Uniform Expected Utility Criteria for Decision Making under Ignorance or Objective Ambiguity*

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Abstract

We provide an axiomatic characterization of a family of criteria for ranking *completely uncertain* and/or *ambiguous* decisions. A completely uncertain decision is described by the set of all its consequences (assumed to be finite). An ambiguous decisions is described as a finite set of possible probability distributions over a finite set of prizes. Every criterion in the characterized family can be thought of as assigning to every consequence (probability distribution) of a decision an *equal probability* of occurence and as comparing decisions on the basis of the *expected utility* of their consequences (probability distributions) for some utility function.

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1 Introduction

It is common to categorize decision problems by the structure of the environment that is assumed to be known to the decision maker. In situations

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of *certainty*, the decision maker is assumed to know the *unique consequence* of every decision which can, therefore, be usefully identified by this unique consequence. In situations of *risk*, studied along the lines of Neumann and Morgenstern (1947), the decision maker knows the *probability distribution* over all consequences that decisions can have so that the problem of choosing the "right" decision amounts to that of choosing the "right" probability distribution over the set of consequences. In situations of *uncertainty*, decisions are described as functions - *acts* in Savage (1954) terminology - from a set of states of nature to a set of consequences. In situations of *complete uncertainty* or *ignorance* as studied in the sizeable literature surveyed by Barberà, Bossert, and Pattanaik (2004), a decision is described even more parsimoniously by the *set* of all its (foreseeable) consequences, without reference to a set of states of nature and to a process that maps states of nature into consequences. A somewhat hybrid category of decision problems is provided by the recent literature on *objective ambiguity* without state space, illustrated by Ahn (2008) and Olszewski (2007), in which a decision is depicted as a *set of probability distributions* over a set of consequences. This paper is concerned with the two last classes of decisions problems.

With the noticeable exception of Baigent and Xu (2004) and Nitzan and Pattanaik (1984), all rankings of decisions under ignorance that have emerged in the literature are based on the *best* and the *worst* consequences of the decisions or on some lexicographic extension thereof. The limitation of such "extremist" rankings for understanding actual decision making under ignorance is clear enough. Suppose we consider an investor facing two alternative investment strategies in some completely uncertain environment. If strategy A is adopted, the investor gains (net of the cost of investing) either one or one million dollars. If strategy B is adopted, then the investor's gain is either nothing, or any (integer) amount between \$900000 and \$999999. Hence, the two investment strategies can be described by:

$A = \{1, 1000000\}$

$B = \{0, 900000, 900001, ..., 999999\}$

Under the assumption that the ranking of certain (singletons) decisions is increasing in money, most rules studied in the literature that are "monotonically increasing" with respect to the worst and the best elements would rank A above B. Yet it is not clear that an actual investor placed in that circumstance would make the same ranking. For instance an investor who would be somehow capable of assigning probabilities of occurrence to consequences - even without being able to identify clearly the states of nature and the mapping that associates consequences to states of nature - could very plausibly rank B above A on the basis that the "expected utility" of the consequences is higher in B than in A. The *median-based* ranking of sets characterized in Nitzan and Pattanaik (1984), and which compares sets in terms of their median consequence with respect to the underlying ranking of certain outcomes, would also consider B to be a better decision than A

in a situation like this.

Another limitation of many rankings of decisions under ignorance considered in the literature, which applies also to the median-based one, is that they do *not* allow for a diversity of attitudes toward ignorance across individuals. Consider again the case of decisions with pecuniary consequences. If all decision makers prefer more money received for sure to less and follow any particular positional rule such as the maximin, the maximax, the median or some lexicographic extension thereof, they will all rank uncertain decisions in the same fashion. This feature of positional rankings is clearly restrictive. After all, the fact that two individuals prefer more money to less and have a choice behavior that obeys the same axioms should not imply that they have the same attitude with respect to ignorance.

The relative scarcity of criteria for comparing sets of consequences in the context of decision making under ignorance is particularly striking when compared with what is observed in classical (Savagian) situations of uncertainty. In the later case one finds, along with "extremist" criteria that compare acts on the basis of their worst or best consequence, as characterized in Arrow and Hurwicz (1972) and Maskin (1979), the well-known (subjective) Expected Utility (EU) criterion characterized in Savage (1954) as well as many other "non-additive" criteria such as "Maximin Expected Utility over a Set of Priors" (characterized in Gilboa and Schmeidler (1989) and Casadesus-Masanell, Klibanoff, and Ozdenoren (2000)) or the "Choquet Expected Utility" criterion characterized in Schmeidler (1989). Contrary to their "extremist" or positional counterparts, individuals whose behavior satisfies a particular additive or non-additive EU criterion and who have the same preferences for the consequences do not need to have the same attitude toward uncertainty.

In this paper, we provide an axiomatic characterization of a family of criteria for ranking sets of objects which, when interpreted in terms of choice under ignorance, is quite close in spirit to the EU family. Any criterion characterized herein can be viewed as ranking decisions (sets) on the basis of the expected utility of their consequences for some utility function, under the assumption that the decision maker assigns to every consequence of a decision an *equal probability* of occurrence. For this reason we refer to a criterion in this family as to a *Uniform Expected Utility* (UEU) criterion. Beside the framework of analysis, the main difference between UEU criteria and standard EU ones lies in the uniformity of the probabilities that our axiomatic structure implies. In our view, uniform probabilities are not unreasonable in the context of choice under ignorance. A decision maker who ignores the mechanism by which consequences are produced as a function of the states of nature, and who is only capable of identifying the set of possible consequences of a decision has *a priori* no reason to believe one consequence to be more likely than another. This principle of *insufficient reason*, renamed "principle of indifference" by Keynes (1921), has been, after all, the main justification given by early probability theorists such as Bernouilli and Laplace, to their assumption of uniform probabilities as applying to "games of chance" (see also Jaynes (2003) for a recent justification of this principle). To that extent, this paper can be seen as an axiomatic characterization of this principle.

The framework used in this paper is similar to that assumed in the literature on choices under ignorance in the sense that we describe decisions as *finite* sets of consequences and we propose axioms that apply to the ranking of these sets. We depart however from many papers in the literature by assuming that the universe of all conceivable consequences is rich - in a sense to be made precise below - and has an Archimedean structure (using the terminology of Krantz, Luce, Suppes, and Tversky (1971)). While we do not, for the main result, endow this universe with topological properties that would enable one to define appropriate continuity conditions on the ranking of decisions, our framework is compatible with such a topological setting. We actually illustrate this by characterizing, in our Theorem 4 below, the UEU family of criteria for decisions having their consequences in some connected separable topological space. Assuming such an environment enables one to eliminate the Richness condition and to replace the Archimedean property by a mild continuity condition imposed on the ranking of decisions.

The special case of our topological framework can be compared with that of Nehring and Puppe (1996). They endow the universe of consequences with a topology and impose a continuity property on the ranking of all finite subsets of the universe. However continuity is not a straightforward notion when applied to rankings of sets of objects (as opposed to a ranking of objects). For instance, a widely used notion of continuity for sets rankings, adopted by Nehring and Puppe (1996), is continuity with respect to the Hausdorff topology. This notion of continuity fails to recognize as continuous a UEU ranking, even though such a ranking is continuous when characterized in a Savagian (uncertainty) framework. This remark explains the difference between our results and those of Nehring and Puppe (1996). These authors characterize rankings that compare sets on the basis of their maximal and minimal elements only using Hausdorff continuity and a mild independence condition (satisfied by UEU criteria). In contrast, we consider an abstract setting that is compatible (as demonstrated by our Theorem 4) with many topological structures. We then characterize a family of rankings that are continuous in a natural sense, albeit not Hausdorff continuous, and that are not based only on the maximal and minimal elements of the sets.

To the best of our knowledge, there have been two other papers that have provided axiomatic characterizations of UEU criteria for ranking sets of objects. The first of them is Fishburn (1972) who characterizes the UEU family of rankings of all non-empty subsets of a finite universe (without Archimedean structure). The Fishburn characterization is a direct adaptation of the *additivity axiom* of Scott (1964), Adams (1965) and Fishburn

(1970). The unappealing nature of this axiom is well-known and is especially striking when adapted to the problem of comparing sets on the basis of their average utility. This axiom involves the construction of arbitrarily long sequences of set comparisons which are both difficult to motivate as primitive axioms as well as hard to verify in practice. By contrast, the structure of our model enables us to characterize the UEU family of rankings of sets by means of axioms that are, in our opinion, considerably easier to interpret and verify. We note that one of our axioms, "Averaging", is identified by Fishburn (1972) as being satisfied by any ranking in the UEU family. We show in this paper that, along with another axiom - "Restricted Independence" - Averaging and the Archimedean axiom actually characterize the UEU family of rankings of sets if the universe is sufficiently rich.

The other paper that contains a characterization of a *particular* UEU criterion (and not the whole family of such criteria) for ranking finite sets, is Baigent and Xu (2004). This paper characterizes, without an Archimedean structure, a ranking of finite sets based on the average *Borda score* of their elements. This ranking is clearly a member of the UEU family for which the utility of a consequence is defined by its Borda score. Baigent and Xu (2004) use the Averaging axiom, along with several others, in their characterization.

While choice under ignorance provides a natural interpretation of our framework, it is not the only one. What is provided in this paper is an axiomatic characterization of a family of rankings of all finite subsets of some universe of objects that have the property that each of these rankings *can* be interpreted *as if* it was assigning utility to every object in the universe and *as if* it was comparing sets on the basis of the (symmetric) average utility of these objects. There are at least two other contexts where such an axiomatic characterization could be useful.

The first is *mechanism design*, where several papers have used UEU criteria to model preferences of individual agents over subsets of some fundamental set of alternatives, with the subsets interpreted as possible results of a *social choice correspondence*. For example Barberà, Dutta, and Sen (2001) have characterized *strategy-proof* social choice correspondences when agents preferences are assumed to belong to the UEU family. Benoît (2002) studies a similar problem and recently, Ozyürt and Sanver (2006) have refined and extended this analysis. UEU criteria have also been considered by Peleg and Peters (2009) in their analysis of Nash consistent representation of effectivity functions

The second is the literature on *objective ambiguity* mentioned above, in which decision makers rank sets of lotteries. These sets of lotteries are interpreted as describing "ambiguous" decisions like those arising in the well-known Ellsberg paradox where the decision maker does not know which probability distribution over a set of outcomes is associated to a particular decision. In this setting, Ahn (2008) characterizes a family of criteria which, if interpreted in the finite setting considered herein, would contain the UEU family as a particular case. Interpreted for finite sets of lotteries, any criterion characterized by Ahn (2008) can be thought of as resulting from the comparison of the expected utility of their lotteries conditional on being in the set but without assuming that the probabilities are uniform. Ahn (2008) analysis bears many formal similarities with the somewhat non-standard Bolker-Jeffrey approach to decision making under uncertainty (see e.g. Bolker (1966), Bolker (1967), Jeffrey (1965) or, for a good discussion of this approach, Broome (1990)). A key aspect of the framework used by both Ahn (2008) and Bolker-Jeffrey, is that it only considers sets (other than singletons, considered in Ahn but excluded in Bolker-Jeffrey) that contain a continuum of elements. By contrast we limit attention to finite sets of lotteries. This makes the two settings *fundamentally* different and, among other things, prevents one from viewing our main result as a particular case of their results specialized to finite subsets and uniform probabilities. We believe that our finite subsets framework is an important one conceptually. It is, for instance, more suitable for addressing the Ellsberg paradox where the ambiguity concerning the number of balls in an urn is clearly of finite nature.

Another related paper on objective ambiguity is Olszewski (2007). It considers a framework in which sets of lotteries can be of any size but where lotteries are restricted to have a finite and given support. It characterizes the family of rankings of sets that can be expressed as a weighted average of the utility of their best and their worst element. This family can be viewed as a salient subset of the family characterized in Nehring and Puppe (1996).

The plan of the rest of this paper is as follows. In the next section, we present the formal framework, the definition of the axioms, and the family of rankings. The third section presents the main results and the fourth section shows how the results can be obtained if topological properties are imposed on the universe. The fifth section comments on the independence of the axioms and the sixth section concludes.

2 Notation and basic concepts

2.1 Notation

The sets of integers, non-negative integers, real numbers and non-negative real numbers are denoted respectively by $\mathbb{N}$, $\mathbb{N}_+$, $\mathbb{R}$ and $\mathbb{R}_+$. The cardinality of any set A is denoted by $\#A$ and the k-fold Cartesian product of a set A with itself is denoted by A^k. By a *binary relation* $\succsim$ on a set Ω, we mean a subset of $\Omega \times \Omega$. Following the convention in economics, we write $x \succsim y$ instead of $(x, y) \in R$. Given a binary relation $\succsim$, we define its *symmetric factor* $\sim$ by $x \sim y \Longleftrightarrow x \succsim y$ and $y \succsim x$ and its *asymmetric* factor $\succ$ by $x \succ y \Longleftrightarrow x \succsim y$ and not $(y \succsim x)$. A binary relation $\succsim$ on Ω is *reflexive* if the statement $x \succsim x$ holds for every x in Ω, is *transitive* if $x \succsim z$ always follows

$x \succsim y$ and $y \succsim z$ for any x, y, $z \in \Omega$ and is *complete* if $x \succsim y$ or $y \succsim x$ holds for every distinct x and y in Ω. A reflexive, transitive and complete binary relation is called an *ordering*.

2.2 Basic concepts

Let X be the set of objects. To motivate the framework in terms of decision making under ignorance, we will refer to these objects as to possible "consequences" of a decision, keeping in mind the "objective ambiguity" interpretation that these consequences be lotteries defined on a more fundamental space of consequences. While we do not make any specific assumptions on X, it will be clear subsequently that the axioms that we impose makes it natural to regard this set as infinite. As an example, further considered in section 4 below, one could think of X as being $\mathbb{R}$, interpreted as the set of all conceivable financial returns (either negative or positive) of some investment decision in a highly uncertain environment. As another (objective ambiguity) example, one could think of X as the set of all conceivable probability distributions on a basic set of k different prizes.

We denote by $\mathcal{P}(X)$ the set of all non-empty *finite* subsets of X (with generic elements A, B, C, D, etc.). Any such a subset is interpreted as a description of all consequences of a *completely uncertain* decision or, for short, as a *decision*. A *certain* decision with consequence $x \in X$ is identified by the singleton $\{x\}$.

Let $\succsim$ (with asymmetric and symmetric factors $\succ$ and $\sim$ respectively) be an *ordering* on $\mathcal{P}(X)$. We interpret the statement $A \succsim B$ as meaning "decision with consequences in A is weakly preferred to decision with consequences in B". A similar interpretation is given to the statements $A \succ B$ ("strictly preferred to") and $A \sim B$ ("equivalent to").

We want to identify the properties (axioms) of the ordering $\succsim$ that are necessary and sufficient for the existence of a function $u : X \to \mathbb{R}$ such that, for every A and B in $\mathcal{P}(X)$:

$$A \succsim B \Longleftrightarrow \sum_{a \in A} \frac{u(a)}{\#A} \geq \sum_{b \in B} \frac{u(b)}{\#B} \qquad (1)$$

An ordering satisfying this property *could* therefore be thought of as resulting from the comparisons of the *expected utility* of the consequences of the decision for some utility function under the assumption that the decision maker assigns to every consequence of a decision an *equal probability* of occurrence. There are obviously many criteria like that, as many as there are logically conceivable utility functions (up to an affine transform) defined on X. We refer to any ranking that satisfies (1) for some function u as to a *Uniform Expected Utility* (UEU) criterion.

We now introduce the three main axioms that characterize the family of

UEU criteria. The first of these axioms, that we call Averaging, is stated as follows.

Axiom 1 *(Averaging) For all disjoint sets* A *and* $B \in \mathcal{P}(X)$*,* $A \succsim B \Leftrightarrow A \succsim A \cup B \Leftrightarrow A \cup B \succsim B$*.*

In words, this axiom asserts that enlarging a set A with a (disjoint) set B is worth doing (viz. not worth doing) if and only if the set B of added consequence is better (viz worse) than the set A to which it is added. We call this axiom "Averaging" because it captures an intuitive property satisfied by calculations of "average" in various settings (e.g. adding a student to a class will increase the average of the class if and only if the grade of the added student is larger than the average of the class). The "only if" part of the axiom is obviously very strong since it asserts that the *only* reason for ranking a set B above (below) a set A is when the addition of B to A is considered a good (bad) thing. The Averaging axiom is a compact version of the four Averaging conditions AC1-AC4 discussed in Fishburn (1972) and shown by him to be implied by the UEU family of criteria (as well as by a variant of the additivity axiom of Scott (1964), Adams (1965) and Fishburn (1970)). The Averaging axiom has been used also by Baigent and Xu (2004) in their characterization of the average Borda ranking of sets as well as by Ahn (2008) and Bolker (1966) (see also Bolker (1967)) in their characterization of an important family of criteria, containing UEU ones, for ranking *atomless* subsets of a universe. A weaker version of Averaging (that only requires the "if" part in its statement) is used by Olszewski (2007) in his characterization of a ranking of sets based on the weighted average of the utility of their best and their worst alternative, and by Gul and Pesendorfer (2001) in their ranking of sets of objects, interpreted as menus of alternatives, in a way that reflects temptation and self-control. It is also worth mentioning that the Averaging axiom *implies* the Gärdenfors (1976) principle discussed at length in the literature on ignorance as surveyed in Barberà, Bossert, and Pattanaik (2004). This Gärdenfors principle can be stated formally as follows.

Condition 1 *(Gärdenfors Principle) for all* $A \in \mathcal{P}(X)$*,* $(x \in X \backslash A$ *and* $\{x\} \succ \{a\}$ *for all* $a \in A) \Rightarrow A \cup \{x\} \succ A$ *and* $(y \in X \backslash A$ *and* $\{a\} \succ \{y\}$ *for all* $a \in A) \Rightarrow A \succ A \cup \{y\}$*.*

This principle says that is always (never) worth adding to a set a consequence which, if certain, is better (worse) than all consequences in the set. For future reference we record (without proof) in the following proposition, the fact that the Averaging axiom implies the Gärdenfors principle when applied to a transitive ranking of $\mathcal{P}(X)$.

Proposition 1 *Let* $\succsim$ *be a transitive binary relation on* $\mathcal{P}(X)$ *that satisfies Averaging. Then* $\succsim$ *satisfies the Gärdenfors principle*

The second axiom that enters into the characterization of the family of UEU rankings is the following *Restricted Independence* axiom.

Axiom 2 *(Restricted Independence) For all* A, B *and* $C \in \mathcal{P}(X)$ *satisfying* $\#A = \#B$ *and* $A \cap C = B \cap C = \emptyset$, $A \succsim B \Leftrightarrow A \cup C \succsim B \cup C$.

This axiom requires that the ranking of sets with the same number of elements be independent of any elements that they have in common. Adding or subtracting these common elements from the two sets should not affect their ranking. A weak form of the Restricted Independence condition, applied only to the case where A and B are singletons, plays an important role in Nehring and Puppe (1996) and Puppe (1995). It is worth noticing that the scope of this *independence* axiom is indeed significantly *restricted* by the fact that it applies only to sets that have the same number of elements.

The third, and last, axiom that characterizes the UEU family of orderings of $\mathcal{P}(X)$ is the following one that we call "Archimedean" by analogy to axioms of a similar spirit discussed in Krantz, Luce, Suppes, and Tversky (1971).

