Condition 5 *(Archimedean) Let* $\{x^h\}$*,* $h = 1, 2...$ *be a sequence of attributes bundles in* $\mathbb{R}^k$ *and let* y *and* z *be attributes bundles in* $\mathbb{R}^k$ *such that* $y \succ z$ *and* $(x^h, y) \sim (x^{h+1}, z)$ *for all* h*,* $h + 1$ *with* $h = 1, 2....$ *If the sequence is strictly bounded by bundles* $\underline{w}$ *and* $\overline{w}$ *in* $\mathbb{R}^k$ *in the sense that* $\underline{w} \succ x^h \succ \overline{w}$ *for every* h*, then the sequence must be finite.*

In words, the first condition says that, for every society s, it is always possible to find an attribute bundle that, if given to a single individual, would be socially equivalent to s. This condition is very closely related (and under averaging, is identical) to the axiom called "existence of critical levels" by Blackorby, Bossert, and Donaldson (2005) (p. 160) in their welfarist framework. For it says indeed that, for every society, there exists an attribute bundle which, if received by an individual, will make the social planner indifferent between adding this individual with the concerned bundle to the society and not bringing the individual into existence. If the bundle is one-dimensional, and interpreted as utility, then this bundle, which is in fact a single number, is called a "critical level" of utility in the population ethics literature. For obvious reasons, this axiom was called "*certainty equivalent*" in our paper on uncertainty.

The second condition says that the universe is sufficiently rich to enable, by suitable addition of one individual with a specific bundle to a society, various kinds of comparisons with the ordering $\succsim$.

Finally, the Archimedean condition has been widely discussed in the measurement theory literature (see e.g. Krantz, Luce, Suppes, and Tversky (1971)). It can be considered to be a mild condition since it "bites" only when there exists sequences of the type described by the antecedent clause of this axiom (such sequences are called "standard sequences" in the measurement theory literature).

We now state formally that these three conditions are implied by monotonicity, anonymity, continuity, averaging and same number existence independence (the later axiom being implied by same number expansion consistency). We provide the proof for completeness but we notice that it mimics quite closely the proof of theorem 3 in Gravel, Marchant, and Sen (2007).

Proposition 1 *Let $\succsim$ be minimally increasing, anonymous and continuous ordering of $\bigcup_{l \in \mathbb{N}_{++}} \mathbb{R}^{lk}$ that satisfies averaging and same number existence independence. Then $\succsim$ satisfies conditions 3, 4 and 5.*

Proof. *We first notice that, under averaging, if the sets $B(s) = \{x \in \mathbb{R}^k : x \succsim s\}$ and $W(s) = \{x \in \mathbb{R}^k : s \succeq x\}$ are closed in $\mathbb{R}^k$ for every society s, then so are the sets $\widetilde{B}(s) = \{x \in \mathbb{R}^k : (s,x) \succsim s\}$ and $\widetilde{W}(s) = \{x \in \mathbb{R}^k : s \succeq (s,x)\}$. To see this, assume by contraposition that, say, $\widetilde{B}(s)$ is not closed (the argument for $\widetilde{W}(s)$ is similar) . Then, there exists a sequence $\{x^t\}$ $t = 1,...$converging to some limit x in $\mathbb{R}^k$ such that:*

$$(s, x^t) \succsim s$$

for all t and

$$s \succ (s, x)$$

where the last strict ranking is obtained from the assumption that $\succsim$ is complete. But by averaging this implies the existence of a sequence $\{x^t\}$ $t = 1,...$converging to some limit x in $\mathbb{R}^k$ such that:

$$x^t \succsim s$$

and:

$$s \succ x \tag{3}$$

in violation of the closedness of the set $B(s)$. Let us now prove that $\succsim$ satisfies conditions 3, 4 and 5.

Condition 3 (existence of critical levels). *Consider any society $s \in \bigcup_{l \in \mathbb{N}_{++}} \mathbb{R}^{lk}$ and, without loss of generality because of anonymity, write it as $s = (s_1, ..., s_{n(s)})$ with $s_h \precsim s_{h+1}$ for $h = 1, ..., n(s) - 1$. By averaging, one has that $s \succsim s_1$ and $s_{n(s)} \succsim s$ so that none of the closed sets $\{x \in \mathbb{R}^k : x \succsim s\}$ and $\{x \in \mathbb{R}^k : s \succeq x\}$ is empty. Since $\succsim$ is complete, $\mathbb{R}^k = \{x \in \mathbb{R}^k : x \succsim s\} \cup \{x \in \mathbb{R}^k : s \succeq x\}$. Since $\mathbb{R}^k$ is an arc connected set, there exists a continuous function $f : [0,1] \to \mathbb{R}^k$ such that $f(0) = s_1$ and $f(1) = s_{n(s)}$. By continuity, given the closedness of $B(s) = \{x \in \mathbb{R}^k : x \succsim s\}$ and $W(s) = \{x \in \mathbb{R}^k : s \succeq \{x\}\}$, there must be some $\alpha \in [0,1]$ such that*

$f(\alpha) \in B(s) \cap W(s)$. By definition $f(\alpha) \sim s$.

