Problem set in Consumer theory, updated version (Solutions)

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Problem 1 For every preference defined below, indicates if it is convex (weakly or strictly), increasing (weakly or strictly), locally non satiable, continuous, complete and transitive (explains and draw the sets $NW \succeq$)

(i) $(x_1, x_2) \succeq (y_1, y_2) \Leftrightarrow x_1 + x_2 \geq y_1 + y_2$ and $\max\,(x_1, x_2) \geq \max\,(y_1, y_2)$

Answer: Representative $NW_{\succsim}$ and $NB_{\succsim}$ sets for this preference are depicted below.

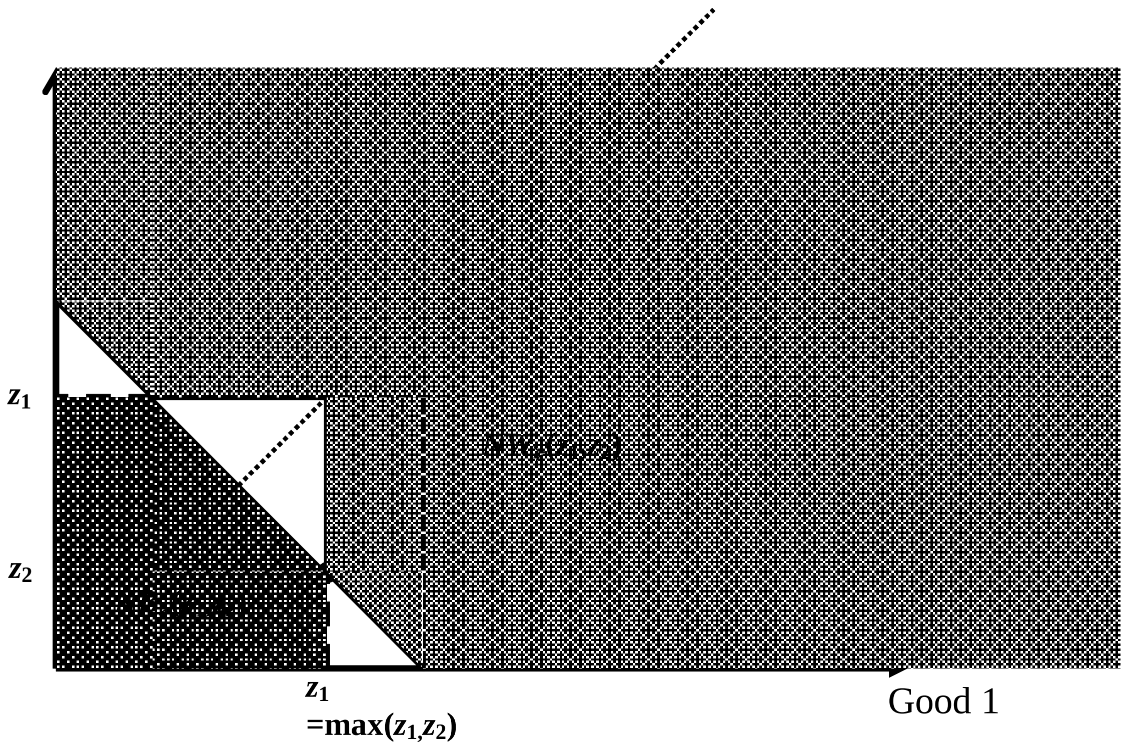

Figure 1

These preferences are not complete (bundle in the white zone in Figure 1 are not comparable to (z_1, z_2)). These preferences are transitive because if $(x_1, x_2) \succeq (y_1, y_2)$ and $(y_1, y_2) \succeq (z_1, z_2)$, then $x_1 + x_2 \geq y_1 + y_2$ and max $(x_1, x_2) \geq$ max (y_1, y_2) and $y_1 + y_2 \geq z_1 + z_2$ and max $(y_1, y_2) \geq$ max (z_1, z_2) which implies, given the transitivity of the binary relation $\geq$ on real numbers, that $x_1 + x_2 \geq z_1 + z_2$ and max $(x_1, x_2) \geq$ max (z_1, z_2). These preferences are weakly monotonic because if $(x_1, x_2) \geq (y_1, y_2)$, one has clearly max $(x_1, x_2) \geq$ max (y_1, y_2) and $x_1 + x_2 \geq y_1 + y_2$ while if $x_i > y_i$ for $i = 1, 2$, one has max $(x_1, x_2) >$ max (y_1, y_2)) and $x_1 + x_2 > y_1 + y_2$. Since they are weakly monotonic, they are locally non-satiable. Notice that the strict (asymmetric) $\succ$ and indifference (symmetric) $\sim$ relation corresponding to these preferences are defined by:

$$
\begin{array}{rcl}
(x_1, x_2) & \succ & (y_1, y_2) \Leftrightarrow (x_1, x_2) \succeq (y_1, y_2) \text{ and } \neg(y_1, y_2) \succeq (x_1, x_2) \\
& \Leftrightarrow & x_1 + x_2 \geq y_1 + y_2 \text{ and } \max(x_1, x_2) \geq \max\ (y_1, y_2) \text{ and} \\
\neg(y_1 + y_2 & \geq & x_1 + x_2 \text{ and } \max(y_1, y_2) \geq \max\ (x_1, x_2)) \\
& \Leftrightarrow & \\
x_1 + x_2 & > & y_1 + y_2 \text{ and } \max(x_1, x_2) \geq \max\ (y_1, y_2) \text{ or} \\
x_1 + x_2 & \geq & y_1 + y_2 \text{ and } \max(x_1, x_2) > \max\ (y_1, y_2)
\end{array}
$$

and

$$
\begin{array}{rcl}
(x_1, x_2) & \sim & (y_1, y_2) \Leftrightarrow (x_1, x_2) \succeq (y_1, y_2) \text{ and } (y_1, y_2) \succeq (x_1, x_2) \\
& \Leftrightarrow & x_1 + x_2 \geq y_1 + y_2,\ \max(x_1, x_2) \geq \max\ (y_1, y_2)\ , \\
y_1 + y_2 & \geq & x_1 + x_2 \text{ and } \max(y_1, y_2) \geq \max\ (x_1, x_2)) \\
& \Leftrightarrow & \\
x_1 + x_2 & = & y_1 + y_2 \text{ and } \max(x_1, x_2) = \max\ (y_1, y_2)
\end{array}
$$

Hence in order for the two bundles to be indifferent, they must contains the same total amount of goods and have the same maximal quantity of either good. Hence only permutations of a vector can be indifferent. This means that the preferences are not continuous because it is possible to continuously moving from a worse zone to a better zone without going through indifference. The picture also clearly shows that these preferences are not convex. For example, $(2, 10)$ and $(10, 2)$ both belong to $NW_{\succsim}(10, 2)$ but the bundle $(6, 6) = (\frac{1}{2}2 + \frac{1}{2}10, \frac{1}{2}10 + \frac{1}{2}2)$ is not comparable to $(10, 2)$.