Axiom 3 *(Archimedean) If a sequence* $\{c_i\}$, *for* $i = 1, 2, ...$ *of consequences* $c_i \in X$ *is such that one has either* $\{c_i, a\} \sim \{c_{i+1}, b\}$ *for all* i, $i+1$ *with* $i = 1, 2, ...$ *or* $\{c_{i+1}, a\} \sim \{c_i, b\}$ *for all* i, $i+1$ *with* $i = 1, 2, ...$ *for some consequences* a *and* b, *distinct from any element of the sequence, and satisfying* $\{a\} \succ \{b\}$, *then, if the sequence is strictly bounded by* x *and* $y \in X$ *in the sense that* $\{x\} \succ \{c_i\} \succ \{y\}$ *for every* i, *the sequence must be finite.*

While appearing somewhat involved, this axiom is quite simple. Suppose we consider two consequences a and b over which the decision maker has definite preference. To be specific, think of a as a "good" consequence and b as a "bad" one. Imagine now that the decision maker is indifferent between an uncertain decision leading to either a or some consequence c_1 and an uncertain decision leading to either b or to some consequence c_2. Intuitively, for such an indifference to arise, c_2 (added to the bad consequence b) must be better than c_1 (added to the good consequence a). Imagine also that the decision maker is indifferent between uncertain decision $\{a, c_2\}$ and uncertain decision $\{b, c_3\}$ for some other consequence c_3. Again such an indifference suggests that c_3 is better than c_2. When combined with the previous indifference (e.g. between $\{a, c_1\}$ and $\{b, c_2\}$), this indifference also suggests that c_1 and c_2 on the one hand and c_2 and c_3 are "equally spaced" in the preference ordering of the decision maker. Indeed both c_1 and c_2 and c_2

and c_3 "calibrate" the decision maker's preference for a over b. Suppose now that one can construct a sequence of arbitrary length (finite or not) of consequences like that (e.g. $c_1, c_2, c_3, ...$) that are increasing (or decreasing) and "equally spaced" in this sense. The axiom requires that if the sequence formed by these consequences is infinite, then it can not be bounded. Such a property would be clearly satisfied if the objects where numbers (since any infinite and monotonic sequence of equidistant numbers is unbounded). This axiom can be considered to be mild since it "bites" only when there exist sequences of the type described above (such sequences are called "standard sequences" in the measurement theory literature). The axiom is, of course, trivially satisfied if X is a finite set. As will be seen in the next section, this axiom can be replaced by a continuity condition if a topological structure is imposed on X.

We now formally state that these three axioms are satisfied by any UEU criterion.

Proposition 2 *Any UEU criterion satisfies Averaging, Restricted Independence and the Archimedean axiom.*

Proof. *We leave to the reader the task of verifying that a UEU criterion satisfies Averaging and Restricted Independence. To prove that it satisfies the Archimedean axiom, let* $\succsim$ *on* $\mathcal{P}(X)$ *be a UEU criterion and consider a sequence of consequences* $\{c_i\}$ *for* $i = 1, 2, ...$ *such that, for some consequences* a *and* b *distinct from every element in the sequence satisfying* $\{a\} \succ \{b\}$*, one has, say,* $\{c_i, a\} \sim \{c_{i+1}, b\}$ *for all* $i = 1, 2, ...$*(the argument is similar if* $\{c_{i+1}, a\} \sim \{c_i, b\}$ *for all* $i = 1, 2, ...$*). Since* $\succsim$ *is a UEU criterion, there exists a function* $u : X \to \mathbb{R}$ *such that* $u(a) > u(b)$ *and* $u(c_i)+u(a) = u(c_{i+1})+u(b) \Leftrightarrow u(a)-u(b) = u(c_{i+1})-u(c_i)$ *for all* i*. Assume that the sequence is strictly bounded by* x *and* $y \in X$ *so that* $\{x\} \succ \{c_i\} \succ \{y\}$ *for all* i*. Since* $\succsim$ *is a UEU criterion, one has* $u(x) > u(c_i) > u(y)$*. Define the numbers* u_i *by* $u_i = u(c_i)$ *and* d *by* $d = u(a)-u(b) > 0$*. We therefore have a sequence of numbers* u_i *(for* $i = 1, ..$*) such that* $u_i = (i-1)d + u_1 > u(y)$ *for every* i *for some strictly positive real number* d*. Clearly one can only have* $u(x) > (i-1)d + u_1$ *for all element* i *of the increasing sequence of numbers* $\{(i-1)d + u_1\}$ *if this sequence is finite.* ■

As shall be seen Averaging, Restricted Independence and the Archimedean axioms characterize the family of UEU rankings of sets if some structure is imposed on the environment. We provide this structure by imposing another axiom on the pair $\langle X, \succsim \rangle$. This axiom, referred to as "*Richness*", imposes richness and smoothness on *both* the set X of alternatives *and* the ordering $\succsim$. Yet this axiom is not specifically tailored to UEU criteria and may even be violated by these criteria if the set X of alternatives is too "sparse". Theorem 4 below establishes that this Richness axiom can be dispensed with if X is taken to be a connected separable topological space. The formal statement of the Richness axiom is as follows.

Axiom 4 *(Richness) For every set $B \in \mathcal{P}(X) \subset X$, and every finite, but possibly empty, subset A of X, if there are consequences c^* and c_* in X such that $A \cup \{c^*\} \succsim B \succsim A \cup \{c_*\}$, then there exists a consequence $c \in X$ such that $A \cup \{c\} \sim B$.*

This axiom states that the universe is sufficiently rich to enable, by adding *single* consequences to sets, various kinds of comparisons with the ordering $\succsim$. Suppose that, starting with two decisions A and B, it is possible to add consequences c^* and c_* to A in such a way that A enlarged with c^* is ranked above B and A enlarged with c_* is ranked below B. Then it must also be possible to add to A a consequence c such that the resulting set of consequences is indifferent to A. In a sense, this axiom is weak since the asserted existence of the consequence c is contingent upon the existence of consequences c^* and c_* that have the required properties. Yet, Richness applies also if the set A to which the consequences c^*, c_* and c are added is empty. Because of this, the Richness axiom has the somewhat strong implication, at least when combined with the Gardenförs principle, that every uncertain decision has a "certainty equivalent". Put differently if a decision maker ranks uncertain decisions by an ordering that satisfies Averaging and Richness, then for any uncertain decision, there must exist a certain decision that the decision maker considers equivalent to it. For further reference, we state formally this "Certainty Equivalence" condition and the fact that it is implied by Richness if the ranking satisfies Averaging as follows.

Condition 2 *(Certainty Equivalence) For every $B \in \mathcal{P}(X)$, there exists a consequence $b \in X$ such that $\{b\} \sim B$.*

Proposition 3 *Let $\succsim$ be an ordering on $\mathcal{P}(X)$ satisfying Averaging and Richness. Then $\succsim$ satisfies the Certainty Equivalence condition.*

Proof. *Let B be any set in $\mathcal{P}(X)$. Because B is non-empty and $\succsim$ is an ordering on $\mathcal{P}(X)$, there exists a consequence $c^* \in B$ such that $\{c^*\} \succsim \{b\}$ for all $b \in B$ and there exists a consequence c_* (not necessarily distinct from c^*) such that $\{b\} \succsim \{c_*\}$ for all $b \in B$. By Averaging one has $\{c^*\} \succsim B \succsim \{c_*\}$ which can be written equivalently as $\varnothing \cup \{c^*\} \succsim B \succsim \varnothing \cup \{c_*\}$. By Richness (for $A = \varnothing$), there exists c such that $\{c\} \cup \varnothing \sim B$, which proves Certainty Equivalence.* ■

It is also worth mentioning that the combination of the Richness and Averaging axioms implies either that the ranking $\succsim$ is trivial or that there are infinitely many consequences in X. Specifically, if X is finite, then a decision maker who compares decisions in $\mathcal{P}(X)$ with an ordering satisfying Averaging and Richness (and therefore Certainty Equivalence, due to Proposition 3) must be indifferent between all such decisions. We state this formally as follows.

Proposition 4 *Suppose $\#X = n$ for some $n \in \mathbb{N}_+$ and let $\succsim$ be an ordering on $\mathcal{P}(X)$ satisfying Averaging. Then $\succsim$ satisfies Certainty Equivalence if and only if $A \sim B$ for all $A, B \in \mathcal{P}(X)$.*

Proof. *It is clear that the trivial ordering defined by $A \sim B$ for all A, $B \in \mathcal{P}(X)$ satisfies Certainty Equivalence (as well as Averaging). To prove the reverse implication, write the finite set X as $X = \{x_1, ..., x_n\}$ and assume without loss of generality (since $\succsim$ is an ordering) that $\{x_i\} \succsim \{x_{i+1}\}$ for $i = 1, ..., \#X - 1$. By Averaging, we must have, for every $i = 1, ..., \#X - 1$:*

$$\{x_i\} \succsim \{x_i, x_{i+1}\} \succsim \{x_{i+1}\}$$

By certainty there exists $x(i) \in X$ such that $x(i) \sim \{x_i, x_{i+1}\}$. Either $\{x(i)\} \succsim \{x_i\}$ or $\{x_{i+1}\} \succsim \{x(i)\}$. In the first case, one has $\{x_i, x_{i+1}\} \sim \{x(i)\} \succsim \{x_i\}$ so that, by Averaging and transitivity, $\{x_i\} \sim \{x_{i+1}\}$. In the other case, one has $\{x_{i+1}\} \succsim \{x(i)\} \sim \{x_i, x_{i+1}\}$ so that, again, the conclusion $\{x_i\} \sim \{x_{i+1}\}$ follows from Averaging and transitivity. Hence all pairs and singletons must be indifferent. Repeated application of Averaging (adding first indifferent singletons to pairs and then indifferent singletons to triples etc.) leads to the conclusion of universal indifference. ■

We note in passing that the trivial ordering that considers all sets to be indifferent is a member of the UEU family (any constant function u having X as domain could serve as a representation as per (1)). Hence, in the rest of the paper, we shall be interested in characterizing the UEU family of orderings of $\mathcal{P}(X)$ in the **non trivial** case where there are at least two sets A and B such $A \succ B$.

Beside forcing X to be infinite (at least when combined with Averaging), the Richness axiom precludes from consideration some "discontinuous" rankings such as the "Leximin" or the "Leximax" rules studied in Pattanaik and Peleg (1984). For instance, the Leximin rule compares sets on the basis of their worst consequences. If a tie in the worst consequence arises, then the second worst consequence is considered and so on until either a strict ranking is obtained or the consequences in at least one of the sets are exhausted. In the latter case the set which contains the largest number of elements is ranked above. It is clear that such a Leximin rule violates Richness. Indeed if we take $X = \mathbb{R}_+$ one has that $\{1, 3\} \prec \{2\} \prec \{2, 3\}$ for the Leximin criterion. Yet, contrary to what is required by Richness, it is impossible to find a non-negative real number x such that $\{x\} \cup \{3\} \sim \{2\}$.

It should be also noted that the Richness axiom is not specifically related to the UEU family of ranking of decisions and may even be violated by a UEU criterion if the set X is not sufficiently rich. For example if $X = \mathbb{N}$ and the function u of (1) is the identity function, we notice that, since $\frac{2+6+7}{3} = 5 < \frac{5+6}{2} = 5.5 < \frac{2+6+10}{3} = 6$, we have $\{2,6,7\} \prec \{5,6\} \prec \{2,6,10\}$. Yet, contrary to what Richness requires, there does *not* exist any *integer* c such that $\frac{2+6+c}{3} = \frac{5+6}{2}$ and, therefore, such that $\{2,6,c\} \sim \{5,6\}$.

3 Main results

In order to prove the main result that the family of UEU rankings of all finite subsets of X is characterized, given Richness, by Averaging, Restricted Independence and the Archimedean axiom, we proceed as follows. We first consider the sets $m(X)$ and $M(X)$ of *minimal* and *maximal* (respectively) elements of X with respect to the restriction of $\succsim$ to singletons defined by:

$$m(X) = \{x \in X : \{x\} \precsim \{y\} \ \forall \ y \in X\} \text{ and}$$
$$M(X) = \{z \in X : \{z\} \succsim \{y\} \ \forall \ y \in X\}$$

The possibility that either (or both) of these two sets be empty is, of course, not ruled out. Let X' be defined by $X' = X \backslash (m(X) \cup M(X))$. Hence, X' is the set of all conceivable consequences that remain after one has removed the worst and the best certain consequences (if any) from X. It is easy to see that, if $\succsim$ is an ordering satisfying Averaging and certainty equivalent, then the set X' is "unbounded" in the sense that, for any consequence $x \in X'$, one can find two consequences w and z in X' such that $\{w\} \succ \{x\} \succ \{z\}$. For later reference, we state formally this fact as follows.

Proposition 5 *Let X be a universe of consequences and let $\succsim$ be a non-trivial ordering on $\mathcal{P}(X)$ satisfying Richness and Averaging. Then, for all consequence $x \in X'$, there are consequences w and z in X' such that $\{w\} \prec \{x\} \prec \{z\}$.*

Proof. *Consider any $x \in X'$. Such an x exists if $\succsim$ is non-trivial. Let us first find a consequence w such that $\{w\} \prec \{x\}$. Suppose first that $m(X) = \varnothing$. This means that $x \notin m(X)$ so that there exists some $t \in X$ such that $\{t\} \prec \{x\}$. By Proposition 3, the ordering $\succsim$ satisfies Certainty Equivalence. Hence there exists some $w \in X$ such that $\{w\} \sim \{t, x\}$. By Averaging and transitivity, $\{x\} \succ \{w\} \succ \{t\}$. Hence $w \notin m(X) \cup M(x)$ so that $w \in X'$. Suppose now that $m(X) \neq \varnothing$ and let $t \in m(X)$. By the definition of $m(X)$, we have $\{t\} \prec \{x\}$ so that, by Certainty Equivalence again, there exists a consequence w such that $\{w\} \sim \{t, x\}$. As before, we can conclude from the Averaging axiom that $\{t\} \prec \{w\} \prec \{x\}$ so that $w \in$*

$X \backslash (m(X) \cup M(X))$, as required. The argument for finding a consequence $z \in X'$ such that $\{z\} \succ \{x\}$ is similar. ■

We proceed by first proving the result on $\mathcal{P}(X')$ defined as the set of all finite subsets of X'. After having obtained that any ordering on $\mathcal{P}(X')$ satisfying Averaging, Restricted Independence, the Archimedean axiom and Richness can be represented as per (1) for some function $u : X' \to \mathbb{R}$, we then show that this numerical representation can be "extended" to the whole set X.

The proof that any ordering on $\mathcal{P}(X')$ satisfying Averaging, Restricted Independence, the Archimedean axiom and Richness can be represented as per (1) for some function $u : X' \to \mathbb{R}$ proceeds itself in two steps.

First, we show that Averaging, Restricted Independence and the Archimedean axiom characterize the family of UEU criteria in an environment satisfying Richness if one restricts attention to subsets of X' that have **at most two** consequences. The proof of this first theorem provided in the Appendix rides on Theorem 2 of Krantz, Luce, Suppes, and Tversky (1971) (p. 257) that enables an additively separable numerical representation of an ordering over a Cartesian product $X \times X$ (see also Debreu (1960) or Adams and Fagot (1959) for an earlier statement framed in a topological setting). An important ingredient in the proof of this theorem is the demonstration that, given our axiomatic structure, the ranking of the Cartesian product $X \times X$ induced by $\succsim$ satisfies the so-called "Thomsen condition" (see Krantz, Luce, Suppes, and Tversky (1971), Definition 3, p. 250). This is shown in Lemma 3 of the Appendix.

The formal statement of this result is as follows.

Theorem 1 *Let X be a set of consequences and $\succsim$ be a non-trivial ordering on $\mathcal{P}(X)$ satisfying Richness. Then if $\succsim$ satisfies Averaging, Restricted Independence and the Archimedean axiom, its restriction to the sets in $\mathcal{P}(X')$ of cardinality no greater than two can be represented as per (1) for some utility function $u : X \to \mathbb{R}$. Furthermore, the utility function u is unique up to a positive affine transformation.*

Our main result extends Theorem 1 to subsets of X' with an arbitrary (but finite) number of consequences using the same axioms. Specifically, we prove that the unique utility function whose expectation (under uniform probabilities) represents the ranking of sets containing at most two elements exhibited in Theorem 1 also represents the ranking of sets of larger cardinality. While the full proof of this extension is done in the Appendix using various auxiliary results, a key step in the argument, provided by the following Lemma 1, also proved in the Appendix, is the ability to approximate the arithmetic mean of a set of n numbers *recursively* from the arithmetic means of *pairs* of those numbers.

Lemma 1 *Let $U = \{u_1, ..., u_n\}$ be a set of n numbers such that $u_1 \leq u_2 \leq ... \leq u_n$ with arithmetic mean $\overline{u}$. Define the $n-1$ sequences $\{b_h^i\}$, $i = 1, 2, ...$ and $h = 1, ..., n-1$ by:*

$$b_{n-1}^0 = (u_n + u_{n-1})/2,$$

$$b_h^0 = (u_h + b_{h+1})/2$$

for $h = 1, ..., n-2$ and for $i = 1, 2, ...$:

$$\begin{aligned} b_1^{2i-1} &= b_1^{2i-2}, \\ b_h^{2i-1} &= \frac{b_{h-1}^{2i-1} + b_h^{2i-2}}{2} \text{ for } h = 2, ..., n-1, \\ b_{n-1}^{2i} &= b_{n-1}^{2i-1} \text{ and} \\ b_h^{2i} &= \frac{b_h^{2i-1} + b_{h+1}^{2i}}{2} \text{ for } h = 1, ..., n-2. \end{aligned}$$

Then:

$$\lim_{i \to \infty} b_h^i = \overline{u} \text{ for all } h = 1, ..., n-1$$

Equipped with this lemma and the other auxiliary results stated and proved in the Appendix, we prove - also in the Appendix - the following result.

Theorem 2 *Let $\succsim$ be a non-trivial ordering on $\mathcal{P}(X')$ satisfying Richness. Then $\succsim$ satisfies Averaging, Restricted Independence and the Archimedean axiom if and only if it is a UEU criterion. Furthermore, the u function in the definition of a UEU criterion is unique up to a positive affine transformation.*

The last step in the proof consists in showing that the numerical representation of $\succsim$ restricted to $\mathcal{P}(X')$ can be extended to the whole set $\mathcal{P}(X)$. This step is provided by the proof, in the Appendix, of the following theorem.

Theorem 3 *Let $\succsim$ be a non-trivial ordering on $\mathcal{P}(X)$ satisfying Richness. Then $\succsim$ satisfies Averaging, Restricted Independence and the Archimedean axiom if and only if it is a UEU criterion. Furthermore, the u function in the definition of a UEU criterion is unique up to a positive affine transformation.*

4 Topological interpretation of the structural environment

We show in this section that if one imposes a topological structure on the set X from the outset, then there is no need to impose the Richness axiom. Moreover, the Archimedean axiom can be replaced by a mild continuity one in the characterization of the UEU family of orderings. This shows that the "algebraic" structure considered in the previous section is more general for the purpose at hand than the standard topological structure assumed in many decision problems (see e.g. Wakker (1988) for similar findings in a different context).

Assume specifically that X is a connected subset of a separable topological space.[1] At least two interpretations could be given to X in that context. First, X could be thought of as the set of all bundles of k goods that could result from any uncertain decision (taking $k = 1$ would obviously cover the case, discussed earlier, of decisions with pecuniary consequences). In that case, it would be natural to take $X = \mathbb{R}^k$ (all bundles of goods or amounts of money - possibly negative - are *a priori* conceivable).

The second interpretation, developed along the line of the literature on objective ambiguity (see e.g. Ahn (2008) or Olszewski (2007)) would be to view X as the set of all *lotteries* yielding k different prizes. A typical element $p \in X$ would then be thought of as a probability vector assigning to every prize i its probability of realization $p_i \in [0, 1]$. A finite set $A \subset X$ of such probability vectors would then be interpreted as an ambiguous decision in which the precise probability distribution over the set of k prizes is not known to the decision maker. A classical instance of decision making under this kind of ambiguity is provided by the so-called Ellsberg paradox in which a decision maker does not know how a certain number of balls are split between two colors (see Olszewski (2007) for further discussion). If this interpretation is favoured, then $X = S^{k-1} = \{p \in [0, 1]^k : \sum_{j=1}^{k} p_j = 1\}$.