Condition 4 (richness). *Consider societies s and s' in $\bigcup_{l \in \mathbb{N}_{++}} \mathbb{R}^{lk}$ and bundles x^* and $x_* \in \mathbb{R}^k$ such that $(s', x^*) \succsim s \succsim (s', x_*)$. If either $(s,' x^*\} \sim s$ or $s \sim (s', x_*)$, then richness is satisfied and there is nothing to be proved. Assume therefore that*

$$(s', x^*) \succ s \succ (s', x_*) \tag{4}$$

holds and let $\mathbf{1}^k$ denote the unit vector in $\mathbb{R}^k$. Since $\succsim$, when restricted to $\mathbb{R}^k$, is continuous and increasing, there exists, for each $x \in \mathbb{R}^k$, a real number $v(x)$ such that $v(x).\mathbf{1}^k \sim x$. Since $\succsim$ satisfies same number existence independence, one has:

$$(s', v(x^*).\mathbf{1}^k) \sim (s', x^*) \succ s \succ (s, x_*) \sim (s, v(x_*).\mathbf{1}^k) \tag{5}$$

Define then $h(x)$ by:

$$h(x).\mathbf{1}^k \sim (s, x.\mathbf{1}^k)$$

The number $h(x)$ exists by virtue of the fact that, as we have just checked, $\succsim$ satisfies existence of critical level. Moreover $h(x)$ is clearly unique if $\succsim$ monotonic. Hence h is a function from $\mathbb{R}^k$ to $\mathbb{R}$. It must also be a continuous function if, as established above, $\widetilde{B}(s)$ and $\widetilde{W}(s)$ are both closed in X for any s. Let now $v(s)$ be defined by:

$$(v(s).\mathbf{1}^k) \sim s$$

Because of (4) and (5), one has that $h(x^*) > v(s) > h(x_*)$. Since $\mathbb{R}^k$ is arc connected, let f be a continuous function from $[0,1]$ to X satisfying $f(0) = x_*$ and $f(1) = x^*$. Consider now the function $\Psi : [0,1] \to [h(x_*), h(x^*)]$ defined by:

$$\Psi(\alpha) = h(f(\alpha))$$

This function, which composes two continuous function is continuous and satisfies $\Psi(0) = h(x_*)$ and $\Psi(1) = h(x^*)$. By the intermediate value theorem, there must be some $\alpha \in [0,1]$ such that $\Psi(\alpha) = h(f(\alpha)) = v(s)$. By definition of f and h, this implies the existence of some $x \in \mathbb{R}^k$ satisfying $h(x) = v(s)$ such that

$$s \sim (h(x).\mathbf{1}^k) \sim (s', x.\mathbf{1}^k)$$

Condition 5 (Archimedean). *If it is impossible to construct a standard sequence as in the antecedent clause of the Archimedean condition, then the proof is (trivially) over. Assume therefore that such a sequence exists and, therefore that y and z be two bundles of k goods such that $y \succ z$ for which one has, for a sequence of bundles $\{x_t\}_{t \in \mathbb{N}_+}$:*

$$(x_t, y) \sim (x_{t+1}, z) \tag{6}$$

for every $t = 0, \ldots$. By Same number existence independence, we must have $x_{t+1} \succ x_t$ for all t. Using again same number existence independence, one has:

$$(z_t, y) \sim (z_{t+1}, z)$$

for every bundles z_t and z_{t+1} in $\mathbb{R}^k_+$ such that $z_t \sim x_t$ and $z_{t+1} \sim x_{t+1}$. Since $\succsim$ restricted to $\mathbb{R}^k_+$ is continuous, there exists, for every $t = 0, \ldots,$, a real number α_t such that:

$$\alpha_t.\mathbf{1}^k \sim x_t$$

Hence the existence of a sequence of bundles $\{x_t\}_{t \in \mathbb{N}_+}$ satisfying (6) implies the existence of a sequence of real numbers $\{\alpha_t\}_{t \in \mathbb{N}_+}$ such that:

$$(\alpha_t.\mathbf{1}^k, y) \sim (\alpha_{t+1}.\mathbf{1}^k, z) \tag{7}$$

Since $x_{t+1} \succ c_t$ for all t, we must have, since the ordering $\succsim$ is increasing, that $\alpha_{t+1} > \alpha_t$ for all t. As the increasing sequence $\{x_t\}_{t \in \mathbb{N}_+}$ is initiated somewhere, it is bounded from below so that there exists some bundle x for which $x_t \succ x_0 \succsim x$. Now, since every increasing sequence of numbers that is bounded from above is either convergent or finite, the only thing we need to check is that the sequence is not convergent. Suppose by contradiction that the sequence $\{\alpha_t\}$ is infinite and converges to some number $\overline{\alpha}$. By same number existence independence, we know that:

$$(\overline{\alpha}.\mathbf{1}^k, y) \succ (\overline{\alpha}.\mathbf{1}^k, z)$$

By continuity, there exists a number $\varepsilon > 0$ such that:

$$(\alpha'.\mathbf{1}^k, y) \succ (\alpha''.\mathbf{1}^k, z)$$

for all α' and $\alpha'' \in N_\varepsilon(\overline{\alpha})$. Assuming the sequence $\{\alpha_t\}$ to be converging to $\overline{\alpha}$ implies the existence of some positive integer s such that, for all $t \geq s$, one has $\alpha_t \in N_\varepsilon(\overline{\alpha})$. By the continuity condition, we must therefore have:

$$(\alpha_t.\mathbf{1}^k, y) \succ (\alpha_{t+1}.\mathbf{1}^k, z)$$

for any such t, which contradicts the definition of α_t provided by (7). Hence the increasing sequence $\{\alpha_t\}$ is not convergent. ■

The next step in establishing our result consists in showing that an analogue of a technical condition, established in lemma 1 of Gravel, Marchant, and Sen (2007), is satisfied in the current setting. The condition we are after is the following.