(ii) $(x_1, x_2) \succeq (y_1, y_2) \Leftrightarrow x_1 \geq y_1 + 2$ ou $x_2 \geq y_2 - 1$

Answer: These preferences **are complete**. Indeed, consider two arbitrary bundles (x_1, x_2) and (y_1, y_2). Since the binary relation $\geq$ defined on the set of real numbers is complete, one of the following four (mutually exclusive) cases must hold:

1) $x_1 \geq y_1 + 2$ and $x_2 \geq y_2 - 1$
2) $x_1 \geq y_1 + 2$ and $x_2 < y_2 - 1$
3) $x_1 < y_1 + 2$ and $x_2 \geq y_2 - 1$
4) $x_1 < y_1 + 2$ and $x_2 < y_2 - 1$.

Cases 1)-3) imply that either one of $x_1 \geq y_1 + 2$ or $x_2 \geq y_2 - 1$ is true and, by definition of $\succeq$, that $(x_1, x_2) \succeq (y_1, y_2)$ is true. Case 4) implies that $x_2 < y_2 - 1$ and, therefore, that $x_2 - 1 < x_2 + 1 < y_2$, which implies that $(y_1, y_2) \succeq (x_1, x_2)$ holds. Hence, for every bundles (x_1, x_2) and (y_1, y_2), one can always write either $(x_1, x_2) \succeq (y_1, y_2)$ or $(y_1, y_2) \succeq (x_1, x_2)$ (or both)..

These preferences **are not transitives** as illustrated by the bundles $(2, 2)$, $(2, 3)$ et $(2, 4)$. We have indeed that $(2, 2) \succeq (2, 3)$ (because $2 \geq 3 - 1 = 2$) and $(2, 3) \succeq (2, 4)$ (because $3 \geq 4 - 1 = 3$). Yet $(2, 2) \succeq (2, 4)$ does **not** hold. These préférences **are not monotonic croissantes** because, for instance, $(2, 3)$ contains strictement more of every good than $(1, 2)$ but is not considered to be strictly better than $(1,2)$. These preferences violat**e local non-satiation** (because if $\varepsilon \leq 1$, bundle $(1 \pm \varepsilon, 2 \pm \varepsilon)$ will not be preferred to $(1,2)$. These preferences are **non convex** (for example: $(3,0) \succeq (1,2)$ and $(1,1) \succeq (1,2)$ but $(2, \frac{1}{2}) \succeq (1,2)$ does not hold. Here is the picture of the "better than", worse than and indifference zones. One can see from this picture that these preferences are continuous (one can not go continuously from worse to better without encountering the (thick) indifference zone).

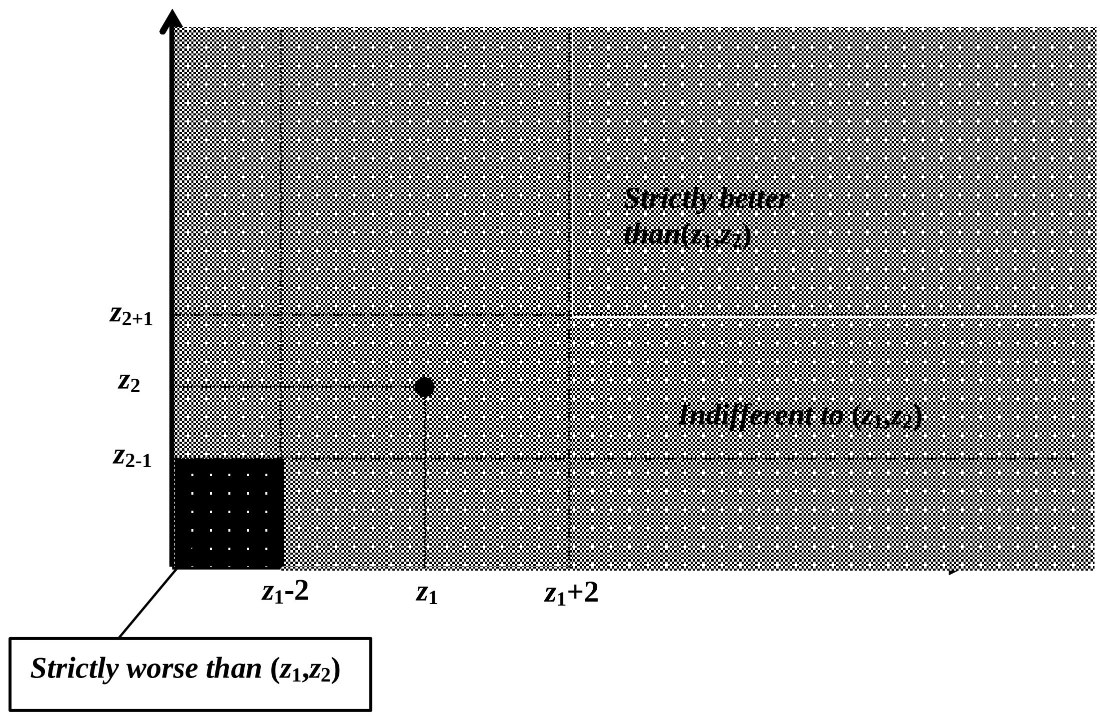

Figure 2

(iii) $(x_1, x_2) \succeq (y_1, y_2) \Leftrightarrow x_1 \geq y_1$ or $x_1 = y_1$ and $x_2 \geq y_2$.