For either of these interpretations, we believe that the following *Continuity* axiom is a plausible one to impose on $\succsim$.

Axiom 5 *(Continuity)For every set $A \in \mathcal{P}(X)$, and consequences $y, z \in X$, the sets $B(A) = \{x \in X : \{x\} \succsim A\}$ and $W(A) = \{x \in X : A \succeq \{x\}\}$ are both closed in X.*

This axiom says that a small change in a consequence - or a lottery - should not have drastic effect on the ranking of this consequence (lottery)

[1] A subset A of a topological space is *connected* if it can not be written as a finite union of pairwise disjoint open sets. A topological space S is separable if it contains a countable subset whose closure is S.

obtained for sure *vis-à-vis* any set. Notice that this Continuity axiom, which only concerns comparisons of sets vis-à-vis singletons is much weaker than the (Vietoris) continuity condition examined in Nehring and Puppe (1996) which restricts the comparisons of any two sets in a way that is not even compatible with the UEU family of set rankings. It is clear that any UEU criterion for which the function u that appears in the representation (1) is continuous satisfies this axiom.

We now establish in the following theorem, that in this environment, the UEU family of rankings of $\mathcal{P}(X)$ is characterized by Averaging, Restricted Independence and Continuity. In order to prove this theorem, we only need to prove that, if X is a connected subset of separable topological space, then an ordering of $\mathcal{P}(X)$ that satisfies the Continuity axiom as well as Averaging and Restricted Independence satisfies the Richness and the Archimedean axioms. As before, the proof of this theorem is relegated to the Appendix.

Theorem 4 *Let X be a connected subset of a separable topological space and let $\succsim$ be an ordering of $\mathcal{P}(X)$. Then $\succsim$ satisfies Continuity, Averaging and Restricted Independence if and only if it is a UEU criterion for which the function u is continuous.*

As mentioned in Introduction, a limitation of UEU criteria is that they impose equal probability of realization to every consequence that a decision can have. A generalization of the UEU family of criteria that would avoid this feature is family of all orderings $\succsim$ of $P(X)$ that can be defined, for every sets A and B in $P(X)$, by:

$$A \succsim B \Longleftrightarrow \frac{\displaystyle\sum_{a \in A} p(a)u(a)}{\displaystyle\sum_{a \in A} p(a)} \geq \frac{\displaystyle\sum_{b \in B} p(b)u(b)}{\displaystyle\sum_{b \in B} p(b)} \tag{2}$$

for some real-valued functions u and p, the latter taking strictly positive values, both having X as domain. Any UEU criterion is a member of this family that satisfies the additional property that, for all consequences $x \in X$, $p(x) = c$ for some strictly positive real number c. Orderings that can be represented as per (2) for some real-valued functions u and p can be thought of as comparing sets on the basis of the expected utility of their consequence, but without imposing the requirement that the probability of all consequences is the same. This interpretation obviously requires that $p(x)$ can be thought of as "the probability of x". Under this interpretation, any ordering of $P(X)$ that can be represented as per (2) can be viewed as comparing sets on the basis of their expected utility *conditional of being in the sets*. It can be checked easily that any ordering that can be represented as per (2) satisfies Averaging but may violate Restricted Independence (see for instance example 2 in section 5).

The family of rankings of $P(X)$ that can be represented as per (2) is somewhat evocative of the family examined by Ahn (2008) (and in a somewhat different paradigm Bolker (1966)) in the specific context of decision under objective ambiguity. Ahn (2008) has indeed characterized all rankings that can be defined, for every two sets A and B on his domain, by:

$$A \succsim B \Longleftrightarrow \frac{\int_A u(a)d\mu}{\mu(A)} \geq \frac{\int_B u(b)d\mu}{\mu(B)} \tag{3}$$

for some function u and some *probability measure* μ defined on an algebra of subsets of X all containing continuously many elements. However the continuous setting assumed in Ahn (2008) is so fundamentally different from ours that we cannot view the UEU family of rankings as a particular subset of the rankings that can be represented as per (3). In the finite setting, the UEU family of rankings is obtained by considering any constant and positive function p in the representation (2). The "probability of a set" in that case is simply given by its cardinality. In Ahn's setting, it is unclear what measure μ could serve as the uniform probability. A plausible candidate could be the Lebesgue measure, but this is by no means the only one.

As a family of criteria for decision making under objective ambiguity, the UEU family is rather abstract in the sense that it does not impose any specific structure on the functions u that appears in the representation as per (1). Yet, if the elements of the sets are interpreted as lotteries, it could make sense to impose some further properties on the utility function (for instance, that it be linear in probabilities). While we do not do this here, it is clear that it could be done easily by means of additional axioms imposed on the ranking of singletons, as done in Ahn (2008) and Olszewski (2007).

5 Independence of the Axioms

We show that Averaging, Restricted Independence and the Archimedean axioms are logically independent when they are imposed on an ordering that satisfies Richness. Since any UEU criterion satisfies Averaging, Restricted Independence and the Archimedean axiom, it can therefore be said that these three axioms provides a minimal characterization of the UEU family of orderings in any environment where these orderings satisfy Richness. It is clear that these examples can also be used to show the logical independence of Averaging, Restricted Independence and Continuity when X is assumed to be a connected subset of a separable topological space.

Example 1 *Let $X = \mathbb{R}$ and, for all A, $B \in \mathcal{P}(X)$, $A \succsim B \iff \sum_{a \in A} a \geq \sum_{b \in B} b$. The reader can check easily that this ranking satisfies Restricted Independence, the Archimedean axiom, and Richness but violates Averaging.*

Indeed, $\{3\} \succsim \{1,2\}$ *but* $\{3\} \prec \{1,2,3\}$. *The reader can also verify that* $\succsim$ *satisfies Continuity.*

Example 2 *Let* $X = \mathbb{R}_{++}$ *and define* $\succsim$ *by:*

$$A \succsim B \Leftrightarrow \frac{\sum_{a \in A} a}{\sum_{a \in A} \frac{1}{a}} \geq \frac{\sum_{b \in B} b}{\sum_{b \in B} \frac{1}{b}}. \tag{4}$$

This ordering is clearly a member of the family represented as per (2) where p *is defined by* $p(x) = \frac{1}{x}$ *and* u *by* $u(x) = x^2$. *It satisfies for this reason the Averaging axiom. It is a continuous ordering of* X *because the set* $B(A) = \{x \in X : x^2 \geq \frac{\sum_{a \in A} a}{\sum_{a \in A} \frac{1}{a}}\}$ *and* $W(A) = \{x \in X : \frac{\sum_{a \in A} a}{\sum_{a \in A} \frac{1}{a}} \geq x^2\}$ *are closed for any set* $A \in \mathcal{P}(X)$. *Moreover it can be seen that it satisfies the Archimedean axiom using an argument that parallels that of Proposition 2. Yet* $\succsim$ *violates Restricted Independence because if we take*

$$A = \{1,7\},\ B = \{2,3\} \text{ and } C = \{4,12\}$$

we have $A \succsim B$ *since, using (4):*

$$\frac{1+7}{1+\frac{1}{7}} = 7 \geq \frac{2+3}{\frac{1}{2}+\frac{1}{3}} = 6$$

but, contrary to what is required by Restricted Independence, one has $A \cup C \prec B \cup C$ *since:*

$$\frac{1+4+7+12}{1+\frac{1}{4}+\frac{1}{7}+\frac{1}{12}} = \frac{24 \times 84}{84+21+12+7} = \frac{6 \times 84}{31} < \frac{2+3+4+12}{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{12}} = 6 \times 3$$

Example 3 [2]*Let* $X = \mathbb{R} \times \mathbb{R}$ *and let, for any element* $a \in X$, a_i *denote the* i*-th component of* a, *for* $i \in \{1,2\}$. *Define the ordering* $\succsim$ *by:*
$A \sim B$ *iff* $\frac{1}{\#A}\sum_{a \in A} a_1 = \frac{1}{\#B}\sum_{b \in B} b_1$ *and* $\frac{1}{\#A}\sum_{a \in A} a_2 = \frac{1}{\#B}\sum_{b \in B} b_2$;
$A \succ B$ *iff either:*
(i) $\frac{1}{\#A}\sum_{a \in A} a_1 > \frac{1}{\#B}\sum_{b \in B} b_1$ *or:*
(ii) $\frac{1}{\#A}\sum_{a \in A} a_1 = \sum_{b \in B} b_1$ *and* $\frac{1}{\#A}\sum_{a \in A} a_2 > \frac{1}{\#B}\sum_{b \in B} b_2$.
We first prove that $\succsim$ *violates the Archimedean axiom. Indeed, the set* $\{(2,i) : i \in \mathbb{Z}\}$ *is a standard sequence because* $\{(2,i),(1,2)\} \sim \{(2,i+$

[2]We are indebted to David Krantz for very useful discussions about this example.

1), (1, 1)} for all $i \in \mathbb{Z}$. This standard sequence is infinite but is bounded. Indeed, for any $i \in N$, we have $\{(3,1)\} \succ \{(2,i)\} \succ \{(1,1)\}$. We leave to the reader the (easy) task of verifying that this ordering, which is a lexicographic combination of two UEU orderings defined on each dimension, is not continuous either. Let us show that $\succsim$ satisfies Averaging. Suppose first that $A \succ B$. Using the definition of $\succsim$, this is either equivalent to:

$$\begin{gathered}\frac{1}{\#A}\sum_{a\in A}a_1 > \frac{1}{\#B}\sum_{b\in B}b_1 \\ \Longleftrightarrow \\ \frac{1}{\#A}\sum_{a\in A}a_1 > \frac{1}{\#A+\#B}\sum_{\alpha\in A\cup B}\alpha_1 \\ \Longleftrightarrow \\ A \succ A\cup B\end{gathered}$$

or to:

$$\begin{aligned}&\frac{1}{\#A}\sum_{a\in A}a_1 = \frac{1}{\#B}\sum_{b\in B}b_1 \text{ and } \frac{1}{\#A}\sum_{a\in A}a_2 > \frac{1}{\#B}\sum_{b\in B}b_2 \\ &\quad\Longleftrightarrow \\ &\frac{1}{\#A}\sum_{a\in A}a_1 = \frac{1}{\#A+\#B}\sum_{\alpha\in A\cup B}\alpha_1 \text{ and } \frac{1}{\#A}\sum_{a\in A}a_2 > \frac{1}{\#A+\#B}\sum_{\alpha\in A\cup B}\alpha_2 \\ &\quad\Longleftrightarrow \\ &\quad A \succ A\cup B.\end{aligned}$$

A similar reasoning holds when $A \sim B$. We leave to the reader the (easy) task of verifying that this ordering violates also Continuity. To show that $\succsim$ satisfies Richness on $X = \mathbb{R}\times\mathbb{R}$, consider any finite subsets A and B of X (with A possibly empty) and define c by means of the following two equations:

$$\frac{c_1 + \sum\limits_{a\in A}a_1}{1+\#A} = \frac{1}{\#B}\sum_{b\in B}b_1 \quad \text{and} \quad \frac{c_2 + \sum\limits_{a\in A}a_2}{1+\#A} = \frac{1}{\#B}\sum_{b\in B}b_2.$$

We then have $A\cup\{c\} \sim B$. We notice that this conclusion holds no matter what is assumed on the ranking of A vis-à-vis B. Hence this conclusion can also be obtained for sets B and A that satisfy the requirements of the Richness axiom. Finally, to show that $\succsim$ satisfies Restricted Independence,

consider finite and non-empty subsets A and B of X such that $\#A = \#B$ and $A \cap C = \emptyset = B \cap C$. We have $A \succ B$ if and only if either:
$\frac{1}{\#A}\sum_{a\in A} a_1 > \frac{1}{\#B}\sum_{b\in B} b_1 \Leftrightarrow \frac{1}{\#A+\#C}\sum_{\alpha\in A\cup C}\alpha_1 > \frac{1}{\#B+\#C}\sum_{\beta\in B\cup C}\beta_1$
$\Leftrightarrow A \cup C \succ B \cup C$ *or*
$\frac{1}{\#A}\sum_{a\in A} a_1 = \frac{1}{\#B}\sum_{b\in B}$ *and* $\frac{1}{\#A}\sum_{a\in A} a_2 > \frac{1}{\#B}\sum_{b\in B} b_2$
$\Leftrightarrow \frac{1}{\#A+\#C}\sum_{\alpha\in A\cup C}\alpha_1 = \frac{1}{\#B+\#C}\sum_{\beta\in B\cup C}\beta_1$ *and* $\frac{1}{\#A+\#C}\sum_{\alpha\in A\cup C}\alpha_2 > \frac{1}{\#B+\#C}\sum_{\beta\in B\cup C}\beta_2$
$\Longleftrightarrow A \cup C \succ B \cup C$. *A similar reasoning holds when $A \sim B$.*

6 Conclusion

This paper characterizes (using three axioms) the family of UEU rankings of completely uncertain decisions, under the assumption that the rankings are used in a rich environment. The axioms used in the characterization are finite and, therefore, verifiable from observation of choice behavior. We have also shown that UEU rankings can be used to compare ambiguous decisions or decisions with financial consequences and is characterized in that setting under the same axioms, but with the Archimedean axiom replaced by a mild Continuity one.

A limitation of UEU criteria is that they assign to every consequence of a decision the same probability of occurrence. A next step in the research agenda is therefore to identify the properties of a more general EU criterion that does not impose this uniform assumption on the probabilities assigned to the consequences of a decision. The family of orderings that can be represented as per (2) for some functions p and u is an obvious first step into that direction. We have seen that any ordering in this family satisfies Averaging and Continuity (or the Archimedean axiom) but may violate Restricted Independence. It would be nice to know the axioms which, along with Averaging and Continuity, characterize this large family of rankings of completely uncertain decisions. While Ahn (2008) has characterized the family of orderings that can be represented as per (3) in the specific ambiguity context where decisions lead to continuously many different probability distributions over a fundamental set of consequences, we think that obtaining a characterization of the family represented as per (2) in a finite context is a high priority for future research.

7 Appendix

Proof of Theorem 1

Before proving Theorems 1 and 2 on the subdomain $P(X')$, we must be sure that all our axioms - formulated for the domain $P(X)$- are also valid for the subdomain $P(X')$. While this is clear for Averaging, Restricted Independence and the Archimedean axiom, it may not be so clear for Richness which, given any two sets in $P(X)$- and therefore in the subdomain $P(X')$- asserts the existence, in X, of a specific consequence c having some property. Yet we must prove that this consequence c can actually be shown to belong to X'. Specifically, we must prove that the following lemma is true.

Lemma 2 *Let $\succsim$ be an ordering on $P(X)$ satisfying Averaging, Restricted Independence, Richness and the Archimedean axiom. Then the restriction of $\succsim$ to $P(X')$ satisfies the same axioms.*

Proof.

We leave to the reader the task of verifying that this is indeed the case for Averaging, Restricted Independence and the Archimedean axiom. For Richness, let A and B be two finite subsets of X' (with A possibly empty) and assume that there are consequences c^* and $c_* \in X'$ such that $A \cup \{c^*\} \succsim B \succsim A \cup \{c_*\}$. By Richness (applied to $P(X)$), there exists a consequence $c \in X$ such that $A \cup \{c\} \sim B$. We need to show that $c \in X'$. By contradiction, assume $c \in X \backslash X' = m(X) \cup M(X)$. If $c \in m(X)$, then $\{c\} \prec \{x\}$ for all $x \in X'$ so that, in particular, $\{c\} \prec \{c_*\}$ and $\{c\} \prec \{a\}$ for all $a \in A$. Hence $c \notin A$. One has therefore, by Restricted Independence (if $c_* \notin A$) or by Averaging (if $c_* \in A$), that $B \succsim A \cup \{c_*\} \succ A \cup \{c\}$, a contradiction (if $\succsim$ is transitive). The argument is symmetric if $c \in M(X)$. QED

An important ingredient in the proof of Theorem 1 is the following lemma, which states that if an ordering $\succsim$ on $P(X')$ satisfies Averaging, Restricted Independence, Richness and the Archimedean axiom, then it satisfies, when restricted to pairs and singletons, the following important condition that is closely related to the so-called "Thomsen condition" in the theory of conjoint measurement (using Krantz, Luce, Suppes, and Tversky (1971) terminology).

Lemma 3 *Let $\succsim$ be an ordering on $P(X')$ satisfying Averaging, Restricted Independence, Richness and the Archimedean axiom. Then for every (not necessarily distinct) consequences a, b, c, d, e and $f \in X'$, $\{a\} \cup \{f\} \sim \{c\} \cup \{e\}$ and $\{c\} \cup \{d\} \sim \{b\} \cup \{f\}$ must imply $\{a\} \cup \{d\} \sim \{b\} \cup \{e\}$.*

Proof.

We consider several cases.

1. $\{a\} \sim \{b\}$ **and** $\{d\} \sim \{e\}$**.** In this case, we conclude that $\{a\} \cup \{d\} \sim \{b\} \cup \{d\}$ by Restricted Independence if both $a \neq d$ and $b \neq d$. If either $a = d$ and $b \neq d$ or $a \neq d$ and $b = d$, the conclusion $\{a\} \cup \{d\} \sim \{b\} \cup \{d\}$ follows from Averaging. Finally, if $a = d$ and $b = d$, the conclusion that

$\{a\} \cup \{d\} \sim \{b\} \cup \{d\}$ follows trivially from the assumption that $\{a\} \sim \{b\}$. By an analogous reasoning we can obtain the conclusion that $\{b\} \cup \{d\} \sim \{b\} \cup \{e\}$. The conclusion that $\{a\} \cup \{d\} \sim \{b\} \cup \{e\}$ follows then at once from transitivity.

2. $\{a\} \succ \{b\}$ **and** $\{d\} \succsim \{e\}$**.** In this case, it follows from Restricted Independence (if $a \neq f \neq b$) or Averaging (if $a = f$ or $b = f$) that $\{a\} \cup \{f\} \succ \{b\} \cup \{f\}$. Analogously, we can conclude from the premises of this case that $\{c\} \cup \{d\} \succsim \{c\} \cup \{e\}$ (using Restricted Independence if $d \neq c \neq e$ and Averaging if $d = c$ or $e = c$). It then follows from transitivity that $\{c\} \cup \{d\} \succsim \{c\} \cup \{e\} \sim \{a\} \cup \{f\} \succ \{b\} \cup \{f\} \sim \{c\} \cup \{d\}$. Since this is a contradiction, we conclude that this case is impossible.

3. $\{a\} \prec \{b\}$ **and** $\{d\} \precsim \{e\}$**.** This case can also shown to be impossible, following a similar reasoning as for case 2.

4. $\{a\} \prec \{b\}$ **and** $\{d\} \succsim \{e\}$**.** We then consider several subcases.

(i) $c = f$. Since $\{a\} \cup \{f\} \sim \{c\} \cup \{e\}$, we conclude that $\{a\} \sim \{e\}$ using Restricted Independence (if $a \neq f$ and $c \neq e$), Averaging (if $a = f$ and $c \neq e$ or $a \neq f$ and $c = e$) or trivially (if $a = f$ and $c = e$). By an analogous reasoning, we conclude from $\{c\} \cup \{d\} \sim \{b\} \cup \{f\}$ that $\{b\} \sim \{d\}$. Hence, we have $\{a\} \sim \{e\}$ and $\{b\} \sim \{d\}$. This implies that $\{a\} \cup \{d\} \sim \{b\} \cup \{e\}$ by Restricted Independence (if $a \neq d$ and $b \neq e$) or by Averaging (in all other cases).