Condition 6 *For all attributes bundles w, w', x, x', y, y', z and z' in $\mathbb{R}^k$ if $(w, w') \sim y$, $(x, x') \sim y'$, $(w, x) \sim z$ and $(w', x') \sim z'$, then $(y, y') \sim (z, z')$.*

This condition is not terribly intuitive. Yet, it is implied by the same number enlargement consistency axiom, at least when the social ordering to which the axiom applies is anonymous. The next claim establishes this.

Claim 2 *Let $\succsim$ be an anonymous ordering on $\bigcup_{l\in\mathbb{N}_{++}} \mathbb{R}^{lk}$ satisfying same number enlargement consistency. Then $\succsim$ satisfies condition 6.*

Proof. *Assume that $\succsim$ is an ordering that satisfies same number enlargement consistency and consider attributes bundles w, w', x, x', y, y', z and z' in $\mathbb{R}^k$ such that $(w, w') \sim y$, $(x, x') \sim y'$, $(w, x) \sim z$ and $(w', x') \sim z'$. By same number enlargement consistency, we must have :*

$$(w, w', x, x') \sim (y, y')$$

and

$$(w, x, w', x') \sim (z, z').$$

Since, by anonymity, $(w, x, w', x') \sim (w, x, w', x')$, we therefore have by transitivity that $(y, y') \sim (z, z')$, as required by condition 6. ■

Endowed with this result, we can now adapt Theorem 1 in Gravel, Marchant, and Sen (2007) to our current context and prove that, for a continuous, monotonic and anonymous ordering defined over the set $\mathbb{R}^k \cup \mathbb{R}^{2k}$ of all one and two-individual societies, the axiom of averaging and same number expansion consistency characterize the AU family of social orderings. As in Gravel, Marchant, and Sen (2007), this result relies on an important representation theorem of Krantz, Luce, Suppes, and Tversky (1971) (Theorem 10, p. 295).

Theorem 1 *Let $\succsim$ be an ordering on $\mathbb{R}^k \cup \mathbb{R}^{2k}$ satisfying minimal increasingness, anonymity, continuity, averaging and same number expansion consistency. Then $\succsim$ is a AU social ordering. Furthermore, the u function in the definition of a AU social ordering is unique up to a positive affine transformation and is an increasing function of its k arguments.*

Proof. *let Y be the quotient of $\mathbb{R}^k$ under the equivalence relation $\sim$. Hence $Y = \{\underline{a} \subset \mathbb{R}^k : x, y \in \underline{a}$ if and only if $x \sim y\}$. Define the binary operation $\circ$ on Y by $\underline{a} \circ \underline{b} = \underline{c}$ iff there are $x \in \underline{a}$, $y \in \underline{b}$ and $z \in \underline{c}$ such that $(x, y) \sim z$. We note first that $\underline{a} \circ \underline{a} = \underline{a}$ by virtue of the replication equivalence condition (that is satisfied by the ordering thanks to the remark made above), which requires that $(x, x) \sim x$. For this reason, $\circ$ is an idempotent binary operation. We first show that the binary operation $\circ$ is well-defined in the sense that, for any two (not necessarily distinct) equivalence classes $\underline{a}$ and $\underline{b}$ in Y, there exists a unique $\underline{c} \in Y$ such that $\underline{a} \circ \underline{b} = \underline{c}$. Assume first that $\underline{a} \neq \underline{b}$. Since, thanks to proposition 1, the ordering $\succsim$ satisfies condition 3,* *there exists a bundle* $z \in \mathbb{R}^k$ *such that, for* $x \in \underline{a}$, $y \in \underline{b}$ *with* $x \not\sim y$ *one has* $(x, y) \sim z$. *Suppose by contradiction that there are also* $x' \in \underline{a}, y' \in \underline{b}$ *and* $z' \notin \underline{c}$ *such that* $(x', y') \sim z'$. *If this was the case, we would have* $(x, y) \sim z \not\sim z' \sim (x', y')$. *By transitivity, we would have* $(x, y) \not\sim (x', y')$ *but by the composition consistency condition, satisfied thanks to claim 2, we should have* $(x, y) \sim (x', y')$, *a contradiction. Consider now the case where* $\underline{a} = \underline{b}$. *Suppose by contradiction, that there are attributes bundles* x, y *and* $z \in \mathbb{R}^k$ *with* $x \sim y$ *such that* $(x, y) \sim z$ *and* $x \not\sim z$. *This obviously contradicts averaging. We note also that, thanks to the anonymity axiom, the binary operation* $\circ$ *is commutative. The next step in the proof consists in verifying that the triple* $\langle Y, \widehat{\succsim}, \circ \rangle$ *(where,* $\widehat{\succsim}$ *is the ordering of the equivalence classes as per the ordering* $\succsim$ *applied to their elements) satisfies all conditions of a bisymmetric structure as defined in Krantz, Luce, Suppes, and Tversky (1971) (p. 294, definition 10). As in Gravel, Marchant, and Sen (2007), we omit the details of the verification but we note that both the properties of monotonicity and bisymmetry in Definition 10 in Krantz, Luce, Suppes, and Tversky (1971) are implied, given the definition of* $\circ$, *by the axiom of same number expansion consistency and composition consistency. We note also that the property of Restricted Solvability in Krantz, Luce, Suppes, and Tversky (1971) is an immediate consequence of condition 4. Hence Theorem 10 of Krantz, Luce, Suppes, and Tversky (1971) (p.295) applies to this structure so that there exists a mapping* $v : Y \to \mathbb{R}$ *such that* $\underline{a} \succsim \underline{b}$ *iff* $v(\underline{a}) \geq v(\underline{b})$ *and* $v(\underline{a} \circ \underline{b}) = \alpha v(\underline{a}) + \beta v(\underline{b}) + \gamma$. *Moreover, by clause (iii) of theorem 10 of Krantz, Luce, Suppes, and Tversky (1971), the function* v *is unique up to a positive affine transform. Since the binary operation* $\circ$ *is commutative, we must have* $\alpha = \beta$. *Since the binary operation is idempotent, we must have* $\gamma = 0$ *and* $\alpha = \beta = 1/2$. *Define now the function* $u : \mathbb{R}^k \to \mathbb{R}$ *by:* $u(x) = v(\underline{a})$ *for all* $x \in \mathbb{R}^k$. *The function* u *obviously numerically represents* $\succsim$ *as per (2). And it must be an increasing function of its* k *argument if the ordering* $\succsim$ *is minimally increasing.* ■