Answer: These preferences are the lexicographic preferences in $\mathbb{R}^2_+$, They are complete, transitive, monotonic and convex (prove it!!) but are not continuous. Consider for instance the bundle $(1,2)$. The sequence of bundles $(1 + \frac{1}{n}, 1)$ for any strictly positive integer n is strictly preferred to bundle. Yet this sequence of bundle converges to $(1,1)$, which is strictly worse than $(1,2)$. Draw a picture of representative $NW_{\succsim}(x_1, x_2)$ and $NW_{\succsim}(x_1, x_2)$ sets for these preferences (we have done it in class!!!).

(iv) $(x_1, x_2) \succeq (y_1, y_2) \Leftrightarrow \min(x_1, x_2) \geq \min(y_1, y_2)$ and $\max\,(x_1, x_2) \geq \max\,(y_1, y_2)$.

Answer:

The picture of representative sets $NW_{\succsim}(x_1, x_2)$ and $NW_{\succsim}(x_1, x_2)$ is as in figure 3. It can be seen from this picture that these preferences are not complete since the union of the sets $NW_{\succsim}(x_1, x_2)$ and $NW_{\succsim}(x_1, x_2)$ is not $\mathbb{R}^2_+$. It can also be seen from this picture that these preferences are not convex. By an analogous reasonning than that used for preferences of exercise (i), one can see that these preferences are not continuous (only permuations of a same bundle are indifferent) but are transitive and weakly monotonic.

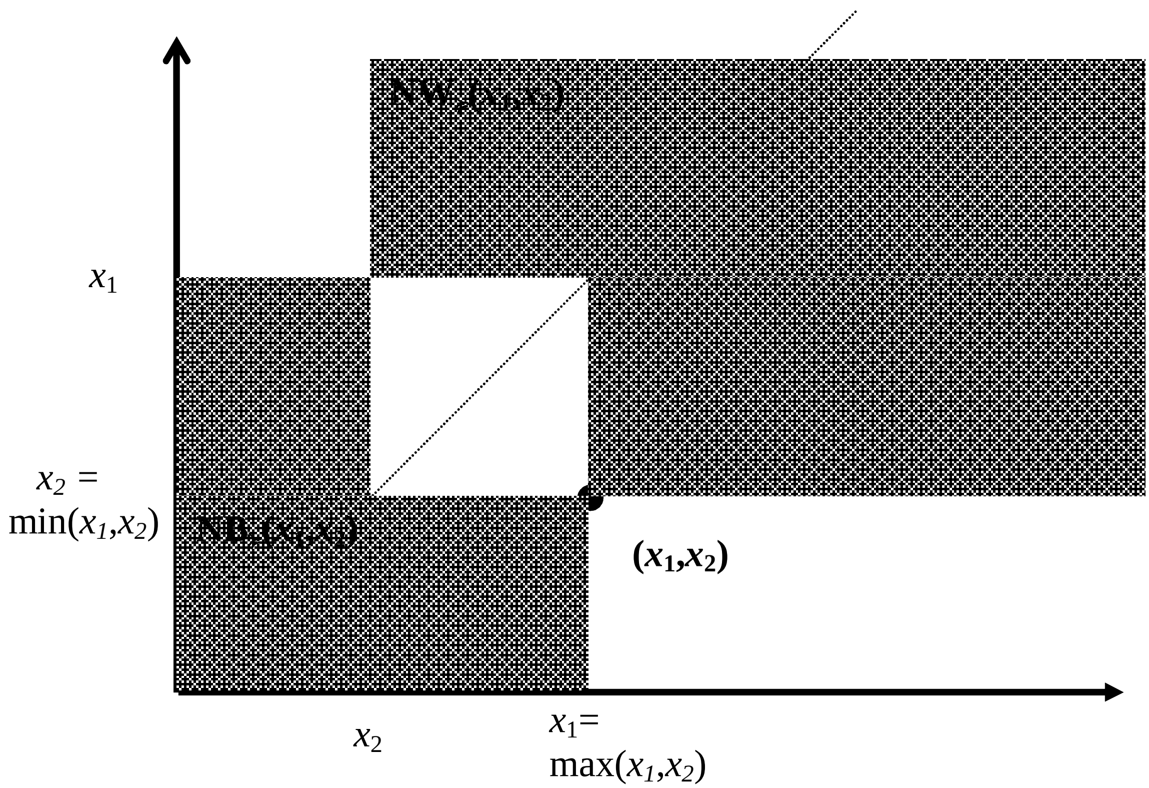

Figure 3

Problem 2 Shasikanta's preferences for consumption bundles of two goods are as follows. For every pair of bundles A and B, Shasikanta considers that A is weakly better than B if A contains weakly more of at least one good than B. Are Shasikanta preferences transitive ? Complete ? Continuous ? Locally non-satiable ? Convex ? Justify.

Answer: The picture of representative sets $NW_{\succsim}(x_1, x_2)$ and $NW_{\succsim}(x_1, x_2)$ is as in figure 4. It is clear from this picture that Shashikanta's preferences are complete (the union of the sets $NW_{\succsim}(x_1, x_2)$ and $NW_{\succsim}(x_1, x_2)$) is the consumption set $\mathbb{R}^2_+$. They are also continous (one can not go continuously from the worse to the best without encountering indiffernce), monotonic (if the quantity of a good is increased, then the bundle will increase in Shasikanta's scale of preference and, for this reason, locally non-satiable. On the other hand, they are not convex ($NW_{\succsim}(x_1, x_2)$ is clearly not a convex set) and not transitive. To prove this last point, just notice that $(2, 3) \succsim (1, 5)$ and $(1, 4) \succsim (2, 3)$ but, contrary to what is required by transitivity, $(1, 4)$ is not weakly preferred to $(1, 5)$.