(ii) Suppose $f \neq c$, $e \neq c \neq d$, $d \neq a \neq f$and $e \neq b \neq f$. We first establish that there are consequences u and $v \in X'$such that $\{u\} \cup \{v\} \cap \{a\} \cup \{b\} \cup \{c\} \cup \{d\} \cup \{e\} \cup \{f\} = \emptyset$ and $\{a, u\} \sim \{c, v\}$. Suppose first $\{a\} \sim \{c\}$. Take then any $u \in X' \backslash (\{a\} \cup \{b\} \cup \{c\} \cup \{d\} \cup \{e\} \cup \{f\})$ (the existence of such a u is secured by Proposition 5) and define $v = u$. We then immediately obtain $\{a, u\} \sim \{f, v\}$ by Restricted Independence. Suppose now that $\{a\} \prec \{c\}$. By Proposition 5 there is a consequence $v \in X'$ such that $\{v\} \prec \{a\}$. By Restricted Independence, one has $\{a, v\} \prec \{c, v\}$ and $\{c, v\} \prec \{c, a\}$ and, by transitivity, $\{a, v\} \prec \{c, v\} \prec \{a, c\}$. It follows from Richness that there is a consequence $u \in X'$ such that $\{a\} \cup \{u\} \sim \{c, v\}$. If $\{u, v\} \cap \{a, b, f, p, q, x\} \neq \emptyset$, one can repeat this procedure, starting with another v. The repetition of the procedure will be finite because the set $\{a\} \cup \{b\} \cup \{c\} \cup \{d\} \cup \{e\} \cup \{f\}$ is finite. Hence one is sure to find a consequence $u \in X' \backslash \{a\} \cup \{b\} \cup \{c\} \cup \{d\} \cup \{e\} \cup \{f\}$ such that $\{a, u\} \sim \{c, v\}$. Using an analogous argument, one can obtain a similar conclusion if $\{a\} \succ \{c\}$ is assumed. Now, by Restricted Independence, one has $\{c, f, v\} \sim \{a, f, u\}$ and $\{a, f, u\} \sim \{c, e, u\}$. It follows from transitivity that $\{c, f, v\} \sim \{c, e, u\}$ so that $\{f, v\} \sim \{e, u\}$ must hold by Restricted Independence. Using Restricted Independence again, we obtain from $\{a, u\} \sim \{c, v\}$ that $\{a, d, u\} \sim \{c, d, v\}$ and from $\{c, d\} \sim \{b, f\}$

that $\{c, d, v\} \sim \{b, f, v\} \sim \{b, e, u\}$. It follows from transitivity that $\{a, d, u\} \sim \{b, e, u\}$ and, by Restricted Independence, $\{a, d\} \sim \{b, e\}$.

(iii) Suppose $c \neq f$, $e \neq c \neq d$, and $e \neq b \neq f$. The only difference with subcase (ii) is that we relax the constraint '$d \neq a \neq f$'. Hence this case is more general than (ii). Suppose, contrary to the asserted implication of the lemma, that $\{a\} \cup \{d\} \nsim \{b\} \cup \{d\}$. Since $\succsim$ is complete, two symmetric cases can arise: $\{a\} \cup \{d\} \prec \{b\} \cup \{e\}$ or $\{a\} \cup \{d\} \succ \{b\} \cup \{e\}$. We only handle the first one. We first show that we can find distinct consequences a' and $f' \in X' \backslash \{a\} \cup \{b\} \cup \{c\} \cup \{d\} \cup \{e\} \cup \{f\}$ such that $\{a'\} \prec \{a\}$, $\{f\} \prec \{f'\}$ and $\{a', f'\} \sim \{a\} \cup \{f\}$. The existence of consequences $a' \in X'$ such that $\{a'\} \prec \{a\}$ is secured by Proposition 5. Assume first $a = f$. Either $\{a', g\} \prec \{a\}$ for all $g \in X'$ or there exists some g' such that $\{a', g'\} \succsim \{a\}$. In the second case, the existence of a consequence f' such that $\{a', f'\} \sim \{a\}$ follows from Richness. In the first case, choose by Proposition 5 some $\widehat{g} \in X'$ such that $\{\widehat{g}\} \succ \{a\}$ and, by Certainty Equivalence, some $\widetilde{g} \in X'$ such that $\{\widetilde{g}\} \sim \{a, \widehat{g}\}$. By Averaging and transitivity, one has $\{a\} \prec \{\widetilde{g}\} \sim \{a, \widehat{g}\} \prec \{\widetilde{g}, \widehat{g}\} \prec \{\widehat{g}\}$. Hence one has $\{\widetilde{g}, \widehat{g}\} \succ \{a\} \succ \{a', \widetilde{g}\}$ so that, by Richness, there exists a consequence $\widetilde{a} \in X'$ such that $\{\widetilde{a}, \widetilde{g}\} \sim \{a\}$. Choosing then $a' = \widetilde{a}$ and $f' = \widetilde{g}$ gives the result. If $a \neq f$, we can do the previous reasoning for the certainty equivalent of $\{a, f\}$ that exists by Certainty Equivalence. If either a' or $f' \in \{a\} \cup \{b\} \cup \{c\} \cup \{d\} \cup \{e\} \cup \{f\}$, we can redo the procedure while starting with another a'. Since the set $\{a\} \cup \{b\} \cup \{c\} \cup \{d\} \cup \{e\} \cup \{f\}$ is finite, we will redo the procedure at most a finite number of times. Hence we are sure to find consequences a' and $f' \in X' \backslash \{a\} \cup \{b\} \cup \{c\} \cup \{d\} \cup \{e\} \cup \{f\}$ such that $\{a'\} \prec \{a\}$, $\{f\} \prec \{f'\}$ and $\{a', f'\} \sim \{a\} \cup \{f\}$ and, by redoing the above procedure as many times as required, we can choose as many distinct pairs of such $a'$and f'as we want. Choose now a consequence b'such that $\{b'\} \cup \{f'\} \sim \{b, f\}$. This is possible because $\{b, f'\} \succ \{b, f\}$ thanks to Restricted Independence. Moreover it is impossible that $\{\widetilde{b}\} \cup \{f'\} \succ \{b, f\}$ for all consequences $\widetilde{b}$ in X'. Indeed, since $\{a', f'\} \sim \{a\} \cup \{f\}$, assuming $\{\widetilde{b}\} \cup \{f'\} \succ \{b, f\}$ for all $\widetilde{b}$ would imply, given transitivity, that $\{a\} \cup \{f\} \succ \{b, f\}$. Yet using Averaging (if $a = f$) or Restricted Independence (if $a \neq f$), this would imply in turn that $\{a\} \succ \{b\}$, contradicting our assumption that $\{a\} \prec \{b\}$. Hence, there are consequences $\widetilde{b}$ such that $\{b, f\} \succsim \{\widetilde{b}, f'\}$ so that, by Richness, one can find a consequence b' such that $\{b', f'\} \sim \{b, f\}$. Given the flexibility we have for choosing a' and f' it is clear that b' can be chosen so that it does not belong to $\{a\} \cup \{a'\} \cup \{b\} \cup \{c\} \cup \{d\} \cup \{e\} \cup \{f\} \cup \{f'\}$. Thanks to case (ii), we know that we can obtain $\{a', d\} \sim \{b', e\}$ if we replace a by a', f by $f'$and b by b' in the antecedent clause of the lemma. Since $\{a'\} \prec \{a\}$, we know that $\{a', d\} \prec \{a\} \cup \{d\}$ by Restricted Independence (if $a \neq d$) or by Averaging (if $a = d$). Hence, it follows from transitivity that $\{b', e\} \sim \{a', d\} \prec \{a\} \cup \{d\} \prec \{b\} \cup \{e\}$. Combine now as before the axioms of Averaging, certainty equivalent and Richness to find a consequence b'' such that $\{a\} \cup \{d\} \prec \{b''\} \cup \{e\} \prec \{b\} \cup \{e\}$. Using Richness, one can also find a consequence f'' such that $\{c\} \cup \{d\} \sim \{b''\} \cup \{f''\}$ and a consequence a'' such that $\{a''\} \cup \{f''\} \sim \{f\} \cup \{e\}$. As before, we have the flexibility to find these consequences b'', f'' or a'' in such a way that they not belong to $\{a\} \cup \{b\} \cup \{c\} \cup \{d\} \cup \{e\} \cup \{f\}$. Thanks to case (ii) again, we can obtain the conclusion that $\{a'', d\} \sim \{b'', e\}$. Since $\{a''\} \prec \{a\}$, we have $\{a''\} \cup \{d\} \prec \{a\} \cup \{d\}$ by Restricted Independence (if $a \neq d$) or by Averaging (if $a = d$). We then obtain from transitivity that $\{a\} \cup \{d\} \prec \{b'', e\} \sim \{a'', d\} \prec \{a\} \cup \{d\}$, a contradiction.

(iv) Suppose $c \neq f$ and $e \neq b \neq f$. The difference with case (iii) is that we relax the constraint '$e \neq c \neq d$'. Hence, as before, this case is more general than case (iii). Suppose by contradiction that the lemma is false and that $\{a\} \cup \{d\} \not\sim \{b\} \cup \{e\}$. As before, the completeness of $\succsim$ implies either $\{a\} \cup \{d\} \prec \{b\} \cup \{e\}$ or $\{a\} \cup \{d\} \succ \{b\} \cup \{e\}$. Since these two cases are symmetric, we only provide the argument for the first one. Using analogous argument than in case (iii), one can find consequences c' and $d' \in X' \backslash \{a\} \cup \{b\} \cup \{c\} \cup \{d\} \cup \{e\} \cup \{f\}$ such that $\{d'\} \prec \{d\}$, $\{c\} \prec \{c'\}$ and $\{c', d'\} \sim \{c\} \cup \{d\}$. As in case (iii) also, one can find a consequence $e' \in X' \backslash \{a\} \cup \{b\} \cup \{c, c'\} \cup \{d, d'\} \cup \{e\} \cup \{f\}$ such that $\{c', e'\} \sim \{c\} \cup \{e\}$. Thanks to case (iii), we know that we can obtain $\{a\} \cup \{d'\} \sim \{b\} \cup \{e'\}$ out of the assumption that $\{a\} \cup \{f\} \sim \{c'\} \cup \{e'\}$ and $\{c'\} \cup \{d'\} \sim \{b\} \cup \{f\}$. Since $\{d'\} \prec \{d\}$, we have $\{a, d'\} \prec \{a\} \cup \{d\}$ by Restricted Independence(if $a \neq d$) or Averaging (if $a = d$). Hence by transitivity we have: $\{b\} \cup \{e'\} \sim \{a\} \cup \{d'\} \prec \{a\} \cup \{d\} \prec \{b\} \cup \{e\}$. Using analogous argument as for the consequences a'', b'' and f'' of case (iii), find now consequences c'', d'' and $e'' \in X' \backslash \{a\} \cup \{b\} \cup \{c, c'\} \cup \{d, d'\} \cup \{e, e'\} \cup \{f\}$ such that 1) $\{a\} \cup \{d\} \prec \{b, e''\} \prec \{b\} \cup \{e\}$, 2) $\{c'', e''\} \sim \{a\} \cup \{f\}$ and 3) $\{c'', d''\} \sim \{b\} \cup \{f\}$. We know from (iii) that $\{a\} \cup \{f\} \sim \{c, e''\}$ and $\{c, d''\} \sim \{b\} \cup \{f\}$ implies $\{a\} \cup \{d''\} \sim \{b\} \cup \{e''\}$. Since $\{d''\} \prec \{d\}$, we have $\{a\} \cup \{d''\} \prec \{a\} \cup \{d\}$ by Restricted Independence (if $a \neq d$) or by Averaging (if $a = d$). Transitivity then yields $\{a\} \cup \{d\} \prec \{b, e''\} \sim \{a, d''\} \prec \{a\} \cup \{d\}$, a contradiction.

(v) Suppose $c \neq f$. The difference with case (iv) is that we relax the constraint '$e \neq b \neq f$. Hence this case is more general than case (iv) and we handle it in an analogous fashion (conditional on (iv)) to what was done for case (iv) conditional to case (iii).

We notice that subcases (i) and (v) are exhaustive, conditional on case 4.

5. $\{a\} \succ \{b\}$ **and** $\{p\} \precsim \{q\}$**.** This case is handled in the same way as case 4

Another result used in the proof of Theorem 1 is the following.

Lemma 4 *Let* $\succsim$ *be an ordering on* $P(X')$ *satisfying Averaging and Restricted Independence. Then for every (not necessarily distinct) consequences* a, b, c, *and* $d \in X'$, $\{a\} \cup \{b\} \succsim \{c\} \cup \{b\} \Leftrightarrow \{a\} \cup \{d\} \succsim \{c\} \cup \{d\}$.

Proof.

We consider several cases.

1) $a \neq b$**,** $c \neq b$**,** $a \neq d$ **and** $c \neq d$**.** The result then follows immediately from Restricted Independence.

2) $a = b$**,** $c \neq b$**,** $a \neq d$ **and** $c \neq d$**.** Assume $\{a\} \cup \{b\} \succsim \{c\} \cup \{b\}$ or, equivalently under our assumption, that $\{a\} \succsim \{a, c\}$. By Averaging this statement is equivalent to $\{a\} \succsim \{c\}$ which is itself equivalent, using Restricted Independence, to $\{a, d\} = \{a\} \cup \{d\} \succsim \{c, d\} = \{c\} \cup \{d\}$.

3) $a \neq b$**,** $c = b$**,** $a \neq d$ **and** $c \neq d$**.** Assume $\{a\} \cup \{b\} \succsim \{c\} \cup \{b\}$ or, equivalently under our assumption, that $\{a, b\} \succsim \{b\}$. By Averaging, this statement is equivalent to $\{a\} \succsim \{b\}$ and, by Restricted Independence, to $\{a, d\} = \{a\} \cup \{d\} \succsim \{b, d\} = \{c\} \cup \{d\}$.

4) $a \neq b$**,** $c \neq b$**,** $a = d$ **and** $c \neq d$**.** Assume $\{a\} \cup \{b\} \succsim \{c\} \cup \{b\}$ or, equivalently under our assumption, that $\{a, b\} \succsim \{c, b\}$. Using Restricted Independence, this is equivalent to $\{a\} = \{d\} \succsim \{c\}$ which, by Averaging, is equivalent to $\{d\} = \{a\} \cup \{d\} \succsim \{d, c\} = \{c\} \cup \{d\}$.

5) $a \neq b$**,** $c \neq b$**,** $a \neq d$ **and** $c = d$**.** Assume $\{a\} \cup \{b\} \succsim \{c\} \cup \{b\}$ or, equivalently under our assumption, that $\{a, b\} \succsim \{c, b\} = \{d, b\}$. Using Restricted Independence, this is equivalent to $\{a\} \succsim \{d\}$ which, by Averaging, is equivalent to $\{a, d\} = \{a\} \cup \{d\} \succsim \{d\} = \{c\} \cup \{d\}$.

6) $a = b = c \neq d$**.** In that case reflexivity ensures that $\{a\} \cup \{b\} \succsim \{c\} \cup \{b\} \Leftrightarrow \{a\} \succsim \{a\} \Leftrightarrow \{a\} \cup \{d\} = \{a, d\} \succsim \{a, d\} = \{c, d\} = \{c\} \cup \{d\}$.

All other cases are handled trivially using reflexivity.

Proof of Theorem 1.

Proposition 1 establishes that any UEU criterion satisfies Averaging and Restricted Independence. To prove the converse implication, consider the restriction of the ordering $\succsim$ to the set of all subsets of X' containing at most two consequences. Define the binary relation $\widehat{\succsim}$ on $X' \times X'$ by $(a, b) \widehat{\succsim} (c, d) \Leftrightarrow \{a\} \cup \{b\} \succsim \{c\} \cup \{d\}$. The binary relation $\widehat{\succsim}$ is well-defined and is clearly an ordering of $X' \times X'$ if $\succsim$ is an ordering of $P(X')$. We also notice that, thanks to Lemma 4, $\widehat{\succsim}$ satisfies the property that if $(a, b) \widehat{\succsim} (c, b)$ holds for some consequence a, b and c, then $(a, d) \widehat{\succsim} (c, d)$ holds for all consequences $d \in X'$. This property is called "independence" by Krantz, Luce, Suppes, and Tversky (1971) (KLST for short) (p. 249, Definition

1). We similarly obtain that $\widehat{\succsim}$ satisfies both Thomsen's condition (see KLST, p. 250, Definition 3) and the "restricted solvability" condition (KLST, p. 250, Definition 5) using, respectively Lemma 3, and the Richness axiom. Finally, we note that our Archimedean axiom implies the property of the same name of KLST (p. 253, Definition 4) while our assumption that $\succsim$ is non trivial implies, thanks to Averaging (and specifically the Gardenförs condition), that each component of $X' \times X'$ is essential as per KLST Definition 6 (p. 256). Hence the triple $(X', X', \widehat{\succsim})$ is an additive conjoint structure in the sense of KLST (Definition 7, p. 256). By virtue of Theorem 2 of KLST (p. 257), there are real-valued functions Φ_i (for $i = 1, 2$) having both X' as domain such that:

$$(a, b) \mathrel{\widehat{\succsim}} (c, d) \Leftrightarrow \Phi_1(a) + \Phi_2(b) \geq \Phi_1(c) + \Phi_2(d)$$

for all consequences a, b, c and $d \in X'$. Since $(a, b) \mathrel{\widehat{\succsim}} (c, d) \Leftrightarrow \{a\} \cup \{b\} \succsim \{c\} \cup \{d\} \Leftrightarrow \{b\} \cup \{a\} \succsim \{d\} \cup \{d\} \Leftrightarrow (b, a) \mathrel{\widehat{\succsim}} (d, c)$, the ordering $\widehat{\succsim}$ is symmetric so that $\Phi_1(x) = \Phi_2(x) = u(x)$ must hold for every consequence $x \in X'$ for some function $u : X' \to \mathbb{R}$. By virtue of the second part of Theorem 2 in KLST, the function u is unique up to an affine transform. Let us now show that, for all subsets A and B of X' containing at most two consequences, one has $A \succsim B \Leftrightarrow \sum_{a \in A} \frac{u(a)}{\#A} \geq \sum_{a \in A} \frac{u(a)}{\#A}$ so that $\succsim$ can be represented as per (1). If $\#A = \#B = 1$, then one has, for all consequences x and $y \in X'$, $\{x\} \succsim \{y\} \Leftrightarrow \{x\} \cup \{x\} \succsim \{y\} \cup \{y\} \Leftrightarrow 2u(x) \geq 2u(y) \Leftrightarrow u(x) \geq u(y)$ so that the numerical representation holds for that case. The argument clearly works just as well if $\#A = \#B = 2$. Suppose now that $\#A = 1$ and $\#B = 2$. Then, for all consequences x, y and $z \in X'$such that $y \neq z$, one has:

$$\begin{aligned}
\{x\} \succsim \{y, z\} &\Longleftrightarrow \{x\} \cup \{x\} \succsim \{y\} \cup \{z\} \\
&\Longleftrightarrow (x, x) \mathrel{\widehat{\succsim}} (y, z) \\
&\Longleftrightarrow u(x) + u(x) \geq u(y) + u(z) \Longleftrightarrow u(x) \geq \frac{u(y) + u(z)}{2}
\end{aligned}$$

so that the numerical representation holds for that case as well. QED

Proof of Lemma 1.

We find useful to represent the sequence defined in this lemma in the following array, with $n - 1$ columns and an infinite number of rows:

	1	2	...	$n-2$	$n-1$
$\mathbf{b}^0$	$(u_1 + b_2^0)/2$	$(u_2 + b_3^0)/2$	...$\leftarrow$	$(u_{n-2} + b_{n-1}^0)/2$	$(u_{n-1} + u_n)/2$
$\mathbf{b}^1$	$(u_1 + b_2^0)/2$	$(b_1^1 + b_2^0)/2$	$\rightarrow$...	$(b_{n-3}^1 + b_{n-2}^0)/2$	$(b_{n-2}^1 + b_{n-1}^0)/2$
$\mathbf{b}^2$	$(b_2^2 + b_1^1)/2$	$(b_3^2 + b_2^1)/2$	...$\leftarrow$	$(b_{n-1}^2 + b_{n-2}^1)/2$	$(b_{n-2}^1 + b_{n-1}^0)/2$
$\mathbf{b}^3$	$(b_2^2 + b_1^1)/2$	$(b_1^3 + b_2^2)/2$	$\rightarrow$...	$(b_{n-3}^3 + b_{n-2}^2)/2$	$(b_{n-2}^3 + b_{n-1}^2)/2$
...	...	...	...	...	...