The last step of our task is to extend Theorem 1 to societies with an arbitrary (but finite) number of individuals. In Gravel, Marchant, and Sen (2007), we did this by means of an extra axiom that we called "attenuation". As it happens, this attenuation condition is satisfied in the current setting by any anonymous, increasing and continuous ordering of $\bigcup_{l \in \mathbb{N}_{++}} \mathbb{R}^{lk}$ satisfying averaging and same number existence independence. The attenuation condition that plays a role in the proof of theorem 2 in Gravel, Marchant, and Sen (2007) can be written, in the current context, as follows.

Condition 7 *(Attenuation) For all societies* s, s' *and* $s'' \in \bigcup_{l \in \mathbb{N}_{++}} \mathbb{R}^{lk}$ *satisfying* $s \sim s'$ *and* $n(s) > n(s')$, $s" \succ s$ *implies* $(s', s'') \succ (s, s'')$ *and* $s \succ s"$

implies $(s, s'') \succ (s', s'')$.

This condition says that the positive or negative impact of bringing in a society a new collection of individuals should decrease with society's size. That is, the size of the society should *attenuate* the impact, positive or negative, of a given population enlargement.

We now prove that attenuation is implied by the axioms presented in the previous section.

Proposition 2 *Let* $\succsim$ *be an ordering on* $\mathbb{R}^k \cup \mathbb{R}^{2k}$ *satisfying minimal increasingness, anonymity, continuity, averaging and same number expansion consistency. Then* $\succsim$ *satisfies attenuation.*

Proof. *Consider societies* s, s' *and* $s'' \in \bigcup_{l \in \mathbb{N}_{++}} \mathbb{R}^{lk}$ *satisfying* $s \sim s'$, $n(s) > n(s')$ *and* $s" \succ s$. *By virtue of proposition 1,* $\succsim$ *satisfies condition 3 and, therefore, there exists some bundle* x *such that* $s' \sim x$. *By replication equivalence and transitivity, we have:*

$$s' \sim x \sim \underbrace{(x, ..., x)}_{n(s)-n(s') \text{ times}}$$

and, by averaging and transitivity, it follows that:

$$s \sim (s', \underbrace{x, ..., x}_{n(s)-n(s') \text{ times}})$$

Now, by the axiom of same number existence independence (implied by same number expansion consistency thanks to claim 1), we must have:

$$(s, s'') \sim (s', \underbrace{x, ..., x,}_{n(s)-n(s') \text{ times}} s'')$$

Now, since $s" \succ s \sim s' \sim (x, ..., x)$, *it follows by averaging and anonymity that:*

$$(s', s'') \succ (s', \underbrace{x, ..., x,}_{n(s)-n(s') \text{ times}} s'') \sim (s, s'')$$

as required, given transitivity, by attenuation. The proof of the other part of attenuation is similar. ■

Given this, we can extend Theorem 1 to societies with an arbitrary (but finite) number of individuals. We do this in the next theorem, which we prove, as in Gravel, Marchant, and Sen (2007), in a couple of steps.

Theorem 2 *Let* $\succsim$ *be an ordering on* $\bigcup_{l \in \mathbb{N}_{++}} \mathbb{R}^{lk}$ *satisfying minimal increasingness, anonymity, continuity, averaging and same number expansion consistency. Then* $\succsim$ *is a AU social ordering. Furthermore, the u function in the definition of a AU social ordering is unique up to a positive affine transformation and is an increasing function of its* k *arguments.*

A key ingredient in the proof of this theorem is the following lemma, proved in the appendix of Gravel, Marchant, and Sen (2007), and which provides a way to write any finite arithmetic mean of k numbers as a recursive combination of arithmetic means of two numbers.