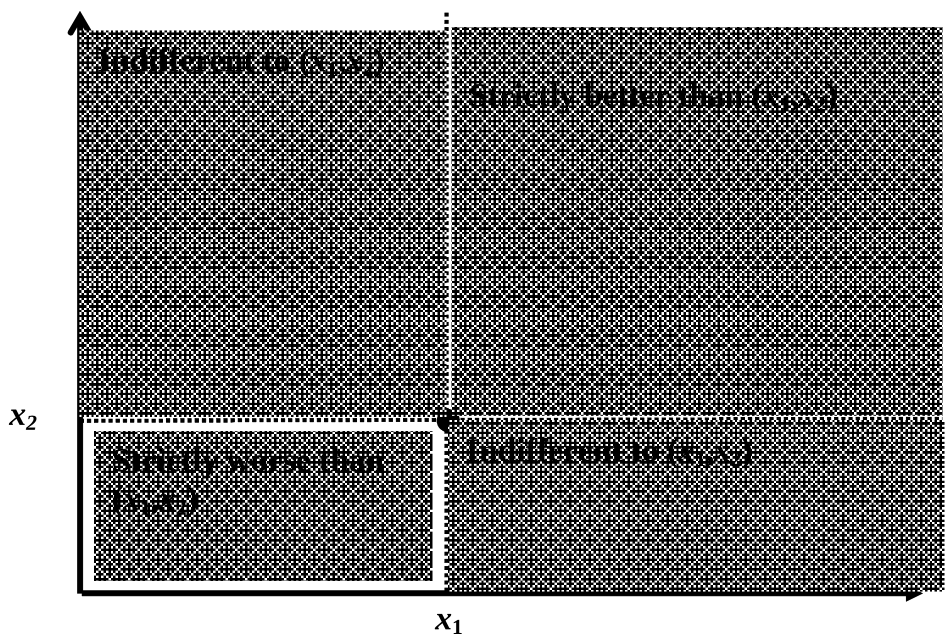

Figure 4

Problem 3 Conrad is an amateur of souvlakis who is endowed with the following preferences. For every pair A and B of bundles of souvlakis and money available to other use than souvlakis, Conrad *weakly prefer* A to B *if and only if*:

1) The quantity of souvlakis available in A exceeds the quantity available in B by at least two units and

2) The money available to other use than souvlakis in A is not lower than that available in B by more than 100 euros.

Are Conrad preferences transitive ? Complete ? Continuous ? Locally non-satiable ? Convex ? Justify and draw a representative $NW_{\succsim}(x_1, x_2)$ set.

Answer: Conrad's preferences are of the same kind than those of problem 1

(ii) (with good 1 being Souvlakis and good 2 being money measured in hundreds in euros).

Problem 4. Kumar has preference for two goods that are numerically represented by the utility function

$$U(x_1, x_2) = \min(x_1, x_2^2)$$

a) Find the Marshallian and Hicsksian demand functions (or correspondances) as well as the indirect utility and the expenditure functions.

Answer: Since Kumar compares bundles on the basis of the minimum of the numbers x_1 and x_2^2, he will always choose bundles for which

$$x_2^2 = x_1 \tag{1}$$

(why consuming more of one of the good than is necessary if these goods are costly to purchase ?). As these preferences are monotonic, Kumar will spend all his wealth W on the two goods (purchased at prices p_1 and p_2). We therefore have, given (1):

$$\begin{aligned} p_1[x_2^M(p_1,p_2,W)]^2 + p_2 x_2^M(p_1,p_2,W) &= W \\ &\Leftrightarrow \\ x_2^M(p_1,p_2,W) &= \frac{-p_2}{2p_1} \pm \frac{\sqrt[2]{p_2^2 + 4p_1 W}}{2p_1} \\ &= \frac{1}{2p_1}[[p_2^2 + 4p_1 W]^{1/2} - p_2] \end{aligned}$$

after keeping only the positive root (negative quantities of good must be ruled out). Substituting $x_2^M(p_1, p_2, W)$ back into (1) yields:

$$\begin{aligned} [x_2^M(p_1,p_2,W)]^2 &= x_1^M(p_1,p_2,W) \\ &\Leftrightarrow \\ x_1^M(p_1,p_2,W) &= \frac{1}{4p_1^2}[[p_2^2 + 4p_1 W] - 2p_2[p_2^2 + 4p_1 W]^{1/2} + p_2^2] \end{aligned}$$

Plugging these Marshallian demands into the direct utility function gives us the indirect utility function $V(p_1, p_2, W)$:

$$\begin{aligned} V(p_1,p_2,W) &= \min(x_1^M(p_1,p_2,W), [x_2^M(p_1,p_2,W)]^2 \\ & \quad \frac{1}{4p_1^2}[[p_2^2 + 4p_1 W] - 2p_2[p_2^2 + 4p_1 W]^{1/2} + p_2^2] \end{aligned}$$

From this, we obtain the expenditure function $E(p_1, p_2, u)$ by the identity:

$$\begin{aligned} V(p_1,p_2,E(p_1,p_2,u)) &\equiv u \\ &\Leftrightarrow \\ \frac{1}{4p_1^2}[[p_2^2 + 4p_1 E(p_1,p_2,u)] - 2p_2[p_2^2 + 4p_1 E(p_1,p_2,u)]^{1/2} + p_2^2] &\equiv u \end{aligned} \tag{2}$$

Define x by

$$x = [p_2^2 + 4p_1 E(p_1, p_2, u)]^{1/2} \tag{3}$$

With this definition, the identity (2) writes:

$$x^2 - 2p_2 x + p_2^2 - 4p_1^2 u \equiv 0 \tag{4}$$

a quadratic equation whose roots are:

$$\begin{aligned} x &= p_2 \pm \frac{1}{2}\sqrt[2]{4p_2^2 + 16p_1^2 u - 4p_2^2} \\ &= p_2 \pm \frac{1}{2}\sqrt[2]{16up_1^2} \\ &= p_2 \pm 2p_1 u^{1/2} \end{aligned}$$

Using (3), we therefore have that

$$\begin{aligned} [p_2^2 + 4p_1 E(p_1, p_2, u)]^{1/2} &= p_2 \pm 2p_1 u^{1/2} \\ &\Leftrightarrow \\ [p_2^2 + 4p_1 E(p_1, p_2, u)] &= p_2^2 \pm 4p_1 p_2 u^{1/2} + 4p_1^2 u \\ &\Longleftrightarrow \\ E(p_1, p_2, u) &= p_2 u^{1/2} + p_1 u \end{aligned}$$

From this, we obtain from Sheppard lemma that the Hicksian (compensated) demands are given by:

$$x_1^H(p_1, p_2, u_)) = \partial E(.)/\partial p_1 = u$$

and

$$x_2^H(p_1, p_2, u_)) = \partial E(.)/\partial p_2 = u^{1/2}$$

Notice that these Hicksian demands do not depend upon prices. In effect, there are no substution effect with these preferences in which the two goods are pure complements.

b) Is good 1 a luxury or a necessity good ?