We are going to show that the "grand" sequence that starts from the "north-east" of the array and follows the arrows up to infinity, converges to $\overline{u}$. Since the sequence $\{b_h^i\}$ is the h^{th}column of this array and therefore, a subsequence of the grand sequence, the conclusion of the lemma would follow immediately. Define accordingly the grand sequence $\{\widehat{b}^t\}$, for $t = i(n-1)+1, ..., (i+1)(n-1)$, and $i = 0, 1, 2, ...$by:

$$\widehat{b}^t = b^i_{n-(t+1-(i(n-1)+1))} \text{ if } i \text{ is even and}$$
$$\widehat{b}^t = b^i_{t+1-(i(n-1)+1)} \text{ if } i \text{ is odd}$$

Any element of the grand sequence can be written as a weighted average of $\{u_1, ..., u_n\}$. In particular, for all $t = 1, ...,$ there exists $n-1$ real numbers $\beta_1^t, ..., \beta_{n-1}^t$ such that:

$$\widehat{b}^t = \beta_1^t u_1 + \beta_2^t u_2 + ... + \beta_{n-1}^t u_{n-1} + \beta_{n-1}^t u_n$$

Moreover inspection reveals that β_h^t is defined by the following recursive formula:

$$\begin{aligned} \beta_h^t &= 0 \text{ if } t \in \{1, ..., n-h-1\} \\ \beta_h^{n-h} &= \frac{1}{2} \text{ and} \\ \beta_h^t &= \frac{1}{2}(\beta_h^{t-1} + \beta_h^{2m(t)-t+1} \text{ if } t \geq n-h+1 \end{aligned} \qquad (5)$$

where $m(t)$ is defined as the largest integer strictly smaller than t that is divisible by $n-1$. In order to prove the lemma, it suffices to prove that $\lim_{t \to \infty} \beta_h^t = \frac{1}{n}$ for all h. In what follows we will fix $h \in \{1, .., n-1\}$ and drop the subscript h from the sequence $\{\beta_h^t\}$ for notational convenience.

Once again, it is convenient to refer to the aforementioned representation of the sequence $\{\beta^t\}$, $t = 1, ..$as an array with $n-1$ columns and an infinite number of rows. We start from the first row with β^1 and move left until we reach β^{n-1}. We then move down to the second row where the first element from the left is β^n. The sequence then increases from left right and the right-most element in the this row is β^{2n-2}. The right-most element in the third row is then β^{2n-1} and the sequence increases as it moves left (like in the first row) so that the left-most element is β^{3n-3} and so on. Let t be an arbitrary integer. If we write $t = m(t) + s$, it follows that β^t lies in the $(m(t)+1)^{th}$ row of this array. If $m(t)$ is even then, the $(m(t)+1)^{th}$ row is increasing from right to left so that β^t is the $(s+1)^{th}$ element from the right in this row. If $m(t)$ is odd, then β^t is the $(s+1)^{th}$ element from the left in the $(m(t)+1)^{th}$ row which increases from left to right. It follows that in this array, β^t for $t > n-1$ is the arithmetic mean of the element which immediately precedes it and the element directly in the row above.

The proof proceeds in two steps. The first is to show that the sequence $\{\beta^t\}$, $t = 1, ...$is convergent and the second is to show that the limit of the sequence is, in fact $\frac{1}{n}$. In order to establish the first step, we first record the two following properties $P1$ and $P2$ of the sequence which can be easily verified.

$P1$. Let $r > 1$ be an odd integer. The sequence strictly increases from $\beta^{(r-1)(n-1)+1}$ to $\beta^{(r-1)(n-1)+h}$ and then strictly decreases from $\beta^{(r-1)(n-1)+h}$ to $\beta^{r(n-1)}$. If r is an even integer, then the sequence strictly increases from $\beta^{(r-1)(n-1)+1}$ to $\beta^{(r-1)(n-1)+n-h}$ and strictly decreases from $\beta^{(r-1)(n-1)+n-h}$ to $\beta^{r(n-1)}$. Thus for every row r in the array, the sequence increases from the right as we move left for h terms and then decreases for the remaining $n-h-1$ terms. Clearly $\beta^{(r-1)(n-1)+h}$ is the largest element of the r^{th} row if r is odd and $\beta^{(r-1)(n-1)+n-h}$ if r is even. Note that the maximal element of any row is in the h^{th}column from the right.

$P2$. Let $t = (n-1)r + s$ where $m(t) = r$ (note that $1 \leq s \leq n-1$). Then $\beta^t = \frac{1}{2}\beta^{(n-1)(r-1)+(n-s)} + \frac{1}{2^2}\beta^{(n-1)(r-1)+(n-s-1)} + ... + \frac{1}{2^{s-1}}\beta^{(n-1)(r-1)+1} + \frac{1}{2^{s-1}}\beta^{(n-1)(r-1)}$. Thus each term of the sequence can be expressed as the weighted sum of the terms of the sequence in the row above.

CLAIM: Let $r > 1$ be an integer. Then:
(i) $\beta^{(r-1)(n-1)+h} - \beta^{r(n-1)} < \gamma_1[\beta^{(r-2)(n-1)+n-h} - \beta^{(r-2)(n-1)+1}]$ and
(ii) $\beta^{(r-1)(n-1)+h} - \beta^{(r-1)(n-1)+1} < \gamma_2[\beta^{(r-2)(n-1)+n-h} - \beta^{(r-1)(n-1)}]$
where $\gamma_1 = \frac{2^{n-h-1}-1}{2^{n-h-1}}$ and $\gamma_2 = \frac{2^{h-1}-1}{2^{h-1}}$ if r is odd and:
(iii) $\beta^{(r-1)(n-1)+n-h} - \beta^{(r-1)(n-1)+1} < \gamma_1[\beta^{(r-2)(n-1)+h} - \beta^{(r-1)(n-1)}]$ and,
(iv) $\beta^{(r-1)(n-1)+n-h} - \beta^{r(n-1)} < \gamma_2[\beta^{(r-2)(n-1)+h} - \beta^{(r-2)(n-1)+1}]$
if r is even.

Proof of the Claim: We first prove (ii). We do that by first noting that, according to $P2$:

$$\beta^{(r-1)(n-1)+h} = \frac{1}{2}\beta^{(r-2)(n-1)+n-h} + ... + \frac{1}{2^{h-1}}\beta^{(r-1)(n-1)+1} + \frac{1}{2^{h-1}}\beta^{(r-1)(n-1)}$$

Since $\beta^{(r-1)(n-1)+1} = \beta^{(r-1)(n-1)}$ and $\beta^{(r-2)(n-1)+n-h}$ is the largest term in the $(r-1)^{th}$ row according to $P1$, we conclude that :

$$\begin{aligned}\beta^{(r-1)(n-1)+h} - \beta^{(r-1)(n-1)+1} &< \beta^{(r-2)(n-1)+h}[\frac{1}{2} + ... + \frac{1}{2^{h-1}}] \\ &\quad -[1 - \frac{1}{2^{h-1}}]\beta^{(r-1)(n-1)} \\ &= [1 - \frac{1}{2^{h-1}}][\beta^{(r-2)(n-1)+n+h} - \beta^{(r-1)(n-1)}]\end{aligned}$$

Since $\gamma_2 = (1 - \frac{1}{2^{h-1}})$, this establishes (ii).

We now prove (iii). According to $P2$:

$$\beta^{(r-1)(n-1)+n-h} = \frac{1}{2}\beta^{(r-2)(n-1)+h} + ... + \frac{1}{2^{n-h-1}}[\beta^{(r-1)(n-1)+1} + \beta^{(r-1)(n-1)}]$$

Since $\beta^{(r-1)(n-1)+1} = \beta^{(r-1)(n-1)}$ and since, from $P1$, we know that $\beta^{(r-2)(n-1)+h}$ is the largest term in the $(r-1)^{th}$ row, we obtain:

$$\begin{aligned}
\beta^{(r-1)(n-1)+n-h} - \beta^{(r-1)(n-1)+1} &< [\frac{1}{2} + ... + \frac{1}{2^{n-h-1}}]\beta^{(r-2)(n-1)+n-h} \\
&- [1 - \frac{1}{2^{n-h-1}}]\beta^{(r-1)(n-1)+1} \\
&= [1 - \frac{1}{2^{n-h-1}}][\beta^{(r-2)(n-1)+n-h} - \beta^{(r-1)(n-1)+1}]
\end{aligned}$$

Since $\gamma_2 = (1 - \frac{1}{2^{n-h-1}})$, this establishes (iii).

We now prove (i). Applying $P2$, we have:

$$\begin{aligned}
\beta^{(r-1)(n-1)+h} &= \frac{1}{2}\beta^{(r-2)(n-1)+n-h} + ... + \frac{1}{2^{h-1}}\beta^{(r-1)(n-1)+1} \\
&+ \frac{1}{2^{h-1}}\beta^{(r-1)(n-1)}
\end{aligned}$$

and:

$$\begin{aligned}
\beta^{r(n-1)} &= \frac{1}{2}\beta^{(r-2)(n-1)+1} + ... + \frac{1}{2^{n-h}}\beta^{(r-1)(n-1)+n-h} \\
&+ .... + \frac{1}{2^{n-2}}\beta^{(r-1)(n-1)+1} + \frac{1}{2^{n-2}}\beta^{(r-1)(n-1)}
\end{aligned}$$

We thus have:

$$\Delta = \beta^{(r-1)(n-1)+h} - \beta^{r(n-1)}$$

$$\begin{aligned}
&= [\frac{1}{2} - \frac{1}{2^{n-h}}]\beta^{(r-1)(n-1)+n-h} + ... + [\frac{1}{2^{h-1}} - \frac{1}{2^{n-2}}]\beta^{(r-1)(n-1)+1} \\
&+ [\frac{1}{2^{h-1}} - \frac{1}{2^{n-2}}]\beta^{(r-1)(n-1)} \\
&- \frac{1}{2}\beta^{(r-1)(n-1)+1} ..... - \frac{1}{2^{n-h-1}}\beta^{(r-1)(n-1)+n-h-1}
\end{aligned}$$

Note that, according to $P1$, $\beta^{(r-1)(n-1)+n-h}$ is the largest element in its row. This, combined to the fact that:

$$\beta^{(r-1)(n-1)+1} < ... < \beta^{(r-2)(n-1)+n-h-1}$$

implies:

$$\begin{aligned}\Delta &< [(\frac{1}{2}-\frac{1}{2^{n-h}})+....+(\frac{1}{2^{h-1}}-\frac{1}{2^{n-2}})\\ &+(\frac{1}{2^{h-1}}-\frac{1}{2^{n-2}})]\beta^{(r-1)(n-1)+n-h}\\ &-(\frac{1}{2}+....+\frac{1}{2^{n-h-1}})\beta^{(r-2)(n-1)+1}\\ &=(\frac{1}{2}-\frac{1}{2^{n-h}})[1+....+\frac{1}{2^{h-2}}+\frac{1}{2^{h-1}}-\frac{1}{2^{n-2}}]\beta^{(r-2)(n-1)+n-h}\\ &+[1-\frac{1}{2^{n-h-1}}]\beta^{(r-1)(n-1)+1}\\ &=[(\frac{1}{2}-\frac{1}{2^{n-h}})(2-\frac{1}{2^{n-h}})+(\frac{1}{2^{h-1}}-\frac{1}{2^{n-2}})]\beta^{(r-2)(n-1)+1}\\ &+(1-\frac{1}{2^{n-h-1}})\beta^{(r-1)(n-1)+1}\\ &=[1-\frac{1}{2^{n-h-1}}][\beta^{(r-2)(n-1)+n-h}-\beta^{(r-1)(n-1)+1}]\\ &=\gamma_1[\beta^{(r-2)(n-1)+n-h}-\beta^{(r-1)(n-1)+1}]\end{aligned}$$

which proves (i).

The proof of (iv) is symmetric to that of (i) and we omit the details.

We will use the inequalities in the Claim to put an upper bound on the distance between terms in the same row of the array. Let $r > 1$ be an odd integer. Applying (i) in the Claim, we have:

$$\beta^{(r-1)(n-1)+h}-\beta^{r(n-1)} < \gamma_1(\beta^{(r-2)(n-1)+n-h}-\beta^{(r-2)(n-1)+1})$$

Observe that $\beta^{(r-2)(n-1)+n-h}-\beta^{(r-2)(n-1)+1}$ can be written as $\beta^{(r'-1)(n-1)+n-h}-\beta^{(r'-1)(n-1)+1}$ where $r' = r-1$. Since r' is an even integer, we can apply (iii) to obtain:

$$\beta^{(r-1)(n-1)+h}-\beta^{r(n-1)} < \gamma_1^2[\beta^{(r-3)(n-1)+n-h}-\beta^{(r-2)(n-1)+1}].$$

Hence applying (i) and (iii) repeatedly, we conclude that:

$$\begin{aligned}\beta^{(r-1)(n-1)+h}-\beta^{r(n-1)} &< \gamma_1^{r-1}(\beta^h-\beta^{n-1})\\ &=\gamma_1^{r-1}(\frac{1}{2}-\frac{1}{2^{n-h-1}})\\ &<\gamma_1^{r-1}(\frac{1}{2}).\end{aligned}$$

By the same argument $\beta^{(r-1)(n-1)+n-h}-\beta^{(r-1)(n-1)+1} < \gamma_1^{r-1}(\frac{1}{2})$ when r is even. Moreover, from analogous arguments, we obtain that:

$$\beta^{(r-1)(n-1)+h}-\beta^{(r-1)(n-1)+1} < \gamma_2^{r-1}(\frac{1}{2})$$

when r is odd and:

$$\beta^{(r-1)(n-1)+n-h} - \beta^{r(n-1)} < \gamma_2^{r-1}(\frac{1}{2})$$

when r is even.

Let r be an odd integer. The left-most and right-most terms in row r are $\beta^{(r(n-1)}$and $\beta^{(r-1)(n-1)+1}$ respectively. Using the triangle inequality and the bounds derived in the previous paragraph, it follows that:

$$\begin{aligned}||\beta^{r(n-1)} - \beta^{(r-1)(n-1)+1}|| \le ||\beta^{r(n-1)} - \beta^{r(n-1)+h}|| + ||\beta^{r(n-1)+h} - \beta^{(r-1)(n-1)+1}|| \\ < \frac{1}{2}(\gamma_1^{r-1} + \gamma_2^{r-1}).\end{aligned}$$

If r is an even integer, and the left-most and right-most terms in row r are $\beta^{(r-1)(n-1)+1}$ and $\beta^{(r(n-1)}$ respectively, one has:

$$\begin{aligned}||\beta^{r(n-1)} - \beta^{(r-1)(n-1)+1}|| &\le ||\beta^{r(n-1)} - \beta^{r(n-1)+n-h}|| \\ &+ ||\beta^{(r(n-1)+n-h} - \beta^{(r-1)(n-1)+1}|| \\ &< \frac{1}{2}(\gamma_1^{r-1} + \gamma_2^{r-1}).\end{aligned}$$

Note that the maximal difference of terms in row r is strictly less than $\frac{1}{2}\max[\gamma_1, \gamma_2]^{r-1}$.

Pick an integer t such that $t = r(n-1)$ where r is an odd integer i.e. β^t is the left-most term in row r and $m(t) = r-1$. Let $q = r'(n-1)$ where $r' > r$. Note that, by repeated application of the triangle inequality, it follows that $||\beta^t - \beta^q||$ is less than the sum of the differences between the left-most and right-most terms of all rows starting from $r+1$. Hence:

$$\begin{aligned}||\beta^t - \beta^q|| &< \frac{1}{2}(\gamma_1^r + \gamma_1^{r+1} + ... + \gamma_2^r + \gamma_2^{r+1}.....) \\ &= \frac{1}{2}(\frac{\gamma_1{}^r}{1-\gamma_1} + \frac{\gamma_2{}^r}{1-\gamma_2}) \\ &\equiv \lambda(r) \\ &\equiv \lambda(m(t))\end{aligned}$$

(note that we critically use the fact that γ_1 and γ_2 are strictly less than 1). Now let β^q be a term in row r' where $r' > r$. Applying the triangle inequality again, we have:

$$\begin{aligned}||\beta^t - \beta^q|| &< \lambda(m(t)) + \frac{1}{2}\max[\gamma_1, \gamma_2]^{r-1} \\ &< \lambda(m(t)) + \frac{1}{2}\max[\gamma_1, \gamma_2]^{m(t)} \\ &\equiv \hat{\lambda}(t).\end{aligned}$$

Observe that $\hat{\lambda}(t) \to 0$ as $t \to \infty$. Pick $\varepsilon > 0$ and let T be such that $\hat{\lambda}(t) < \varepsilon$ for all $t > T$. We have shown that $||\beta^T - \beta^q|| < \varepsilon$ for all $q > T$. Hence the sequence β^t is a Cauchy sequence and is convergent.

We now show that the sequence converges to $\frac{1}{n}$. Suppose it converges to α. Let t and k be positive integers such that $t+1 = k(n-1)$ and consider the following sequence of differences.

$$\beta^{t+1} - \beta^{t} = \frac{1}{2}(\beta^{(k-2)(n-1)+1} - \beta^{t}) \tag{6}$$

$$\beta^{t} - \beta^{t-1} = \frac{1}{2}(\beta^{(k-2)(n-1)+2} - \beta^{t-1}) \tag{7}$$

$$... = ...$$

$$\beta^{t-(n-3)} - \beta^{t-(n-2)} = \frac{1}{2}(\beta^{(k-1)(n-1)} - \beta^{(k-1)(n-1)}) \tag{8}$$

$$\beta^{t-(n-2)} - \beta^{t-(n-1)} = \frac{1}{2}(\beta^{(k-2)(n-1)+1} - \beta^{(k-1)(n-1)-1}) \tag{9}$$

$$... = ...$$

$$\beta^{n-h+1} - \beta^{n-h} = \frac{1}{2}(\beta^{0} - \beta^{n-h}) \tag{10}$$

It is clear from these $t-n+h$ equalities that, except for the first $n-2$ negative terms of the right hand sides, every positive term of the first $n-1$ lines has an identical negative term in one of the lines $n+1$,...,$2n$. Hence, if we sum the equalities (6)-(10), we get:

$$\beta^{t+1} - \beta^{n-h} = \frac{1}{2}(\sum_{i=1}^{n-2} \beta^{k(n-1)+i})$$

Observe that $\beta^{n-h} = 1/2$. Also, $\{\beta^{k(n-1)+i}\}$, for $k = 1, ...$, is a subsequence of the original sequence for all $i = 1, ..., n-2$. Since the original sequence converges to α, these subsequences must also converge to α. Therefore by taking limits on both sides of the equation above, we obtain $\alpha - 1/2 = -1/2(n-2)\alpha$, so that $\alpha = \frac{1}{n}$, as required. QED

An important preliminary step in the proof of Theorem 2 is the proof that if the ordering $\succsim$ of $P(X')$ satisfies Restricted Independence and Averaging, then it satisfies, given Richness, the following property of *Attenuation*.

Definition 1 *The ordering* $\succsim$ *of* $P(X')$ *satisfies Attenuation if for all sets* A, B *and* $C \in P(X)$ *satisfying* $A \sim B$, $A \cap C = B \cap C = \emptyset$ *and* $\#A > \#B$, $C \succ A$ *implies* $A \cup C \prec B \cup C$ *and* $A \succ C$ *implies* $A \cup C \succ B \cup C$.