Lemma 1 *Let* $U = \{u_1, ..., u_n\}$ *be a set of* n *numbers such that* $u_1 \leq u_2 \leq ... \leq u_n$ *with arithmetic mean* $\overline{u}$*. Define the* $n-1$ *sequences* $\{b_h^i\}$, $i = 1, 2, ...$ *and* $h = 1, ..., n-1$ *by:*

$$b_{n-1}^0 = (u_n + u_{n-1})/2,$$

$$b_h^0 = (u_h + b_{h+1})/2$$

for $h = 1, ..., n-2$ *and for* $i = 1, 2, ...$*:*

$$\begin{aligned} b_1^{2i-1} &= b_1^{2i-2}, \\ b_h^{2i-1} &= \frac{b_{h-1}^{2i-1} + b_h^{2i-2}}{2} \text{ for } h = 2, ..., n-1, \\ b_{n-1}^{2i} &= b_{n-1}^{2i-1} \text{ and} \\ b_h^{2i} &= \frac{b_h^{2i-1} + b_{h+1}^{2i}}{2} \text{ for } h = 1, ..., n-2. \end{aligned}$$

Then:

$$\lim_{i \to \infty} b_h^i = \overline{u} \text{ for all } h = 1, ..., n-1$$

Endowed with this lemma, we can proceed just as in Gravel, Marchant, and Sen (2007), by proving the analogues, in the current context, of lemmas 3-6 in Gravel, Marchant, and Sen (2007). We remark that the proof of these lemmas in Gravel, Marchant, and Sen (2007) uses an axiom called "unboundedness". In words, this axiom says that the domain of objects ranked by the ordering $\succsim$ is such that, for any object in it, it is always possible to find another object that is strictly preferred. It is immediate to verify that the ordering examined in this chapter verifies this axiom if it is minimally increasing and admits $\bigcup_{l \in \mathbb{N}_{++}} \mathbb{R}^{lk}$ as a domain. We do not provide the details of the translation of the lemmas 3-6 in Gravel, Marchant, and Sen (2007) and their proof as this translation is straightforward.

We end this section by establishing, in the following remark, that, in the class of anonymous, continuous and monotonic orderings of $\bigcup_{l \in \mathbb{N}_{++}} \mathbb{R}^{lk}$, the axioms of averaging and same number expansion consistency are independent.

Remark 1 *When imposed on an anonymous, monotonic and continuous ordering $\succsim$ on $\bigcup_{l \in \mathbb{N}_{++}} \mathbb{R}^{lk}$, the axioms of averaging and same number expansion consistency are independent.*

Proof. *Let $\succsim$ be defined by:*

$$s \succsim s' \Leftrightarrow \frac{\sum_{i=1}^{n(s)} (\sum_{j=1}^{k} s_{ij})}{\sum_{i=1}^{n(s)} \frac{1}{\sum_{j=1}^{k} s_{ij}}} \geq \frac{\sum_{i=1}^{n(s')} (\sum_{j=1}^{k} s'_{ij})}{\sum_{i=1}^{n(s')} \frac{1}{\sum_{j=1}^{k} s'_{ij}}}. \tag{8}$$

This ordering is clearly continuous, monotonic and anonymous. In order to prove that it satisfies averaging, we notice that this ordering is a member of the family of orderings of $\bigcup_{l \in \mathbb{N}_{++}} \mathbb{R}^{lk}$ that can be written as:

$$s \succsim s' \Leftrightarrow \frac{\sum_{i=1}^{n(s)} \rho(s_i) u(s_i)}{\sum_{i=1}^{n(s)} \rho(s_i)} \geq \frac{\sum_{i=1}^{n(s')} \rho(s'_i) u(s'_i)}{\sum_{i=1}^{n(s')} \rho(s_i)}. \tag{9}$$

for some real valued functions ρ and u having $\mathbb{R}^k$ as domain. This family of orderings is a generalization of the average utilitarian one in which individual utilities are weighted by a function ρ that depends upon the attributes bundle before being summed. The average utilitarian family corresponds to the case where the function ρ is defined by $\rho(x) = 1$ for every $x \in \mathbb{R}^k$. The case considered in (8) is that where, for every x in $\mathbb{R}^k$:

$$u(x) = [\sum_{j=1}^{k} x_j]^2 \text{ and}$$

$$\rho(x) = \frac{1}{\sum_{j=1}^{k} x_j}$$

Let us check that any ordering that can be written as per (9) for some functions ρ and u having $\mathbb{R}^k$ as domain satisfies averaging. Let s and s' be two societies in $\bigcup_{l\in\mathbb{N}_{++}} \mathbb{R}^{lk}$ and assume first that $s \succsim s'$. Then, by definition of $\succsim$, one has:

$$\frac{\sum_{i=1}^{n(s)}\rho(s_i)u(s_i)}{\sum_{i=1}^{n(s)}\rho(s_i)} \geq \frac{\sum_{i=1}^{n(s')}\rho(s_i')u(s_i')}{\sum_{i=1}^{n(s')}\rho(s_i')}$$

and, by reflexivity:

$$\frac{\sum_{i=1}^{n(s)}\rho(s_i)u(s_i)}{\sum_{i=1}^{n(s)}\rho(s_i)} \geq \frac{\sum_{i=1}^{n(s)}\rho(s_i)u(s_i)}{\sum_{i=1}^{n(s)}\rho(s_i)}$$