Answer: Recall from basic microeconomics that a necessity good is a good for which the share in the consumer's budget decreases with wealth. That is to say, good 1 is a necessity good if the function $s_1(p_1, p_2, W) = \frac{p_1 x_1^M(p_1, p_2, W)}{W}$ is decreasing in wealth. Using the Marshallian demand find above this means that good 1 is a necessity good if

$$\frac{1}{4p_1 W}[[p_2^2 + 4p_1 W]^{1/2} - p_2]^2$$

is decreasing in wealth. Differentiating this expression with respect to wealth yields

$$-\frac{1}{4p_1 W^2}[[p_2^2 + 4p_1 W]^{1/2} - p_2]^2 + \frac{4p_1[p_2^2 + 4p_1 W]^{-1/2}[[p_2^2 + 4p_1 W]^{1/2} - p_2]}{4p_1 W} =$$

$$-\frac{1}{4p_1W^2}[[p_2^2+4p_1W]^{1/2}-p_2]^2+4p_1W[1-p_2[p_2^2+4p_1W]^{-1/2}]]=$$

$$-\frac{1}{4p_1W^2}[[p_2^2+4p_1W]^{1/2}-p_2]^2+4p_1W[1-p_2/[p_2^2+4p_1W]^{1/2}]]<0$$

$$p_2/[p_2^2+4p_1W]^{1/2}<1$$

if $4p_1W>0$. Hence good 1 is a necessity good..

Problem 5. The rugby team A has beaten rugby team B. Rugby team B has beaten rugby team C while teams A and C have done a dead-heat. .

a) Is the binary relation "has not lost against" defined on the set of teams $\{A,B,C\}$ complete and transitive ? Justify

Answer: As all teams have played at least once with each other, an outcome of the game has arised and, for this reason, any two teams can be compared by the binary relation "has not lost against". On the other hand, this binary relation is not transitive because C has not lost against A and A has not lost against B but, contrary to what is required by transitivity, C has lost against B.

b) Is the binary relation "is the brother of" defined on the set of all individuals living in Marseille complete and transitive ?

Answer: This binary relation is certainly not complete because there are several individuals i and j in Marseille between for which i is not the brother of j and j is not the brother of i. This relation is also not transitive because Jim can be the brother of Bob and Bob can be the brother of Jim but Jim is not, in the usual sense of the word, the brother of himself.Of course if, in the definition of transitivity, we require the three alternatives a, b and c for which $a \succsim b$ and $b \succsim c$ imply $a \succsim c$ to be all distinct, then "being the brother of" is transitive.

Problem 6. True or false ? (justify) If leisure is an inferior good for some household, then the number of hours that this household would wish to supply on the market will be an increasing function of the hourly wage.

Answer: **True**. This question refers to the classical model of labour supply as obtained from a demand of leisure. A household is assumed to have available an endowment T of time (measured in hours per period, say per week) that it can allocate between leisure, noted L, and labour, noted l so that

$$L+l=T \tag{5}$$

. This household has also, possibly, some non labour income, noted A and faces an exogenous wage rate w. The wage rate gives the amount of consumption (in euros) that the household can obtain if it works one hours. Hence, the total amount of consumption C that the household can achieve per week if it works l hours is given by

$$C \leq wl+A$$

or, using (5):

$$C + wL \leq Tw + A \tag{6}$$

This inequality is just a standard budget constraint for a household consuming two goods, leisure and consumption, having a wealth $W = Tw + A$, and facing a price of w for leisure and a price of 1 for consumption. In this model therefore, leisure is implicitely purchased at the wage rate w, in the sense that the household can be seen as "buying" its leisure by foregoning the money it would have earned had it not taken this leisure. The novelty here is that the wealth W depends upon prices. The consumption set relevant for this problem is the set $X = \{(L, C) \in \mathbb{R}^2_+ : L \leq T\}$ because a household can not consume more leisure per week than the total time it has available. The household is assumed to have preferences on the various combination of leisure and consumption it can consume and these preferences are represented by a utility function $U(L, C)$. Hence we have a standard consumer's problem. Forget for a moment the fact that wealth depends upon prices. From the Slutsky equations, denoting by $L^M(w, 1, W)$ and $L^H(w, 1, u)$ the Marshallian and Hicksian (respectively) demand for leisure of this consumer, we know that:

$$\partial L^M(w, 1, W)/\partial w \equiv \partial L^H(w, 1, W)/\partial w - L^M(w, 1, W)\partial L^M(w, 1, W)/\partial W \tag{7}$$

We obtain labour supply $l^s(w, 1, W)$ from (Marshallian) leisure demand using (5) by: $l^s(w, 1, W) = T - L^M(w, 1, W)$. Define now the function $\lambda(w, A)$ by

$$\lambda(w, A) = L^M(w, 1, Tw + A) \tag{8}$$

This function gives you the demand of the household for leisure as a function only of the wage rate and non-labour income. Using (7), one can obtain from (8) the effect of a small change on the wage rate on labour demand by:

$$\begin{aligned}
\partial \lambda(w, A)/\partial w &\equiv \partial L^H(w, 1, W)/\partial w - L^M(w, 1, W)\partial L^M(w, 1, W)/\partial W + T\partial L^M(w, 1, W)/\partial W \\
&\equiv \partial L^H(w, 1, W)/\partial w + [T - L^M(w, 1, W)]\partial L^M(w, 1, W)/\partial W \\
&\equiv \partial L^H(w, 1, W)/\partial w + l^s(w, 1, W)\partial L^M(w, 1, W)/\partial W
\end{aligned}$$

We know that $\partial L^H(w, 1, W)/\partial w \leq 0$ by the concavity of the expenditure function and Sheppard's lemma. If leisure is an inferior good, we have also that $\partial L^M(w, 1, W)/\partial W \leq 0$. Because of this, we have clearly that an increase in a wage rate will lead the housold to reduce its demand for leisure and, therefore, increase its labour supply.