The next two lemmas establish that any ordering $\succsim$ of $\mathcal{P}(X')$ satisfying Averaging, Restricted Independence and Richness satisfies Attenuation.

Lemma 5 *Let* $\succsim$ *be an ordering of* $\mathcal{P}(X)$ *satisfying Averaging, Restricted Independence and Richness. Then, for all finite sets* A $B \in P(X')$, *such that* $\#A - \#B \geq 2$, *and for all sets* $C \in \mathcal{P}(X')$ *such that* $C \cap (A \cup B) = \varnothing$, *there are consequences* $x_1, \ldots, x_n \in X' \backslash (A \cup B \cup C)$ *such that* $B \sim B \cup \{x_1, \ldots, x_n\}$.

Proof. Define $n = \#A - \#B$. We distinguish three cases.

$n = 2$. Using Proposition 5 and Certainty Equivalence, choose a consequence a such that $B \prec \{a\}$. By Averaging, $B \prec B \cup \{a\}$. Using again Proposition 5 and Certainty Equivalence, find a consequence $e \in X'$ such that $\{e\} \succ B \cup \{a\}$. By Averaging and transitivity, we have $B \prec B \cup \{a\} \prec B \cup \{a, e\}$.

- If there is a consequence $b \in X'$such that $B \cup \{a, b\} \precsim B$, then, by Richness, there is a consequence $c \in X'$ such that $\{a, c\} \cup B \sim B$. If a or c belongs to $A \cup B$, we then repeat the same procedure starting with another a. Since $A \cup B \cup C$ is finite, we can do this repetition at most a finite number of times so that, at the end, we are sure to find consequences a and $c \in X'$such that $\{a, c\} \cup B \sim B$ and $\{a, c\} \cap (A \cup B) = \emptyset$.
- If $B \cup \{a, b\} \succ B$ for all consequences $b \in X'$, choose (thanks to Proposition 5 and Certainty Equivalence), consequences b and $e \in X'$ such that $\{e\} \prec \{b\} \prec B$. By Averaging, one has $B \cup \{b, e\} \prec B \cup \{b\} \prec B$ and, by assumption, $B \cup \{a, b\} \succ B$. Hence, by Richness, there is a consequence $c \in X'$ such that $B \cup \{b, c\} \sim B$. If b or c belongs to $A \cup B$, we can repeat the same reasoning starting with another b. Again, the finiteness of $A \cup B \cup C$ guarantees that the repetition of the procedure will be finite and will lead, eventually, to b and c such that $\{b, c\} \cup B \sim B$ and $\{b, c\} \cap (A \cup B) = \emptyset$.

$n = 3$. We have just proved that we can find consequences a and $c \in X'$ such that $B \cup \{a, c\} \sim B$ and $\{a, c\} \cap (A \cup B) = \emptyset$. It can be noticed that $\{a\} \not\sim \{c\}$. Choose now (thanks again to Proposition 5 and Certainty Equivalence), a consequence $d \in X'$such that $\{d\} \prec B \cup \{a, c\}$. By Averaging and transitivity, one has $B \cup \{a, c, d\} \prec B \cup \{a, c\} \sim B$. Choose also (Proposition 5 and Certainty Equivalence) a consequence $e \in X'$ such that $\{e\} \succ B \cup \{a, c\}$. By Averaging and transitivity, $B \cup \{a, c, e\} \succ B \cup \{a, c\} \sim B$. By Richness, there is a consequence $f \in X'$ such that $B \cup \{a, c, f\} \sim B \cup \{a, c\} \sim B$. By Restricted Independence and transitivity, we must have either $\{a\} \succ \{f\} \succ \{c\}$ or $\{a\} \prec \{f\} \prec \{c\}$. If $f \in A \cup B \cup C$, then we can redo the procedure as many (finite) times as required starting with another a or c.

$n > 3$. If $n = 2m$ for some integer $m > 1$, then we apply m times the reasoning of the case $n = 2$. If $n = 2m + 1$ for some integer $m > 1$, then we apply $(m - 1)$ times the reasoning of the case $n = 2$ and once the reasoning of case $n = 3$. QED

Lemma 6 *Let $\succsim$ be an ordering on $P(X')$ satisfying Averaging, Restricted Independence and Richness. Then $\succsim$ satisfies the property of Attenuation.*

Proof.

Let A and B be sets in $P(X)$ such that $A \sim B$ and $\#A > \#B$ and let $n = \#A - \#B$ and let C be a set in $P(X)$ such that $A \nsim C$. Since the argument works symmetrically for $A \succ C$ or $A \prec C$, we only provide it for the later case. The argument requires that we distinguish 3 cases.

$n \geq 2$**.** By Lemma 5, there are consequences $x_1, \ldots, x_n$ such that $B \cup \{x_1, \ldots, x_n\} \sim B$ and $\{x_1, \ldots, x_n\} \cap (B \cup C) = \emptyset$. By Restricted Independence, $A \cup C \sim B \cup \{x_1, \ldots, x_n\} \cup C$. By Averaging, $\{x_1, \ldots, x_n\} \sim B \sim A$. Suppose $A \prec C$. By Averaging, $B \prec B \cup C$. Hence $\{x_1, \ldots, x_n\} \prec B \cup C$. By Averaging again, $B \cup \{x_1, \ldots, x_n\} \cup C \prec B \cup C$. By transitivity, $A \cup C \prec B \cup C$.

$n = 1$, $\#B \geq 2$**.** We first show that there exists a consequence $x \in X'$ and a set $B' \in P(X')$ such that $x \notin B'$, $\{x\} \sim B' \sim B$, $\#B' = \#B$ and $B' \cap C = \emptyset$. Indeed, use Certainty Equivalence to define x by $\{x\} \sim B$. If $x \notin B$, then define $B' = B$ and the proof is done. If $x \in B$, choose a consequence $c \in X'$ such that $\{c\} \prec \{x\}$ and $\{c\} \succsim \{y\}$ for all $y \in B \cup C$ (if any) such that $\{y\} \prec \{x\}$. The finiteness of $B \cup D$ as well as Proposition 5 guarantees the existence of such a c. Using similar arguments, one can also find a consequence $d \in X'$ such that $\{x\} \prec \{d\}$ and $\{d\} \precsim \{z\}$ for all $z \in B \cup C$ (if any) such that $\{x\} \prec \{z\}$. Moreover, c and d can be chosen in such a way that $\{x\} \precsim \{c, d\}$. Indeed, if $\{x\} \succ \{c, d\}$ for some initial choice of c and d, then, we know from Averaging that $\{x, d\} \succ \{x\} \succ \{c, d\}$. Hence by Richness, there exists a c'such that $\{c', d\} \sim \{x\}$. Since $\{c', d\} \sim \{x\} \succ \{c, d\}$, we must have from Restricted Independence that $\{c'\} \succ \{c\}$ and, since $\{d\} \succ \{x\}$ and $\{c', d\} \sim \{x\}$, it follows from Averaging and transitivity that $\{x\} \succ \{c'\}$. We then have $\{x\} \succ \{c'\} \succ \{c\} \succsim \{y\}$ for all $y \in B \cup C$ (if any) such that $\{y\} \prec \{x\}$. Hence replacing c by c' leads immediately to the statement that $\{x\} \precsim \{c', d\}$. Assuming therefore $\{x\} \precsim \{c, d\}$, we consider two cases.

1: $\#B = 2m$, for some strictly positive integer m. Choose m different consequences $z_1, \ldots, z_m \in X'$ such that$\{c\} \prec \{z_1\} \prec \ldots \prec \{z_m\} \prec \{x\}$. This is clearly possible thanks to Certainty Equivalence. By assumption, $z_i \notin B \cup C$. For $i = 1 \ldots m$, define z_i' by $\{x\} \sim \{z_i, z_i'\}$. This is possible thanks to Richness and the fact that $\{x\} \precsim \{c, d\} \precsim \{z_i, d\}$ (by Restricted Independence and transitivity) and that $\{x\} \succ \{z_i\} \succ \{z_i, c\} \succ \{c\}$ (by Averaging and transitivity). By Averaging and transitivity, one has $\{z_i, z_i'\} \sim \{x\} \succ \{z_i, x\}$. It then follows from Restricted Independence that $\{z_i'\} \succ \{x\}$. We now prove that $\{z_i'\} \prec \{d\}$, for $i = 1 \ldots m$. Suppose by contradiction, using the completeness of $\succsim$, that $\{d\} \precsim \{z_i'\}$ for some i. By Restricted Independence and transitivity, we would then have

$\{z_i, d\} \precsim \{z_i, z_i'\} \sim \{x\}$ Yet, since $\{c\} \prec \{z_i\}$, we have by Restricted Independence and transitivity that $\{c, d\} \prec \{z_i, d\} \precsim \{z_i, z_i'\} \sim \{x\}$, in violation of $\{x\} \precsim \{c, d\}$. Hence, since $\{x\} \prec \{z_i'\} \prec \{d\}$, we know that $z_i' \notin B \cup C$. Define then $B' = \{z_1, \ldots, z_m, z_1', \ldots, z_m'\}$. By repeated application of Averaging and transitivity, one obtains that $B' \sim \{x\} \sim B$ and, by construction, that $B' \cap C = \emptyset$ and $x \notin B'$.
2: $\#B = 2m + 1$, for some strictly positive integer m. Using Certainty Equivalence, define $c'$and c'' by $\{c'\} \sim \{c, x\}$ and $\{c''\} \sim \{c', c\}$. We know that $\{x\} \precsim \{c, d\}$ and, by Restricted Independence and transitivity, $\{x\} \precsim \{c, d\} \prec \{c'', d\} \prec \{c', d\}$. Moreover, by Averaging and transitivity, we have that $\{x\} \succ \{c, x\} \sim \{c'\} \succ \{c', c\} \sim \{c''\} \succ \{c\}$. Hence, using Richness, one can define d' and d'' by $\{c', d'\} \sim \{x\}$ and $\{c'', d''\} \sim \{x\}$. As in the case '$\#B = 2m$', we can show that $\{d\} \succ \{d''\} \succ \{d'\} \succ \{x\}$. Hence, $\{c'', c', d', d''\} \cap (B \cup C) = \emptyset$. Since $\{d'\} \prec \{d''\}$, we have $\{x\} \sim \{c', d'\} \prec \{c', d''\}$ (by Restricted Independence and transitivity) and, by Averaging and transitivity, that $\{x\} \prec \{c', d'', d\}$. Now, by Averaging again, $\{c'', c', d', d''\} \sim \{x\}$. Hence, since $\{c\} \prec \{c''\}$, we have $\{c, c', d', d''\} \prec \{x\}$ (by Restricted Independence and transitivity). Moreover, since $\{x\} \prec \{d'\}$, Averaging implies $\{c', c, d''\} \prec \{x\}$. Hence, $\{c', c, d''\} \prec \{x\} \prec \{c', d'', d\}$. By Richness, there is a consequence e such that $\{x\} \sim \{c', d'', e\}$. One can not have $\{c\} \succsim \{e\}$ because this would imply, using Restricted Independence and transitivity, that $\{x\} \sim \{c', d'', e\} \precsim \{c', d'', c\}$, in contradiction of $\{x\} \succ \{c', d'', c\}$. Analogously $\{e\} \succsim \{x\}$ can not hold because, if it did, one would have, using Restricted Independence and transitivity, that $\{x\} \sim \{c', d'', e\} \succsim \{c', d'', x\}$ and, using Averaging, that $\{x\} \succsim \{c', d''\}$, a contradiction. Hence, since $\succsim$ is complete, $\{c\} \prec \{e\} \prec \{x\}$. We therefore conclude that $\{c', d'', e\} \cap B \cup C = \emptyset$. Choose now $(m-1)$ different consequences $z_1, \ldots, z_{m-1} \in X'$ in such a way that $\{c'\} \prec \{z_1\} \prec \ldots \prec \{z_{m-1}\} \prec \{x\}$. It is always possible to choose them different from c', $d''$and e. For $i = 1 \ldots m-1$, define as in the previous case z_i' by $\{x\} \sim \{z_i, z_i'\}$. As in the previous case also, we can show that z_i' is such that $\{x\} \prec \{z_i'\} \prec \{d\}$for $i = 1 \ldots m-1$. Define therefore B' by $B' = \{c', d'', e, z_1, \ldots, z_{m-1}, z_1', \ldots, z_{m-1}'\}$. By Averaging, $B' \sim \{x\} \sim B$ and, by construction, $B' \cap B \cup C = \emptyset$ and $x \notin B'$.
Given the existence of the set $B'$and the consequence x with the required property, we consider two cases.

$x \notin C$. By Averaging, $B' \cup \{x\} \sim A$. By Restricted Independence, $B \cup C \sim B' \cup C$and $B' \cup \{x\} \cup C \sim A \cup C$. Suppose $A \prec C$. Hence, $B' \prec C$and, by Averaging, $B' \prec B' \cup C \prec C$. We also have $\{x\} \prec B' \cup C$and, by Averaging, $\{x\} \prec B' \cup \{x\} \cup C \prec B' \cup C$. By transitivity, $A \cup C \prec B \cup C$.

$x \in C$. We must then have that $\#C > 1$, as assuming otherwise would imply that $C = \{x\} \sim B \sim A$). Using the same argument as above, there is a set $C' \in P(X)$ satisfying $C' \sim C$, $\#C' = \#C, x \notin C'$, $B \cap C' = \emptyset$, $B' \cap C' = \emptyset$, $A \cap C' = \emptyset$. By Restricted Independence, $B \cup C \sim B' \cup C$, $B' \cup C \sim B' \cup C'$,$A \cup C \sim A \cup C'$, and $B' \cup \{x\} \cup C' \sim A \cup C'$. Suppose $A \prec C$. Hence, $B' \prec C'$and, by Averaging, $B' \prec B' \cup C' \prec C'$. We also have $\{x\} \prec B' \cup \{x\} \cup C' \prec B' \cup C'$. By transitivity, $A \cup C \prec B \cup C$.

$n = 1$, $\#B = 1$. Suppose first that $\#C = 1$. Write A, B and C as: $A = \{a, b\}$, $B = \{x\}$ and $C = \{c\}$ and assume that $\{x\} \sim \{a, b\} \prec \{c\}$ but, contrary to what is required by Attenuation, that $\{a, b, c\} \succsim \{x, c\}$. By Certainty Equivalence, there exists a consequence $z \in X'$ such that $\{z\} \sim \{x, c\}$. Since $\{x\} \prec \{c\}$, $\{x\} \prec \{z\} \prec \{c\}$ by Averaging so that z is distinct from either x or c. We therefore have (using Averaging and transitivity) $\{a, b, c\} \succsim \{x, c\} \sim \{z\} \sim \{x, c, z\}$. It then follows from Restricted Independence and transitivity that $\{a, b\} \succsim \{x, z\} \succ \{x\}$, contradicting $\{x\} \sim \{a, b\}$. Suppose now that $\#C > 1$. Suppose $\{x\} \sim\{a, b\} \prec C$ but, contrary to what is required by Attenuation, $\{a, b\} \cup C \succsim \{x\} \cup C$. By Certainty Equivalence, there is a consequence $z \in X'$such $\{z\} \sim \{x\} \cup C$. By Averaging (since $x \notin C$), one has $\{x\} \prec \{z\} \prec C$. One has therefore $\{a, b\} \cup C \succsim \{x\} \cup C \sim \{z\}$. If $z \notin C$, then Averaging and transitivity entails that $\{z\} \sim \{x, z\} \cup C$. Using then Restricted Independence and transitivity, one obtains that $\{a, b\} \succsim \{x, z\} \succ \{x\}$, a contradiction. If $z \in C$, then apply Certainty Equivalence recursively to find a sequence of z_t such that $\{z_t\} \sim \{z_{t-1}, x\}$ for $t = 1, ...$starting with $z_0 = z$. Since there are only finitely many elements in C, one will eventually find some t for which $z_t \notin C$and $\{x\} \prec \{z_t\} \prec ... \prec \{z\} \prec C$. By Averaging $\{x\} \cup C \sim \{x\} \cup C \backslash \{z\} \sim \{z\}$. Since $\{z\} \succ \{z_t\}$, we have, by transitivity and Averaging, that $\{z\} \sim \{x\} \cup C' \succ \{x, z_t\} \cup C \succ \{z_t\}$. We therefore have $\{a, b\} \cup C \succsim \{x\} \cup C \sim \{z\} \sim \{x\} \cup C \succ \{x, z_t\} \cup C$ which implies, thanks to Restricted Independence and transitivity, that $\{a, b\} \succ \{z_t, x\}$ and, by Averaging and transitivity, that $\{a, b\} \succ \{x\}$, again a contradiction.

We next establish some further auxiliary lemmas.

Lemma 7 *Let $\succsim$ be an ordering on $\mathcal{P}(X')$ satisfying Averaging, Restricted Independence, and Richness. Then, if A and B are subsets of X' and c is a consequence in X' such that $A \prec B \cup \{c\}$,then there exists some $e \in X$ such that $\{e\} \prec \{c\}$ and $A \prec B \cup \{e\}$. Dually, if A and B are sets and c is a consequence in X' such that $A \succ B \cup \{c\}$, then there exists $e \in X'$such that $\{e\} \succ \{c\}$ and $A \succ B \cup \{e\}$.*

Proof.

We only prove the first statement and distinguish three cases.

(a) $A \prec B \cup \{d\}$, in which case the proof is done.

(b) $A \sim B \cup \{d\}$. Then, by Certainty Equivalence, there exists e such that $\{e\} \sim \{d, c\}$. By Averaging, $\{d\} \prec \{e\} \prec \{c\}$. By Restricted Independence, $B \cup \{d\} \prec B \cup \{e\}$ so that the statement $A \prec B \cup \{e\}$ follows.

(c) $A \succ B \cup \{d\}$. In that case the Richness axiom applies and there is a consequence f such that $A \sim B \cup \{f\}$ and we proceed as in case (b).

We next establish that if $\succsim$ is an ordering of $P(X')$ satisfying Averaging, Restricted Independence, Richness and, by Lemma 6, Attenuation, then it satisfies the following condition.

Condition 3 *(C) For all distinct consequences* a, b, c *and* $d \in X$ *and every set* $B \in P(X)$ *such that* $\{b\} \sim \{c, d\}$ *and* $B \cap \{b, c, d\} = \emptyset$*, we must have:*
(i) $\{a\} \succsim B \cup \{b\}$ *and* $\{b\} \succsim \{a\}$ *with at least one strict ranking imply* $\{a, b\} \succ B \cup \{c, d\}$*, and*
(ii)$\{a\} \precsim B \cup \{b\}$ *and* $\{b\} \prec \{a\}$ *with at least one strict ranking imply* $\{a, b\} \prec B \cup \{c, d\}$.

Three auxiliary lemmas are needed in order to establish this. The first of them is the following.

Lemma 8 *Let* $\succsim$ *be an ordering on* $P(X')$ *satisfying Averaging, Restricted Independence and Richness. Let* A *and* B *be two finite subsets of* X' *and let* a*,* b*, cand* d *be consequences in* X' *satisfying* $A \cup \{a\} \sim B \cup \{b\}$*,* $\#A = \#B$*,* $\{b\} \sim \{c, d\}$*,* $a \neq b$*,* $c \neq d$*,* $\{a, b\} \cap A = \{c, d\} \cap B = \emptyset$ *and* $b \notin B$*. Then* $A \cup \{a, b\} \sim B \cup \{c, d\}$*.*

Proof.

Suppose first $\{c\} \sim \{d\}$. By Averaging, $\{b\} \sim \{c\} \sim \{d\}$. Since $c \neq d$, we have $c \neq b$ or $d \neq b$. Assume without loss of generality that $c \neq b$. By Restricted Independence, $B \cup \{b\} \sim B \cup \{c\}$. Therefore $A \cup \{a\} \sim B \cup \{b\} \sim B \cup \{c\}$ and, by Restricted Independence, $A \cup \{a, b\} \sim B \cup \{c, b\}$. By Restricted Independence again, $B \cup \{c, b\} \sim B \cup \{c, d\}$. Finally, by transitivity, $A \cup \{a, b\} \sim B \cup \{c, d\}$.