Multiplying the first inequality by $(1-\lambda)$, the second by λ (for $0 < \lambda < 1$) and summing the two yields:

$$\frac{\sum_{i=1}^{n(s)}\rho(s_i)u(s_i)}{\sum_{i=1}^{n(s)}\rho(s_i)} \geq \lambda\frac{\sum_{i=1}^{n(s)}\rho(s_i)u(s_i)}{\sum_{i=1}^{n(s)}\rho(s_i)} + (1-\lambda)\frac{\sum_{i=1}^{n(s')}\rho(s_i')u(s_i')}{\sum_{i=1}^{n(s')}\rho(s_i')}$$

If one chooses now λ such that

$$\lambda = \frac{\sum_{i=1}^{n(s)}\rho(s_i)}{\sum_{i=1}^{n(s)}\rho(s_i) + \sum_{i=1}^{n(s')}\rho(s_i')}$$

one is immediately led to the conclusion that:

$$\frac{\sum_{i=1}^{n(s)}\rho(s_i)u(s_i)}{\sum_{i=1}^{n(s)}\rho(s_i)} \geq \frac{\sum_{i=1}^{n(s)}\rho(s_i)u(s_i) + \sum_{i=1}^{n(s')}\rho(s_i')u(s_i')}{\sum_{i=1}^{n(s)}\rho(s_i) + \sum_{i=1}^{n(s')}\rho(s_i')}$$

and, therefore, that $s \succsim (s, s')$*. Similarly, it is immediate to see that* $s \succsim s'$ *implies* $(s, s') \succsim s'$*. Assume now that* $s \succsim (s, s')$ *and, therefore, that:*

$$\frac{\sum_{i=1}^{n(s)} \rho(s_i) u(s_i)}{\sum_{i=1}^{n(s)} \rho(s_i)} \geq \frac{\sum_{i=1}^{n(s)} \rho(s_i) u(s_i) + \sum_{i=1}^{n(s')} \rho(s'_i) u(s'_i)}{\sum_{i=1}^{n(s)} \rho(s_i) + \sum_{i=1}^{n(s')} \rho(s'_i)}$$

or:

$$\frac{\sum_{i=1}^{n(s)} \rho(s_i) u(s_i)}{\sum_{i=1}^{n(s)} \rho(s_i)} [1 - \frac{\sum_{i=1}^{n(s)} \rho(s_i)}{\sum_{i=1}^{n(s)} \rho(s_i) + \sum_{i=1}^{n(s')} \rho(s'_i)}] \geq \frac{\sum_{i=1}^{n(s')} \rho(s'_i) u(s'_i)}{\sum_{i=1}^{n(s)} \rho(s_i) + \sum_{i=1}^{n(s')} \rho(s'_i)}$$

$$\Leftrightarrow$$

$$\frac{\sum_{i=1}^{n(s)} \rho(s_i) u(s_i)}{\sum_{i=1}^{n(s)} \rho(s_i)} \geq \frac{\sum_{i=1}^{n(s')} \rho(s'_i) u(s'_i)}{\sum_{i=1}^{n(s')} \rho(s'_i)}.$$

which is equivalent, given the definition of $\succsim$*, to* $s \succsim s'$*. The fact that* $(s, s') \succsim s'$ *implies* $s \succsim s'$ *can be obtained in the same fashion. Let us now show that* $\succsim$ *as defined by (8) violates same number expansion consistency. Assume for this purpose that* $k = 1$ *(one attribute) and that* s*,* s'*, and* s'' *are 2-individuals societies defined by:*

$$s = (1, 7),\ s' = (2, 3) \text{ and } s'' = (4, 12)$$

Clearly, we have $s \succsim s'$ *since, using (8):*

$$\frac{1 + 7}{1 + \frac{1}{7}} = 7 \geq \frac{2 + 3}{\frac{1}{2} + \frac{1}{3}} = 6$$

Yet, in contradiction with the requirement of same number expansion consistency, one has $(s, s'') \prec (s', s'')$ *since:*

$$\frac{1 + 4 + 7 + 12}{1 + \frac{1}{4} + \frac{1}{7} + \frac{1}{12}} = \frac{24 \times 84}{84 + 21 + 12 + 7} = \frac{6 \times 84}{31} < \frac{2 + 3 + 4 + 12}{\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{12}} = \frac{21 \times 12}{6 + 4 + 3 + 1} = 6 \times 3$$

As an example of an ordering that violates averaging but that satisfies same number expansion consistency, consider the "classical utilitarian" ranking

$\succsim$ of societies defined, for every societies s and s', by:

$$s \succsim s' \Leftrightarrow \sum_{i=1}^{n(s)} u(s_i) \geq \sum_{i=1}^{n(s')} u(s'_i)$$

for some increasing and continuous real valued function u having $\mathbb{R}^k$ as domain. It is straightforward to check that this anonymous, continuous and monotonic ranking violates averaging but satisfies same number expansion consistency. ■

4 Inequality aversion

The social orderings characterized in theorem 2 do not exhibit specific attitude toward attributes inequalities. It is quite easy to incorporate such attitudes in the framework if they are deemed appropriate. From a technical point of view, requiring the social ordering to exhibit a specific form of inequality aversion leads to additional restrictions on the "utility function" whose average defines the social ordering as per (2).