Problem 7 True of false? (justify) Preeti has a yearly income of ω_1 euros this year. She is certain to receive ω_2 euros next year. If Preeti decides to borrow this year at some interest rate r, she would certainly decide to borrow if the interest rate was lower.

Answer: True. Preeti's budget set is the set of all combinations of consumption today (x_1) and consumption tomorow (x_2) that satisfy (using future consumption as numéraire):

$$(1 + r)x_1 + x_2 \leq (1 + r)\omega_1 + \omega_2$$

If Preeti decides to borrow at interest rate of r, this means that $x_1^M(1+r,1,(1+r)\omega_1+\omega_2) > \omega_1$ and $x_2^M(1+r,1,(1+r)\omega_1+\omega_2) = (1+r)\omega_1+\omega_2-(1+r)x_1^M(1+r,1,(1+r)\omega_1+\omega_2) < \omega_2$. Let r' be an interest rate that is lower than r and consider any bundle x_1' and x_2' in which Preeti does not borrow at interest rate r'. We can see that, at interest rate r, the bundle (x_1', x_2') is available to Preeti. Indeed:

$$(1+r)x_1' + x_2' < (1+r)\omega_1 + \omega_2$$

if $r' < r'$ and $x_1' \leq \omega_1$ (prove it!!!). Similarly, one can show, under the same assumption that

$$(1+r')x_1^M(1+r,1,(1+r)\omega_1+\omega_2)+x_2^M(1+r,1,(1+r)\omega_1+\omega_2) \leq (1+r')\omega_1+\omega_2$$

so that the bundle choosen at interest rate r could have been choosen at interest rate r'. By choosing $x_1^M(1+r,1,(1+r)\omega_1+\omega_2)$ and $x_2^M(1+r,1,(1+r)\omega_1+\omega_2)$ at interest rate r, Preeti is revealing to us a strict preference for the bundle $x_1^M(1+r,1,(1+r)\omega_1+\omega_2)$, $x_2^M(1+r,1,(1+r)\omega_1+\omega_2)$ over the bundle x_1', x_2' while a choice of x_1', x_2' at interest rate r' would reveal from her part a strict preference for x_1', x_2' over $x_1^M(1+r,1,(1+r)\omega_1+\omega_2)$, $x_2^M(1+r,1,(1+r)\omega_1+\omega_2)$, in violation of the weak axiom of revealed preference.

Problem 8 Suppose that at prices $(p_1, p_2) = (5, 10)$, a rational consumer endowed with a wealth of 100 consumes the bundle $(6, 7)$. Suppose that an econometrician has measured the following derivatives:

$$\begin{aligned}
\frac{\partial x_1^H(5,10,V(5,10,100))}{\partial p_1} &= -2 \\
\frac{\partial x_1^H(5,10,V(5,10,100))}{\partial p_2} &= +1 \\
\frac{\partial x_1^M(5,10,100)}{\partial R} &= 2/7
\end{aligned}$$

Estimate the bundle that the consumer would have choosen if he or she had faced prices $(p_1, p_2) = (5, 11)$.

Answer: The price of good 1 and the wealth do not change. Hence, we need to estimate the impact of the (small) change in the price of good 2 on the demand of the two goods:

Change in demand of good 1, Δx_1 is given by:

$$\begin{aligned}
\Delta x_1 &= \frac{\partial x_1^M(5,10,100)}{\partial p_2}\Delta p_2 \\
&= \frac{\partial x_1^M(5,10,100)}{\partial p_2} \\
&= \frac{\partial x_1^H(5,10,V(5,10,100))}{\partial p_2} - x_2\frac{\partial x_1^M(5,10,100)}{\partial R} \text{ (Slutsky)} \\
&= 1 - 7(2/7) = -1
\end{aligned}$$

Hence the new consumption of good 1 will be 5. To estimate the new consumption of good 2, use the budget constraint:

$$\begin{aligned} x_2 &= 100/11 - 25/11 \\ &= 75/11 \end{aligned}$$

Problem 9 An individual consumes housing and money available for other things than housing. Housing is available in two different cities: city A and city B. The individual can not live in the two cities in the same time. The preferences of the individual for the three goods are measured by the utility function:

$$U(x_1, x_2, x_3) = (x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}})x_3$$

where x_1, x_2 and x_3 denote, respectively, the number of squared meters occuppied in city A, the number of squared meters occuppied in city B and the number of euros available for other things than housing.

a) Define the consumption set of this individual. Is this set convex ?

Answer: The consumption set X is defined by

$$X = \{(x_1, x_2, x_3) \in \mathbb{R}^3_+ : x_1 = 0,\ x_2 \geq 0 \text{ and } x_3 \geq 0 \text{ or } x_1 \geq 0,\ x_2 = 0 \text{ and } x_3 \geq 0$$

This set is clearly non-convex. For instance $(0, 2, 3)$ and $(2, 0, 3)$ both belong to X but $(0/2 + 2/2, 2/2 + 0/2, 3/2 + 3/2) = (1, 1, 3)$ does not belong to X.

b) Define the Marshallian demand for each of these three goods. Interpret your results. Are these Marshallian demand continuous ? Why ?

Answer: Since the individual can not live at more than one place, he or she will choose the most preferred place, depending upon prices. If the individual decides to live in A, he or she will choose housing consumption in such a way as to solve:

$$V^A(p_1, p_3) = \max_{x_1, x_2} U(x_1, 0, x_3) = x_1^{\frac{1}{2}} x_3 \text{ s.t. } p_1 x_1 + p_3 x_3 \leq R$$

The solution of this program enables us to find Marshallian demand *conditional upon the fact of living in A*

$$\begin{aligned} x_1^{MA}(p_1, p_3) &= \frac{R}{3p_1} \text{ and} \\ x_3^{MA}(p_1, p_3) &= \frac{2R}{3p_3} \end{aligned}$$