Suppose now $\{c\} \not\sim \{d\}$ and assume, without loss of generality, that $\{c\} \prec \{d\}$. Two cases need to be considered.

1. Assume by contradiction that $A \cup \{a, b\} \prec B \cup \{c, d\}$. Let us show that there is a consequence $\underline{d}$ such that $A \cup \{a, b\} \prec B \cup \{c, \underline{d}\} \prec B \cup \{c, d\}$. Choose a consequence u distinct from c such that $\{u\} \prec \{d\}$. The existence

of such a consequence is guaranteed by the fact that $\{c\} \prec \{d\}$ and, using Certainty Equivalence, that one can always define u by $\{u\} \sim \{c, d\}$. By Averaging, one must have $\{c\} \prec \{u\} \prec \{d\}$ which, given the reflexivity of $\succsim$, implies that u is distinct from both c and d. By Restricted Independence, one has $B \cup \{c, u\} \prec B \cup \{c, d\}$. Two mutually exclusive cases can occur.

- $B \cup \{c, u\} \precsim A \cup \{a, b\}$. By Averaging and Certainty Equivalence, one can find a consequence e such that $A \cup \{a, b\} \prec e \prec B \cup \{c, d\}$. By Richness, there is $\underline{d} : B \cup \{c, d\} \sim \{e\}$. Hence $A \cup \{a, b\} \prec B \cup \{c, \underline{d}\} \prec B \cup \{c, d\}$
- $A \cup \{a, b\} \prec B \cup \{c, u\}$. In this case, let $\underline{d} = u$.

By Certainty Equivalence, there is a consequence $\underline{b}$ such that $\{\underline{b}\} \sim \{c, \underline{d}\}$. Notice that we can always choose $\underline{d}$ so that $\underline{d}$ and $\underline{b}$ do not belong to $B \cup \{c\} \cup A \cup \{a\}$. By Restricted Independence, $\{\underline{b}\} \prec \{b\}$. By Averaging, $\{\underline{b}\} \sim \{c, \underline{d}\} \prec \{b, \underline{b}\} \prec \{b\}$. By Restricted Independence, $B \cup \{c, \underline{d}\} \prec B \cup \{b, \underline{b}\}$. By Restricted Independence, $A \cup \{a, \underline{b}\} \prec A \cup \{a, b\}$ and $A \cup \{a, \underline{b}\} \sim B \cup \{b, \underline{b}\}$. By transitivity, $B \cup \{c, \underline{d}\} \prec A \cup \{a, b\}$. But we have previously shown that $A \cup \{a, b\} \prec B \cup \{c, \underline{d}\}$. A contradiction.

2. Assume by contradiction that $A \cup \{a, b\} \succ B \cup \{c, d\}$. This case is treated like the previous one.

The next lemma provides the second step in the proof that Averaging, Restricted Independence and Richness imply Condition C.

Lemma 9 *Let $\succsim$ be an ordering on $\mathcal{P}(X')$ satisfying Averaging, Restricted Independence and Richness and let a, b, c and d be consequences in X' and B be a finite subset of X' such that $\{a\} \succsim B \cup \{b\}$, $\{b\} \sim \{c, d\}$, $\{b\} \succ \{a\}$, $b \notin B$ and $\{c, d\} \cap B = \emptyset$. Then there exists a finite subset A' of X' and a consequence $a' \in X'$ such that $A' \cup \{a'\} \sim B \cup \{b\}$, $a' \notin A'$ and $\#A' = \#B$.*

Proof.

Start with $\{b\} \succ \{a\} \succsim B \cup \{b\}$. By Averaging, $\{b\} \succ B$. Write B as $B = \{b_1, \ldots, b_r\}$ with $\{b_1\} \precsim \{b_2\} \precsim \ldots \precsim \{b_r\}$. Let b_j be such that $\{b_j\} \prec \{b\}$ and $\{b\} \precsim b_i$ for all $i > j$. The existence of such a b_j is guaranteed by Averaging. By Certainty Equivalence, one can find a consequence b'_j in X' such that $b'_j \sim \{b, b_j\}$. By Averaging, $b_j \prec b'_j \prec b$. Define A' by $A' = B \cup \{b'_j\} \setminus \{b_j\}$. By Averaging and transitivity, one has $A' \succ B$. By Restricted Independence, $A' \cup \{b\} \succ B \cup \{b\}$. By construction, $A' \cup \{b_j\} = B \cup \{b'_j\}$. By Restricted Independence, $B \cup \{b\} \succ B \cup \{b'_j\}$. Hence $A' \cup \{b\} \succ B \cup \{b\} \succ B \cup \{b'_j\} = A' \cup \{b_j\}$. By Richness, there exists a consequence a' such that $A' \cup \{a'\} \sim B \cup \{b\}$. By Restricted Independence, one has $b \succ a' \succ b'_j$, which, given the definition of A', establishes that $a' \notin A'$.

Combining these two lemmas, we can establish the following.

Lemma 10 *Let* $\succsim$ *be an ordering on* $P(X')$ *satisfying Averaging, Restricted Independence and Richness. Then* $\succsim$ *satisfies condition C.*

Proof.

We prove only part (i) of condition C, the proof of the other part being similar. Suppose that we have $\{a\} \succsim B \cup \{b\}$, $\{b\} \sim \{c, d\}$, $\{b\} \succ \{a\}$, $b \notin B$ and $\{c, d\} \cap B = \emptyset$ for consequences a, b, c, d in X' and some finite subset B of X'. By Lemma 9, there exists a finite set A' and a consequence a' such that $A' \cup \{a'\} \sim B \cup \{b\}$, $a' \notin A'$and $\#A' = \#B$. By Lemma 8, we must have $A' \cup \{a', b\} \sim B \cup \{c, d\}$. By Certainty Equivalence, there exists a consequence a'' such that $a'' \sim A' \cup \{a'\}$. By transitivity, $\{b\} \succ \{a\} \succsim A' \cup \{a'\} \sim \{a''\}$. Since, by Lemma 6, the ordering $\succsim$ satisfies Attenuation, one has $A' \cup \{a', b\} \prec \{a'', b\}$. By transitivity, $\{a'', b\} \succ B \cup \{c, d\}$. Restricted independence and $\{a\} \succsim \{a''\}$ imply $\{b, a\} \succsim \{b, a''\}$. Transitivity finally yields $\{a, b\} \succ B \cup \{c, d\}$.

We are now equipped to prove Theorem 2.

Proof of Theorem 2.

Using Theorem 1, we find a function u that uniquely represents (up to an affine transform) $\succsim$ as per (1) on the subset of $\mathcal{P}(X')$ containing sets of cardinality no greater than 2. We want to prove that the same function u can also be used to represent $\succsim$ on the whole set $\mathcal{P}(X')$. We must prove specifically that, for any $A \in \mathcal{P}(X')$ and $g \in X'$,

$$A \succsim \{g\} \iff \sum_{a \in A} \frac{u(a)}{\#A} \geq u(g).$$

where u is the (unique up to an affine transform) utility function identified in Theorem 1. Since $\succsim$ is complete, it is sufficient to prove $\Rightarrow$. Suppose $\#A = m$ and write $A = \{a_1, a_2, \ldots, a_m\}$ with $\{a_1\} \precsim \ldots \precsim \{a_m\}$. By Certainty Equivalence, there exists $b^0_{m-1} \in X'$ such that $b^0_{m-1} \sim \{a_{m-1}, a_m\}$. Similarly, for $i = m-2, \ldots, 1$, we can find, by Certainty Equivalence, a consequence b^0_i such that $b^0_i \sim \{a_i, b^0_{i+1}\}$. Using Certainty Equivalence repeatedly, one can define this way for $j = 1, 2, 3, \ldots$the sequence of consequences b^j_i by:

$$b_1^{2j-1} = b_1^{2j-2},$$

$$b_i^{2j-1} \sim \{b_{i-1}^{2j-1}, b_i^{2j-2}\}$$

for $i = 2, \ldots, m-1$,

$$b_{m-1}^{2j} = b_{m-1}^{2j-1}$$

and

$$b_i^{2j} \sim \{b_i^{2j-1}, b_{i+1}^{2j}\}$$

for $i = m-2, \dots, 1$. We first show that:
(i) $\{b_1^j\} \precsim \{b_2^j\} \precsim \dots \precsim \{b_{m-1}^j\}$,
(ii) $\{a_1\} \precsim \{b_1^i\} \precsim \{b_1^{i+1}\} \precsim \{b_{m-1}^{i+1}\} \precsim \{b_{m-1}^i\} \precsim \{a_m\}$ and
(iii) $\{b_1^i\} \precsim A \precsim \{b_{m-1}^i\}$.
If $\{a_1\} \sim \{a_m\}$, then, by Averaging, $\{a_1\} \sim A$, $\{b_j^i\} \sim \{a_1\} \sim A$ for all $i \in \mathbb{N}$ and $j \in \{1, \dots, m-1\}$ and the implications (i)-(iii) are immediately established. If $\{a_1\} \prec \{a_m\}$, let k be the largest integer such that $\{a_k\} \prec \{a_{k+1}\}$. We first prove implications (i) and (ii). By Averaging, $\{a_{m-1}\} \precsim \{b_{m-1}^0\} \precsim \{a_m\}$. By transitivity, $\{a_{m-2}\} \precsim \{b_{m-1}^0\}$. By Averaging again, $\{a_{m-2}\} \precsim \{b_{m-2}^0\} \precsim \{b_{m-1}^0\}$. By repeated use of transitivity and Averaging, one is led to the conclusion that $\{a_{k+1}\} \precsim \{b_{k+1}^0\} \precsim \{b_{k+2}^0\}$. Now, by transitivity $\{a_k\} \prec \{b_{k+1}^0\}$ and, by Averaging, $\{a_k\} \prec \{b_k^0\} \prec \{b_{k+1}^0\}$. Analogously, a repeated combination of Averaging and transitivity leads to the conclusion that $\{a_1\} \prec \{b_1^0\} \prec \{b_2^0\}$. Hence, we have established that $\{a_1\} \prec \{b_1^0\} \prec \{b_{k+1}^0\} \precsim \{b_{k+2}^0\} \precsim \dots \precsim \{b_{m-1}^0\} \precsim \{a_m\}$. Now, by Averaging, $\{b_1^0\} \prec \{b_2^1\} \prec \{b_2^0\}$ and, by transitivity, $\{b_2^1\} \prec \{b_3^0\}$. Combining in this way Averaging and transitivity leads us to $\{b_{m-2}^1\} \prec \{b_{m-1}^1\} \prec \{b_{m-1}^0\}$ and, therefore, to $\{a_1\} \prec \{b_1^0\} \sim \{b_1^1\} \prec \{b_2^1\} \prec \dots \prec \{b_{m-1}^1\} \prec \{b_{m-1}^0\} \precsim \{a_m\}$. Repeatedly using the same reasoning, one finds that, for all $i \in N$, $\{b_1^i\} \prec \{b_2^i\} \prec \dots \prec \{b_{m-1}^i\}$ and $\{a_1\} \precsim \{b_1^i\} \precsim \{b_1^{i+1}\} \precsim \{b_{m-1}^{i+1}\} \precsim \{b_{m-1}^i\} \precsim \{a_m\}$. We now turn to implication (iii) that we prove in the following infinite number of steps.

Step 1. We notice that by virtue of the Gärdenfors principle, $\{b_{m-1}^0\} \succ A$.

Step 2. We prove that $\{b_1^0\} \prec A$. Since by assumption $a_l = a_{l+1}$for all $l = k+1, ..., m-1$, we have by Averaging that $\{b_{m-1}^0\} \sim \{a_m\} \sim \{a_{m-1}, a_m\} \sim \{a_{m-2}\} \sim \{a_{m-2}, a_{m-1}, a_m\} \sim ... \sim \{a_{k+1}, \dots, a_{m-1}, a_m\}$. We therefore have $\{b_{k+1}^0\} \sim \{a_{k+1}, \dots, a_{m-1}, a_m\} \sim \{a_{k+1}\}$. Now, since $\{a_k\} \prec \{b_{k+1}^0\} \sim \{a_{k+1}, \dots, a_{m-1}, a_m\}$, it follows from Attenuation property (satisfied thanks to Lemma 6) that $\{a_k, b_{k+1}^0\} \prec \{a_k, a_{k+1}, \dots, a_{m-1}, a_m\}$ and, since $\{b_k^0\} \sim \{a_k, b_{k+1}^0\}$ and $\succsim$ is transitive, that $\{b_k^0\} \prec \{a_k, \dots, a_{m-1}, a_m\}$. Applying the same reasoning below k enables us to reach the conclusion that $\{b_1^0\} \prec \{a_1, \dots, a_2, a_m\} = A$.

Step 3. Since $b_1^1 = b_0^1$, we trivially have that $\{b_1^1\} \prec A$.

Step 4. We prove that $\{b_{m-1}^1\} \succ A$. Notice that $\{b_1^1\} \sim \{a_1, b_2^0\}$, $\{b_2^1\} \sim \{b_1^1, b_2^0\}$, $\{b_2^0\} \sim \{a_2, b_3^0\}$, $\{b_1^1\} \prec \{b_2^0\}$ and clause (i) of condition C (satisfied thanks to Lemma 10) imply that $\{b_2^1\} \succ \{a_1, a_2, b_3^0\}$. Similarly, $\{b_2^1\} \sim \{b_1^1, b_2^0\}$, $\{b_3^1\} \sim \{b_2^1, b_3^0\}$ and clause (i) of the condition C imply that $\{b_3^1\} \succ \{a_1, a_2, a_3, b_4^0\}$. Repeating this reasoning, we obtain $\{b_{m-2}^1\} \succ \{a_1, \dots, a_{m-2}, b_{m-1}^0\}$

and, finally, $\{b^1_{m-1}\} \succ \{a_1, \dots, a_m\} = A$.

Step 5. Trivially, $\{b^2_{m-1}\} = \{b^1_{m-1}\} \succ A$.

Step 6. We prove that $\{b^2_1\} \prec A$. We have $\{b^2_{m-1}\} \sim \{b^0_{m-1}, b^0_{m-2}\}$, $\{b^2_{m-2}\} \sim \{b^2_{m-1}, b^1_{m-2}\}$, $\{b^1_{m-2}\} \sim \{b^1_{m-3}, b^0_{m-2}\}$ and $\{b^1_{m-2}\} \prec \{b^2_{m-1}\}$. Hence, by clause (*ii*) of condition *C*, $\{b^2_{m-2}\} \prec \{b^1_{m-3}, b^0_{m-2}, b^0_{m-1}\}$. We have also $\{b^2_{m-2}\} \prec \{b^1_{m-3}, b^0_{m-2}, b^0_{m-1}\}$, $\{b^2_{m-3}\} \sim \{b^2_{m-2}, b^1_{m-3}\}$, $\{b^1_{m-3}\} \sim \{b^1_{m-4}, b^0_{m-3}\}$ and $\{b^1_{m-3}\} \prec \{b^2_{m-2}\}$. Hence, by clause (*ii*) of condition *C*:
$\{b^2_{m-3}\} \prec \{b^1_{m-4}, b^0_{m-3}, b^0_{m-2}, b^0_{m-1}\}$. This process can be repeated until we obtain:

$$\{b^2_2\} \prec \{b^1_1, b^0_2, b^0_3, \dots, b^0_{m-1}\} = \{b^0_1, b^0_2, b^0_3, \dots, b^0_{m-1}\}.$$

By Lemma 7, there exists $\{c^0_2\} \prec \{b^0_2\}$ such that $\{b^2_2\} \prec \{b^0_1, c^0_2, b^0_3, \dots, b^0_{m-1}\}$. Repeatedly applying Lemma 7, we find $\{c^0_i\} \prec \{b^0_i\}$, for $i = 3 \dots m-1$ such that $\{b^2_2\} \prec \{b^0_1, c^0_2, c^0_3, \dots, c^0_{m-1}\}$. This, combined with $\{b^2_1\} \sim \{b^2_2, b^1_1\}$, $\{b^1_1\} \sim \{a_1, b^0_2\}$, $\{b^1_1\} \prec \{b^2_2\}$ and clause (*ii*) of condition *C*, implies:
$\{b^2_1\} \prec \{a_1, c^0_2, b^0_2, c^0_3, \dots, c^0_{m-1}\}$. By Averaging, it follows that $\{c^0_2, b^0_2\} \prec \{b^0_2\} \sim \{a_2, b^0_3\}$. By Restricted Independence, one has $\{b^2_1\} \prec \{a_1, a_2, c^0_3, b^0_3, \dots, c^0_{m-1}\}$. By Averaging, $\{c^0_3, b^0_3\} \prec \{b^0_3\} \sim \{a_3, b^0_4\}$. By Restricted Independence:

$$\{b^2_1\} \prec \{a_1, a_2, a_3, c^0_4, b^0_4, \dots, c^0_{m-1}\}.$$

Repeating this process leads us to the conclusion that:

$$\{b^2_1\} \prec \{a_1, a_2, \dots, a_{m-2}, c^0_{m-1}, b^0_{m-1}\}.$$

By Averaging, $\{c^0_{m-1}, b^0_{m-1}\} \prec \{b^0_{m-1}\} \sim \{a_{m-1}, a_m\}$. By Restricted Independence:

$$\{b^2_1\} \prec \{a_1, a_2, \dots, a_{m-2}, a_{m-1}, a_m\} = A.$$

Step 7. Trivially, $\{b^3_1\} = \{b^2_1\} \prec A$.

Step 8. We prove that $\{b^3_{m-1}\} \succ A$. We have $\{b^3_1\} \sim \{b^1_1, b^2_2\}$, $\{b^3_2\} \sim \{b^3_1, b^2_2\}$, $\{b^2_2\} \sim \{b^1_2, b^2_3\}$ and $\{b^2_2\} \succ \{b^3_1\}$. Hence, by clause (*i*) of condition *C*, $\{b^3_2\} \succ \{b^1_1, b^1_2, b^2_3\}$. We also have $\{b^3_3\} \sim \{b^3_2, b^2_3\}$, $\{b^2_3\} \sim \{b^1_3, b^2_4\}$ and $\{b^2_3\} \succ \{b^3_2\}$. Hence, by clause (*i*) of condition *C*, $\{b^3_3\} \succ \{b^1_1, b^1_2, b^1_3, b^2_4\}$. Continuing this process, we obtain $\{b^3_{m-2}\} \succ \{b^1_1, b^1_2, \dots, b^1_{m-2}, b^2_{m-1}\}$. Repeatedly applying Lemma 7, we find c^1_i such that $\{c^1_i\} \succ \{b^1_i\}$, for $i = 1 \dots m-2$ such that $\{b^3_{m-2}\} \succ \{c^1_1, c^1_2, \dots, c^1_{m-2}, b^2_{m-1}\}$. This, combined with $\{b^3_{m-1}\} \sim \{b^2_{m-1}, b^3_{m-2}\}$, $\{b^2_{m-1}\} \sim \{b^0_1, b^1_{m-2}\}$, $\{b^2_{m-1}\} \succ \{b^3_{m-2}\}$and clause (*i*) of condition *C*, implies that:

$$\{b^3_{m-1}\} \succ \{c^1_1, c^1_2, \dots, c^1_{m-2}, b^1_{m-2}, b^0_{m-1}\}.$$

By Averaging and Restricted Independence:

$$\{b_{m-1}^3\} \succ \{c_1^1, \ldots, c_{m-3}^1, b_{m-3}^1, b_{m-2}^0, b_{m-1}^0\}.$$

By repeatedly combining Averaging and Restricted Independence in this way, one is led to the conclusion that:

$$\{b_{m-1}^3\} \succ \{c_1^1, b_1^1, b_2^0, b_3^0, \ldots, b_{m-1}^0\}.$$

Repeatedly applying Lemma 7, one finds d_i^1 such that $\{d_i^1\} \prec \{b_i^0\}$, for $i = 2 \ldots m-1$ for which:

$$\{b_{m-1}^3\} \succ \{c_1^1, b_1^1, d_2^0, d_3^0, \ldots, d_{m-1}^0\}.$$

Repeatedly applying Averaging and Restricted Independence, we obtain $\{b_{m-1}^3\} \succ \{a_1, a_2, \ldots, a_{m-2}, \ldots, a_{m-1}, a_m\} = A$.