For instance, a widely discussed notion of inequality aversion in a multidimensional context is that underlying the principle of progressive transfer (see e.g. Ebert (1997), Fleurbaey, Hagneré, and Trannoy (2003), Fleurbaey and Trannoy (2003) and, in this volume, Gravel and Moyes (2007)). According to this principle, which applies to societies of identical size, any transfer of attributes between two individuals must be seen as a social improvement if:

1) the person from which the transfer originates has initially a weakly better endowment of every attribute than the beneficiary of the transfer and,

2) for each attribute, the amount transferred does not exceed the difference between the donator's and the receiver's endowment of the attribute.

This notion of transfer, illustrated on figure 1, can be seen as a generalization, to several attributes, of the conventional notion of Pigou-Dalton transfer used to define one-dimensional inequality. Yet we should notice that the notion of transfer that is defined here allows the amount of attribute to be transferred between two individuals to be as large as the difference in attributes between them. For instance, if Mary has 3 units of attribute 1 and 2 units of attribute 1 while Kumar has 1 unit of each, then transferring 2 units of attribute 1 from Mary to Kumar is considered to be equalizing here. Yet, after giving 2 units of attribute 1 to Kumar, Mary is poorer than Kumar in attribute 1 (even though she remains richer than Kumar in attribute 2). For this reason, our notion of transfer embodies what is called a favorable permutation in Gravel and Moyes (2007) and a correlation decreasing transfer in Tsui (1999).

We define formally this notion of transfer as follows.

Definition 1 *For societies s and s' with $n(s) = n(s') = n$, we say that society s is obtained from society s' by a progressive transfer if there exist individuals h and i and real numbers Δ_j, for $j = 1, ..., k$, such that:*
1) $s_g = s'_g$ for all $g \neq h$, i and,
2) $s_{hj} = s'_{hj} + \Delta_j \leq s'_{ij}$ and $s_{ij} = s'_{ij} - \Delta_j \geq s'_{hj}$ for all $j = 1, ..., k$.

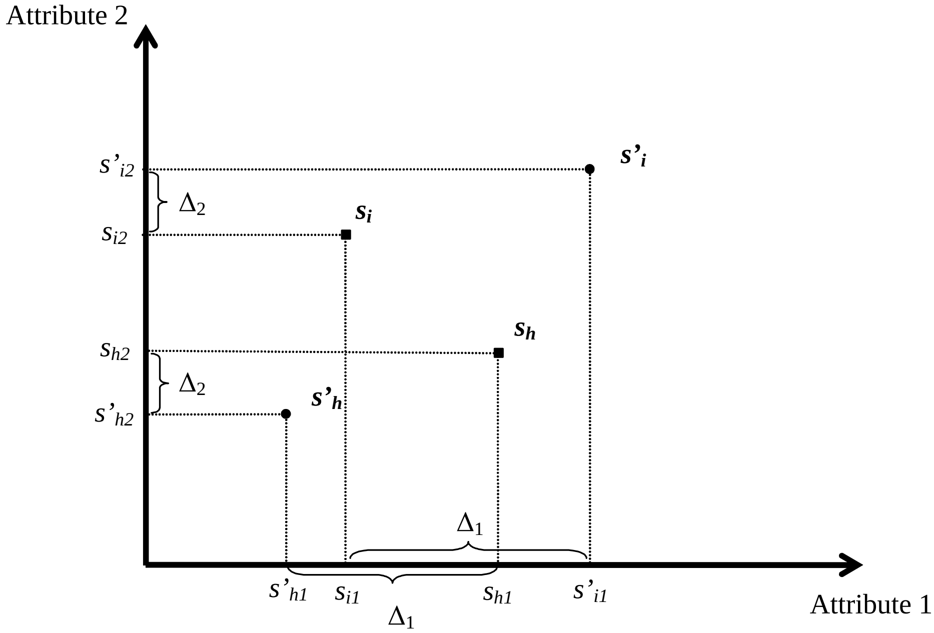

Figure 1: a progressive transfer from i to h.

Say that an ordering $\succsim$ exhibits *attribute-inequality aversion* if it always rank a society s above a society s' when s has been obtained from s' by a progressive transfer. It is of interest to identify the restriction on the utility function u in formula (2) that are implied by the requirement that the social ordering $\succsim$ exhibits inequality aversion.

As it happens, the property of the utility function that is implied by this concept of inequality is that of *decreasing increasingness.* This property, also known as ALEP substitutability in the literature (see e.g. Chipman (1977)), is formally defined as follows.

Definition 2 ***(Decreasing increasingness).*** *A function Φ from $\mathbb{R}^k$ to $\mathbb{R}$ is decreasingly increasing if, for every x and $x' \in \mathbb{R}^k$ such that $x \geq x'$ and every strictly positive real number Δ, one has $0 \leq \Phi_j^\Delta(x) \leq \Phi_j^\Delta(x')$ for all $j = 1, ..., k$.*

In words, a decreasingly increasing function is a function that is increasing in all its arguments at a decreasing rate. That is to say, if the "utility function" is decreasingly increasing with respect to the attributes, then it must have a marginal utility of each attribute that is decreasing with respect to all attributes. In the next proposition, we establish that if $\succsim$ is an anonymous, continuous and attribute-inequality averse ordering that satisfies the averaging and the same number expansion consistency axioms, then it can be thought of as resulting from the comparisons of the average utility of the attributes for some decreasingly increasing utility function.