Similarly, one finds Marhallian demands conditionnally upon living in B by solving

$$V^B(p_2, p_3) = \max_{x_2, x_2} U(0, x_2, x_3) = x_2^{\frac{1}{2}} x_3 \text{ s.t. } p_2 x_2 + p_3 x_3 \leq R$$

which yields:

$$x_2^{MB}(p_2,p_3) = \frac{R}{3p_2} \text{ and}$$
$$x_3^{MB}(p_2,p_3) = \frac{2R}{3p_3}$$

The individual chooses A over B if and only if :

$$\begin{array}{ccc} V^A(p_1,p_3) & \geq & V^B(p_2,p_3) \\ & \Longleftrightarrow & \\ \frac{2R^{3/2}}{9p_1p_3} & \geq & \frac{2R^{3/2}}{9p_2p_3} \\ & \Leftrightarrow & \\ p_1 & \leq & p_2 \end{array}$$

We have therefore the Marshallian demand correspondance $X^M : \mathbb{R}^3_{++} \to \mathbb{R}^3_+$ definine by:e par

$$\begin{aligned} X^M(p_1,p_2,p_3) &= \{(\frac{R}{3p_1}, 0, \frac{2R}{3p_3})\} \text{ if } p_1 < p_2 \\ &= (\frac{R}{3p_1}, 0, \frac{2R}{3p_3}), (0, \frac{R}{3p_1}, \frac{2R}{3p_1})\} \text{ if } p_1 = p_2 \\ &= \{(0, \frac{R}{3p_2}, \frac{2R}{3p_3})\} \text{ if } p_1 > p_2 \end{aligned}$$

The demands are clearly discontinuous (draw a picture of a demand as a function of its own price, the othe price being constant). This discontinuity comes from the non-convexity of the consumption set, a problem that is recurrent in spatial economics.

Problem 10 A preference is said to be *homothetic* if it can be measured by a utility function that is homogenous of degree 1. Show that if a preference is homothetic, then the Marshallian cross-price effects should all be equal (that is, $\frac{\partial x_i^M(.)}{\partial p_j} = \frac{\partial x_j^M(.)}{\partial p_i}$ must hold for every pair of goods i and j).

Answer: A preference relation is homothetic if it can be represented by a fonction that is homogenous of degree 1. We have shown in problem 9 on the first problem set on production that in that case, the expenditure function $E(p,u)$ ($p \in \mathbb{R}^n_+$) can be written as

$$E(p,u) = \Phi(p)u \text{ with } \Phi(p) = E(p,1)$$

One can therefore write the indirect utility $V(p,R) = \frac{R}{\Phi(p)}$. By Roy's identity

$$\frac{-\frac{\partial V(p,R)}{\partial p_i}}{\frac{\partial V(p,R)}{\partial R}} = x_i^M(p,R) = \frac{R\frac{\partial \Phi(p)}{\partial p_i}}{\Phi(p)}$$

From which, we obtain that:

$$\frac{\partial x_i^M(p,R)}{\partial p_j} = \frac{R\frac{\partial^2\Phi(p)}{\partial p_i \partial p_j}\Phi(p) - R\frac{\partial\Phi(p)}{\partial p_j}\frac{\partial\Phi(p)}{\partial p_i}}{\Phi(p)^2} =$$
$$\frac{\partial x_j^M(p,R)}{\partial p_i} = \frac{R\frac{\partial^2\Phi(p)}{\partial p_j \partial p_i}\Phi(p) - R\frac{\partial\Phi(p)}{\partial p_j}\frac{\partial\Phi(p)}{\partial p_i}}{\Phi(p)^2}$$

QED.

Problem 11 The expenditure function of some consumer is given by:

$$E(p_1, p_2, u) = (\frac{2}{3}p_1 + p_1^{\frac{1}{2}}p_2^{\frac{1}{2}} + \frac{1}{3}p_2)u$$

Find the Marshallian demand of this consumer.

Answer: We know that

$$E(p_1, p_2, V(p_1, p_2, R)) \equiv R$$

and therefore that

$$(\frac{2}{3}p_1 + p_1^{\frac{1}{2}}p_2^{\frac{1}{2}} + \frac{1}{3}p_2)V(p_1, p_2, R) \equiv R$$
$$\Leftrightarrow$$
$$V(p_1, p_2, R) = \frac{R}{(\frac{2}{3}p_1 + p_1^{\frac{1}{2}}p_2^{\frac{1}{2}} + \frac{1}{3}p_2)}$$

We therefore obtain Marshallian demand by Roy's identity:

$$x_j^M(p_1, p_2, R) = -\frac{\partial V(p_1, p_2, R)/\partial p_j}{\partial V(p_1, p_2, R)/R}$$

for $j = 1, 2$ which gives us here

$$x_1^M(p_1, p_2, R) = \frac{\frac{R(2/3+\frac{1}{2}p_1^{-1/2}p_2^{1/2})}{(\frac{2}{3}p_1+p_1^{\frac{1}{2}}p_2^{\frac{1}{2}}+\frac{1}{3}p_2)^2}}{\frac{1}{(\frac{2}{3}p_1+p_1^{\frac{1}{2}}p_2^{\frac{1}{2}}+\frac{1}{3}p_2)}} = \frac{R(2/3 + \frac{1}{2}p_1^{-1/2}p_2^{1/2})}{(\frac{2}{3}p_1 + p_1^{\frac{1}{2}}p_2^{\frac{1}{2}} + \frac{1}{3}p_2)}$$

and

$$x_2^M(p_1, p_2, R) = \frac{\frac{R(1/3+\frac{1}{2}p_1^{1/2}p_2^{-1/2})}{(\frac{2}{3}p_1+p_1^{\frac{1}{2}}p_2^{\frac{1}{2}}+\frac{1}{3}p_2)^2}}{\frac{1}{(\frac{2}{3}p_1+p_1^{\frac{1}{2}}p_2^{\frac{1}{2}}+\frac{1}{3}p_2)}} = \frac{R(1/3 + \frac{1}{2}p_1^{1/2}p_2^{-1/2})}{(\frac{2}{3}p_1 + p_1^{\frac{1}{2}}p_2^{\frac{1}{2}} + \frac{1}{3}p_2)}$$

Problem 12 Sylvester has preferences for Pop Corn (good 1) and Pepsi (good 2) that are represented by the utility function $U(x_1, x_2) = -\frac{1}{x_1} - \frac{1}{x_2}$. Find the Marshallian demand correspondance and indicates whether or not they are single-valued (that is to say, whether or not they are functions).