Step 9. Trivially, $\{b_{m-1}^4\} = \{b_{m-1}^3\} \succ A$.

Steps 6 to 9 can clearly be repeated for ever using the same argument and this remark completes the proof of (*iii*). Now, using Certainty Equivalence, let x be a consequence such that $A \sim \{x\}$. Since the function u found in Theorem 1 represents $\succsim$ as per (1), one has $u(b_1^i) \leq u(x) \leq u(b_{m-1}^i)$ for every i. Now it is easy to check that the sequences $\{u(b_h^i)\}$ for every h are just like the sequences in Lemma 1. Because of this lemma, one has:

$$\lim_{t\to\infty} u(b_1^t) = \lim_{t\to\infty} u(b_{m-1}^t) = \sum_{a\in A} \frac{u(a)}{\#A}.$$

Hence, $u(x) = \sum\limits_{a\in A} \frac{u(a)}{\#A}$. By transitivity, $A \succsim \{g\} \Longleftrightarrow \{x\} \succsim \{g\} \Longleftrightarrow \sum_{a\in A} \frac{u(a)}{\#A} \geq u(g)$.

Proof of Theorem 3.

From Theorem 2, we know that if $\succsim$ is an ordering on $P(X)$ satisfying Averaging, Restricted Independence, Richness and the Archimedean axiom, then there exists a function $u : X' \to \mathbb{R}$ such that, for all sets A and $B \in P(X')$, one has $A \succsim B \Longleftrightarrow \sum\limits_{a\in A} \frac{u(a)}{\#A} \geq \sum\limits_{b\in B} \frac{u(b)}{\#B}$. If $M(X) = m(X) = \varnothing$ so that $X = X'$, then the proof is done. Assume first that $M(X) \neq \varnothing$ and let t be a consequence in $M(X)$. We have $\{t\} \succ \{x\}$ for every $x \in X'$. We first show that the image of X'under u, denoted $u(X')$, is a set of real numbers that is bounded from above. That is, there is a real number $\overline{b}$ such that $u(x) \leq \overline{b}$ for all $x \in X'$. Suppose indeed by contradiction that $u(X')$ is not bounded and consider, thanks to Proposition 5, consequences a, c_0 and $b \in X'$such that $\{a\} \prec \{c_0\} \prec \{b\}$. By Averaging and transitivity, one has $\{a\} \prec \{a, c_0\} \prec \{c_0\} \prec \{c_0, b\}$ and, by Theorem 2, one has $u(a) < \frac{u(c_0)+u(b)}{2} \iff u(a) + u(a) < u(c_0) + u(b)$. Since $u(X')$ is unbounded, there is a real number $u' \in u(X')$ such that $u' + u(a) \geq u(c_0) + u(b)$. Since $u' \in X'$, there is a consequence $c \in X'$ such that $u(c) = u'$. By Theorem 2, one has $\{c, a\} \succsim \{c_0, b\} \succ \{c_0, a\}$. If follows from Richness that there is a consequence c_1 such that $\{c_1, a\} \sim \{c_0, b\}$. Since $\{a\} \prec \{b\}$, it follows by Restricted Independence that $\{c_1\} \succ \{c_0\}$. This procedure, initiated by finding c_0 and c_1, can clearly be iterated at infinitum. It therefore generates a sequence c_k, for $k = 0, ...$, of consequences in X'such that $\{c_k, a\} \sim \{c_{k-1}, b\}$ for $k = 1, ...$ By assumption, $\{t\} \succ \{c_k\}$ for every k so that the sequence is bounded by t. Hence the fact that the sequence $\{c_k\}$ $k = 0, 1...$is infinite contradicts the Archimedean axiom. Analogously, starting from the assumption that $m(X) \neq \varnothing$, one can deduce that $u(X')$ is bounded from below. Since the set of real numbers $u(X')$ is either bounded from above and/or from below, it has a least upper bound and/or a greatest lower bound. We therefore extend u to X by defining, for every $t \in M(X)$ (if any), $u(t) = \sup_{x \in X'} u(x)$ and, for every $s \in m(X)$ (if any), $u(s) = \inf_{x \in X'} u(x)$. We now show that u so extended represents $\succsim$ as per (1) on the whole set X (and not only on X'). By definition $u(t) > u(x) > u(s)$ for all $t \in M(X)$, $x \in X'$and $s \in m(X)$, and u represents $\succsim$ as per (1) on X' by Theorem 2. Take any $x \in X'$. By Certainty Equivalence, there are consequences b and $c \in X$ such that $\{b\} \sim \{x, t\}$ and $\{c\} \sim \{x, s\}$. By Averaging and transitivity, we have $\{s\} \prec \{c\}$ $\sim \{x, s\} \prec \{x\} \prec \{x, t\} \sim \{b\} \prec \{t\}$ so that both b and c belong to X'. We therefore only need to show that $\frac{u(x)+u(t)}{2} = u(b)$ and $\frac{u(x)+u(s)}{2} = u(c)$. The argument being symmetric, we only prove that $\frac{u(x)+u(t)}{2} = u(b)$. By contradiction, suppose first that $\frac{u(x)+u(t)}{2} < u(b)$. By Certainty Equivalence, there exists a consequence $b_1 \in X$ such that $\{b_1\} \sim \{x, b\}$. By Averaging $\{x\} \prec \{b_1\} \prec \{b\}$ and, therefore, $b_1 \in X'$. By Theorem 2, the statement $\{b_1\} \sim \{x, b\}$ can be written as $u(b_1) = \frac{u(x)+u(b)}{2}$. Define recursively b_n by $\{b_n\} \sim \{b_{n-1}, b\}$ for $n = 2, ...$Since $\{t\} \succ \{b\} \succ \{b_n\} \succ \{b_{n-1}\}$ by Averaging and transitivity, we have that b and $b_n \in X'$ so that, by Theorem 2, $u(b_n) = \frac{u(b_{n-1})+u(b)}{2} = \frac{\frac{1}{2}[u(b_{n-2})+u(b)]+u(b)}{2} =$ $... = \frac{u(x)}{2^{n-1}} + \frac{2^{n-1}-1}{2^{n-1}} u(b)$. Hence, for n large enough, $u(b_n) \in]\frac{u(x)+u(t)}{2}, u(b)[$. Now, we know that $\{b\} \sim \{x, t\} \succ \{b_n\} \succ \{b_{n-1}, x\}$. By Richness, there exists t' such that $\{x, t'\} \sim \{b_n\}$. Since $\{x, t\} \succ \{b_n\} \sim \{x, t'\}$, it follows from Restricted Independence that $\{t'\} \prec \{t\}$. Hence x, b_n and $t' \in X'$ so that, by Theorem 2, $\frac{u(t')+u(x)}{2} = u(b_n) > \frac{u(t)+u(x)}{2}$. Yet this inequality is incompatible with the definition of $u(t)$ as $u(t) = \sup_{x \in X'} u(x)$.

Proof of Theorem 4

We know from Proposition 2 that a UEU criterion satisfies Averaging and Restricted Independence on any environment. Conversely, let X be a connected separable topological space and let $\succsim$ be an ordering of $P(X)$ satisfying the Continuity axiom as well as Averaging and Restricted Independence. We will prove that, under these conditions, $\succsim$ satisfies Richness and the Archimedean axiom. Using Theorems

1, 2 and 3, the conclusion that $\succsim$ is a UEU criterion will then follow immediately. We first notice that, under Averaging, if the sets $B(A) = \{x \in X : \{x\} \succsim A\}$ and $W(A) = \{x \in X : A \succeq \{x\}\}$ are closed in X for every A, then so are the sets $\widetilde{B}(A) = \{x \in X : A \cup \{x\} \succsim A\}$ and $\widetilde{W}(A) = \{x \in X : A \succeq A \cup \{x\}\}$. To see that, assume by contraposition that, say, $\widetilde{B}(A)$ is not closed (the argument for $\widetilde{W}(A)$ is similar). Then, there exists a sequence $\{x^t\}$, $t = 1, ...$converging to some limit x such that:

$$A \cup \{x^t\} \succsim A$$

for all t and

$$A \succ A \cup \{x\}$$

where the last strict ranking is obtained from the assumption that $\succsim$ is complete. Since $\succsim$ is also reflexive, this strict ranking implies therefore that $x \notin A$. By Averaging one has therefore:

$$A \succ \{x\} \tag{11}$$

Now, since A is finite, and x^t is a sequence converging to x, either x^t is a finite sequence or x^t is infinite. If x^t is finite, then, by the definition of a sequence converging to x, there exists some $s \leq t$ for which $x^s = x \notin X$. But given Averaging, this is incompatible with the definition of the sequence x^t as satisfying $A \cup \{x^t\} \succsim A$ for every t. Hence we must conclude that x^t is infinite. If this is the case, there must exists, since A is finite, an infinite subsequence $\widetilde{x}^t$ of x^t converging to x and such that $\widetilde{x}^t \notin A$ for every t. Since for every t, we have:

$$A \cup \{\widetilde{x}^t\} \succsim A$$

it follows from Averaging that we also have:

$$\{\widetilde{x}^t\} \succsim A$$

Given (11), this gives us the required contradiction of the closedness of the set $B(A)$. We now prove that $\succsim$ satisfies Richness and the Archimedean axiom.

Richness: Consider any set $B \in P(X)$ and, without loss of generality, write it as $B = \{b_1, ..., b_{\#B}\}$ with $\{b_h\} \precsim \{b_{h+1}\}$ for $h = 1, ..., \#B - 1$. By Averaging (and specifically the Gardenförs principle) one has that $B \succsim \{b_1\}$ and $\{b_{\#B}\} \succsim B$ so that none of the (closed under Continuity) sets $\{x \in X : \{x\} \succsim B\}$ and $\{x \in X : B \succeq \{x\}\}$ is empty. Since $\succsim$ is complete, $X = \{x \in X : \{x\} \succsim B\} \cup \{x \in X : B \succeq \{x\}\}$. Since X is connected and can therefore not be written as the union of two disjoint non-empty closed sets, there exists $x \in \{x \in X : \{x\} \succsim B\} \cap \{x \in X : B \succeq \{x\}\}$. By definition such a x verifies $\{x\} \sim B$. Hence $\succsim$ satisfies the Certainty Equivalence condition. Consider now any sets A and B in $P(X)$ and consequences $c^*$and $c_* \in X$ such that $A \cup \{c^*\} \succsim B \succsim A \cup \{c_*\}$ holds. Since, as was just shown, $\succsim$ satisfies Certainty Equivalence, there are consequences b and $b(c) \in X$ (for all $c \in X$) such

that $\{b(c)\} \sim A \cup \{c\}$ and $B \sim \{b\}$. By Continuity, the restriction of the ordering $\succsim$ to singletons is continuous. Hence, by Debreu (1954) Theorem 1, there exists a continuous function $f : X \longrightarrow \mathbb{R}$ such that $f(x) \geq f(y)$ if and only if $\{x\} \succsim \{y\}$ for every $x,\ y \in X$. Define therefore the function $h : X \rightarrow \mathbb{R}$ by $h(c) = f(b(c)) - f(b)$. The function h is continuous if f is. (because even if there are several $b(c)$, they all yield the same number $f(b(c))$). By assumption, we have $h(c^{*}) \geq 0$ and $h(c_{*}) \leq 0$. Since X is connected, we can appeal to the version of the Intermediate Value Theorem provided in Royden (1988) (Proposition 8-11, p. 182) to exhibit a consequence $\overline{c}$ such that $h(\overline{c}) = f(b(\overline{c})) - f(b) = 0$. By definition, $\overline{c}$ is such that $A \cup \{\overline{c}\} \sim \{b\} \sim B$, as required.
Archimedean axiom: If it is impossible to construct one of the standard sequence as in the antecedent clause of the Archimedean axiom, then the proof is (trivially) over. Assume therefore that such a sequence exists (we only provide the argument for the ascending sequence) and, therefore, that a and b are two consequences in X satisfying $\{a\} \succ \{b\}$ for which one has, for a sequence of consequences $\{c_t\}_{t \in \mathbb{N}_+}$:

$$\{c_t, a\} \sim \{c_{t+1}, b\} \tag{12}$$

We notice that, by Restricted Independence, one has $\{c_{t+1}\} \succ \{c_t\}$ for every $t \in \mathbb{N}_{++}$. We will show that if such a sequence is infinite, then it is unbounded from above. For this task, let $C_t = \{x \in X : \{x\} \prec \{c_t\}\}$. By Continuity, C_t is open because it is the complement in X (by completeness) of a closed set. Hence the set $C = \bigcup_{t \in \mathbb{N}_{++}} C_t$ is open and if $z \in X \backslash C$, then $\{z\} \succsim \{c_t\}$ for every t. We will show that this set $X \backslash C$ of upper bounds for the sequence $\{c_t\}_{t \in \mathbb{N}_+}$ is empty. By contradiction, let $z \in X/C$. By Restricted Independence, one has (using (12):

$$\{z, a\} \succ \{z, b\} \succsim \{c_{t+1}, b\} \sim \{c_t, a\}$$

for every t. Since, as was proved above, Richness holds, there exists some consequence $w \in X$ such that $\{w, a\} \sim \{z, b\}$. By Restricted Independence, $\{z\} \succ \{w\}$ and $w \in X/C$ because $\{w\} \succsim \{c_t\}$ for all t. Now the set $\{v : \{v\} \succ \{w\}\}$ is open by Continuity, contains z and is is a subset of X/C by transitivity. This means that that X/C is open because it is a union of open sets. But this contradicts the assumption that X is connected and, as a result, can not be written as a union of two disjoint open sets.

References

ADAMS, E. W. (1965): "Elements of a Theory of Inexact Measurement," *Philosophy of Science*, 32, 205–228.

ADAMS, E. W., AND R. F. FAGOT (1959): "A Model of Riskless Choice," *Behavioral Science*, 4, 1–10.

AHN, D. S. (2008): "Ambiguity without a State Space," *Review of Economic Studies*, 75, 3–28.

ARROW, K. J., AND L. HURWICZ (1972): "An Optimality Criterion for decision-making under ignorance," in *Uncertainty and Expectations in Economics: Essays in honour of G. L. S. Shackle*, ed. by C. F. Carter, and J. L. Ford, pp. 1–11. Basil Blackwell, Oxford.

BAIGENT, N., AND Y. XU (2004): "Decisions under complete ignorance," Mimeo,Institute of public economics, Graz university.

BARBERÀ, S., W. BOSSERT, AND P. K. PATTANAIK (2004): "Ranking Sets of Objects," in *Handbook of Utility Theory, vol. 2: Extensions*, ed. by S. Barberà, P. Hammond, and C. Seidl, pp. 893–977. Kluwer, Dordrecht.

BARBERÀ, S., B. DUTTA, AND A. SEN (2001): "Strategy-Proof Social Choice Correspondences," *Journal of Economic Theory*, 101, 374–394.

BENOÎT, J. P. (2002): "Strategic Manipulation in Voting Games When Lotteries and Ties are Permitted," *Journal of Economic Theory*, 102, 421–436.

BOLKER, E. D. (1966): "Functions Resembling Quotient of Measures," *Transaction of the American Mathematical Society*, 124, 292–312.

——— (1967): "A Simultaneous Axiomatization of Utility and Subjective Probabilities," *Philosophy of Sciences*, 34, 333–340.

BROOME, J. (1990): "Bolker-Jeffrey Expected Utility Theory and Axiomatic Utilitarianism," *Review of Economic Studies*, 57, 477–502.

CASADESUS-MASANELL, R., P. KLIBANOFF, AND E. OZDENOREN (2000): "Maxmin expected utility over Savage acts with a set of priors," *Journal of Economic Theory*, 92, 35–65.

DEBREU, G. (1954): "Representation of a Preference Ordering by a Numerical Function," in *Decision Processes*, ed. by R. L. D. R. M. Thrall, C. H. Coombs, pp. 159–165. Wiley, New York.

——— (1960): "Topological methods in cardinal utility theory," in *Mathematical methods in the social sciences, 1959; proceedings*, ed. by S. K. K. J. Arrow, and P. Suppes, pp. 16–26. Stanford University Press, Stanford.

FISHBURN, P. (1970): *Utility Theory for Decision Making.* John Wiley, New York.

FISHBURN, P. C. (1972): "Even-Chance Lotteries in Social Choice Theory," *Theory and Decision*, 3, 18–40.

GÄRDENFORS, P. A. (1976): "On Definitions of Manipulation of Social Choice Function," in *Aggregation and Revelation of Preferences*, ed. by J. Laffont. North Holland, Amsterdam.

GILBOA, I., AND D. SCHMEIDLER (1989): "Maxmin Expected Utility with non-unique prior," *Journal of Mathematical Economics*, 18, 99–110.

GUL, F., AND W. PESENDORFER (2001): "Temptation and Self-Control," *Econometrica*, 69, 1403–1435.

JAYNES, E. T. (2003): *Probability Theory: the Logic of Science*. Cambridge University Press, Cambridge UK.

JEFFREY, R. (1965): *The Logic of Decision*. University of Chicago Press, Chicago, reprint 1983.

KEYNES, J. M. (1921): *A treatise on probabilities*. Macmillan, New York.

KRANTZ, D., R. D. LUCE, P. SUPPES, AND A. TVERSKY (1971): *Foundations of Measurement, vol. 1*. Academic Press, New York London.

MASKIN, E. (1979): "Decision Making under Ignorance with Implications for Social Choice," *Theory and Decision*, 11, 319–337.

NEHRING, K., AND C. PUPPE (1996): "Continous extension of an order on a set to the power set," *Journal of Economic Theory*, 68, 456–479.

NEUMANN, J. V., AND O. MORGENSTERN (1947): *Theory of Game and Economic Behavior*. Princeton University Press, Princeton, NJ, USA.

NITZAN, S., AND P. K. PATTANAIK (1984): "Median-Based extensions of an Ordering over a Set to the Power Set: An Axiomatic Characterization," *Journal of Economic Theory*, 34, 252–261.

OLSZEWSKI, W. (2007): "Preferences over Sets of Lotteries," *Review of Economic Studies*, 74, 567–595.

OZYÜRT, S., AND R. SANVER (2006): "A General Impossibility Result on Strategy-Proof Social Choice Hyperfunctions," Mimeo,Department of Economics,New York University.

PATTANAIK, P. K., AND B. PELEG (1984): "An Axiomatic Characterization of the Lexicographic Maximin Extension of an Ordering over a Set to the Power Set," *Social Choice and Welfare*, 1, 113–122.

PELEG, B., AND H. PETERS (2009): "Nash Consistent Effectivity Functions through Lottery Models," *Games and Economic Behavior*, 65, 503–515.

PUPPE, C. (1995): "Freedom of Choice and Rational Decisions," *Social Choice and Welfare*, 12, 137–154.

Royden, H. L. (1988): *Real Analysis, 3rd Edition.* Macmillan, New York.

Savage, J. L. (1954): *The Foundation of Statistics.* Wiley, New York.

Schmeidler, D. (1989): "Subjective probability and expected utility without additivity," *Econometrica*, 57, 571–587.

Scott, D. (1964): "Measurement Structure and Linear Inequalities," *Journal of Mathematical Psychology*, 1, 233–247.

Wakker, P. (1988): "The algebraic versus the topological approach to additive representations," *Journal of Mathematical Psychology*, 32, 421–435.