Proposition 3 *Let $\succsim$ be an attribute-inequality averse ordering on $\bigcup_{l\in\mathbb{N}_{++}} \mathbb{R}^{lk}$ satisfying monotonicity, anonymity, continuity, averaging and same number expansion consistency. Then $\succsim$ is a AU social ordering. Furthermore, the u function in the definition of a AU social ordering is unique up to a positive affine transformation and is a decreasingly increasing function.*

Proof. *From theorem 2, we know that any ordering on $\bigcup_{l\in\mathbb{N}_{++}} \mathbb{R}^{lk}$ satisfying monotonicity, anonymity, continuity, averaging and same number expansion consistency is a AU social ordering that is represented, as per (2), by some increasing utility function. Let us show that if the ordering is further required to exhibit aversion to attribute-inequality, then the utility function must be decreasingly increasing. By contraposition, assume that the utility function is not decreasingly increasing and, therefore, that there are attributes bundles x and $x' \in \mathbb{R}^k$ with $x \geq x'$ for which, for one attribute j and a strictly positive real number Δ, one has $0 \leq u_j^{\Delta}(x') < u_j^{\Delta}(x)$. Consider then two societies s and $s' \in \mathbb{R}^{nk}$ for some $n \in \mathbb{N}_{++}$ with $n \geq 2$ for which there exists two individuals h and i such that:*
1) $s_g = s'_g$ for all $g \neq h, i$,
2) $s_{hj} = x'_j + \Delta$, $s'_{hj} = x'_j$, $s_{ij} = x_j$ and $s'_{ij} = x_j + \Delta$ and,
3) $s_{he} = s'_{he} = x'_e$ and $s_{ie} = s'_{ie} = x_e$ for all $e \neq j$.
From the definition given above, s has been obtained from s' by a progressive transfer of attribute (in fact only attribute j has been transferred). Now, notice that

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}u(s_i) - \frac{1}{n}\sum_{i=1}^{n}u(s'_i) &= \frac{1}{n}[u(x'_1, ..., x'_j + \Delta, ..., x'_n) - u(x'_1, ..., x'_j, ..., x'_n)] \\ &\quad - [u(x_1, ..., x_j + \Delta, ..., x'_n) - u(x_1, ..., x_j, ..., x_n)] \\ &< 0\end{aligned}$$

because $0 \leq u_j^{\Delta}(x') < u_j^{\Delta}(x)$. Hence the AU ranking does not exhibit attribute-inequality aversion because it ranks s strictly below s'.. ■

5 Conclusion

This chapter provides axiomatic justification for using, when ranking societies described as finite lists of attributes bundles, a ranking that can be thought of as resulting from a comparison of the average utility achieved by individuals out of their attribute bundle, for some utility function. While our approach applies to the welfarist case where the only normatively relevant individual attribute is taken to be utility, its main interest is, in our view, that it can be applied to multi-dimensional and non-welfarist contexts where many individual attributes may matter for normative evaluation. To that extent, the utility function that appears in the representation of the social ordering should not be interpreted as a measure of the individual's well-being but should, instead, be thought of as the valuation of the attributes made by the theoretician of justice. We have also illustrated that the approach is flexible and can incorporate important normative considerations like, for instance, attributes-inequality aversion.

It seems to us that assuming the social ordering to be anonymous, monotonic and continuous is quite natural. Anonymity is a mild requirement here since it says that the individual name does not matter for normative evaluation once the list of attributes that describes an individual's situation is sufficiently comprehensive. This line of reasoning is, of course, reminiscent of the one, made famous by Serge-Christophe Kolm (1972), in order to justify his notion of "fundamental preference". If there is something in the individual's name that matters for normative evaluation, then we should put this "something" in the list of relevant attributes so that, ultimately, the name of the individual should not matter. Monotonicity is also a very natural requirement if we view attributes as "primary goods" or as things that "any reasonable person" would desire. Continuity is probably not as natural since, among other things, it rules out lexicographic rankings of societies such as the Leximin one that have been advocated, among others, by Kolm. Yet we content that an ability to make continuous trade off between individuals when making collective decisions has clear practical advantages and is, after all, quite defensible from an ethical point of view. If we accept this, and we therefore accept to restrict ourselves to the class of anonymous, monotonic and continuous orderings of societies, then there are only two axioms that single out the AU family of social orderings in this class: averaging, and same number expansion consistency. Any difficulty in accepting to rank societies on the basis of their average utility, for some utility function defined on the set of all attributes bundles, must come from a difficulty in accepting either, or both, of these axioms.

AU rankings and their generalized counterparts have been criticized in the welfarist population ethics literature for the fact that they may fail to recommend societies' enlargement even when the new individuals that are brought in will have a "valuable" existence (see e.g. Blackorby, Bossert, and

Donaldson (2005)). This criticism rides on the existence of an "absolute" norm of what constitutes a "valuable" existence. The existence of such a norm may be plausible in a welfarist context in which individual's utility is given cardinally meaningful significance. Yet we believe that it is much more difficult to come up with such an absolute norm of "valuable" existence in a multidimensional and non-welfarist context. The average utilitarian family of rankings examined in this paper adopts the "relativist" view point that it is worth bringing in a new individual in the society when, and only when, we can provide this individual with an attributes bundle that gives him or her a utility level at least as large as that achieved in average in the society. The value of adding an individual to a society is therefore relative to the society in which the individual is added.

Of course, the AU family of ordering of societies, described as vectors of attributes bundles, is not the only conceivable class. It is our hope that further work in this area will provide us with other social orderings whose axiomatic properties could then be usefully compared with that identified in this chapter.

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