Answer: After having checked that Sylvester's preferences are convex and locally non-satiable, one can characterize locally a solution of the maximization program solved by Marshallian demands $x_1^M(p_1, p_2, R)$, $x_2^M(p_1, p_2, R)$ by the equalities

$$MRS(x_1^M(.), x_2^M(.)) = \frac{\frac{\partial U(x_1^M(.), x_2^M(.))}{\partial x_1}}{\frac{\partial U(x_1^M(.), x_2^M(.))}{\partial x2}} = \frac{\frac{1}{x_1^M(.)^2}}{\frac{1}{x_2^M(.)^2}} = \frac{x_2^M(.)^2}{x_1^M(.)^2} = \frac{p_1}{p_2} \qquad (9)$$

and

$$p_1 x_1^M(.) + p_2 x_2^M(.) = R \qquad (10)$$

Writing equation (9) in the form:

$$x_2^M(.) = (\frac{p_1}{p_2})^{\frac{1}{2}} x_1^M(.)$$

and substituting into (10), one gets:

$$p_1 x_1^M(.) + p_2[(\frac{p_1}{p_2})^{\frac{1}{2}} x_1^M(.)] = R$$

from which we obtain (after simplification)

$$x_1^M(p_1, p_2, R) = \frac{R}{p_1^{\frac{1}{2}}(p_1^{\frac{1}{2}} + p_2^{\frac{1}{2}})}$$

Substituting back into (9) (or using symmetry of the preferences with respect to the two goods), one obtains:

$$x_2^M(p_1, p_2, R) = \frac{R}{p_2^{\frac{1}{2}}(p_1^{\frac{1}{2}} + p_2^{\frac{1}{2}})}$$

Problem 13 Crocodile Dundee has preferences for hamburgers (good 1) and champagne (good 2) that are represented by the utility function $U(x_1, x_2) = x_1 + x_2^{\frac{1}{2}}$. What conditions (if any) must be imposed on prices and Crocodile's wealth for Croco dile to find optimal to abstain from eating hamburgers?

Answer: Proceeding as in the previous problem, one writes, noting the Marshallian demands of good 1 (Hambuger) and 2 (champagne) respectively by $x_1^M(p_1, p_2, R)$ and $x_2^M(p_1, p_2, R)$):

$$TMS(x_1^M(.), x_2^M(.)) = \frac{\frac{\partial U(x_1^M(.), x_2^M(.))}{\partial x_1}}{\frac{\partial U(x_1^M(.), x_2^M(.))}{\partial x2}} = \frac{1}{\frac{1}{2x_2^M(.)^{\frac{1}{2}}}} = 2x_2^M(.)^{\frac{1}{2}} = \frac{p_1}{p_2} \qquad (11)$$

and

$$p_1 x_1^M(.) + p_2 x_2^M(.) = R \qquad (12)$$

One therefore obtains:

$$x_2^M(.) = 4(\frac{p_1}{p_2})^2$$

and, after substitution in (12),

$$\begin{aligned} p_1 x_1^M(.) + \frac{4p_1^2}{p_2} &= R \Leftrightarrow \\ x_1^M(.) &= \frac{R}{p_1} - 4\frac{p_1}{p_2} \end{aligned}$$

Of course this methodology supposes that the optimal (for the consumer) quantities of goods 1 and 2 belong to the domain of definition of these variables, that is $\mathbb{R}_+^2$. One needs therefore that $\frac{R}{p_1} - 4\frac{p_1}{p_2} \geq 0$. If $\frac{R}{p_1} < 4\frac{p_1}{p_2}$, Crocodile Dundee will not consume any Hamburger and will spend all of his wealth on the consumption of good 2. The condition we are looking for is therefore that $R \leq 4\frac{p_1^2}{p_2}$.

Problem 14 Find the Marshallian demand behaviour of a consumer's whose preferences for two goods are represented by the utility function

$$U(x_1, x_2) = (1 + x_1)(2 + x_2)$$

(a) Are the Marshallian demand correspondance in fact functions ?

Answer:
One finds Marshallian demands $x_1^M(p_1, p_2, R)$ and $x_2^M(p_1, p_2, R))$ by solving:

$$TMS(x_1^M(.), x_2^M(.)) = \frac{\frac{\partial U(x_1^M(.), x_2^M(.))}{\partial x_1}}{\frac{\partial U(x_1^M(.), x_2^M(.))}{\partial x2}} = \frac{2 + x_2^M(.)}{1 + x_1^M(.)} = \frac{p_1}{p_2} \qquad (13)$$

and

$$p_1 x_1^M(.) + p_2 x_2^M(.) = R \qquad (14)$$

Equation (13) can be rewritten as:

$$x_2^M(.) = \frac{p_1}{p_2}(1 + x_1^M(.)) - 2$$

which gives us, after substitution in (14) and rearranging:

$$x_1^M(.) = \frac{R - p_1 + 2p_2}{2p_1}$$

and thus:

$$x_2^M(.) = \frac{R + p_1 - 2p_2}{2p_2}$$

One can see that Marshallian demands are functions (only one bundle is associate to any prices and wealth configuration).

(b) Find the indirect utility function, the expenditure function and the Hicksian demand correspondances.

Answer: Indirect utility:

$$\begin{aligned} V(p_1, p_2, R) &= U(x_1^M(p_1, p_2, R), x_2^M(p_1, p_2, R)) \\ &= (1 + x_1^M(p_1, p_2, R)(2 + x_2^M(p_1, p_2, R) \\ &= (\frac{R + 2p_2}{2p_1} + \frac{1}{2})(1 + \frac{R + p_1}{2p_2}) \end{aligned}$$

Expenditure: Using $V(p_1, p_2, E(p_1, p_2, u)) \equiv u$, one has:

$$u = (\frac{E(p_1, p_2, u) + 2p_2}{2p_1} + \frac{1}{2})(1 + \frac{E(p_1, p_2, u) + p_1}{2p_2})$$

Solving this equation, one obtains (after choosing the appropriate root) $E(p_1, p_2, u) = 2p_1^{1/2}p_2^{1/2}u^{1/2} - 2p_2 - p_1$.