Problems set 1 (production)

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Problem 1: Let f be a production function for a technology with two inputs having constant returns to scale. Show that if the average product of one of the factor is increasing with respect to the level of use of this factor, then the marginal product of the other factor must be negative.

Answer:

Since the technology has constant returns to scale, the function $f(.)$ is homogenous of degree 1. We therefore have:

$$\frac{f(x_i, x_j)}{x_i} = f(1, \frac{x_j}{x_i}) = g(\frac{x_j}{x_i}) = g(z)$$

for $i = 1, 2$. The condition that the average product of factor i be increasing with respect to the level of use of this factor writes:

$$\frac{\partial(\frac{f(x_1,x_2)}{x_i})}{\partial x_i} = -\frac{\partial g()}{\partial z}\frac{x_j}{x_i^2} > 0$$

which implies of course that:

$$\frac{\partial g()}{\partial z} = \frac{\partial f()}{\partial x_j} < 0$$

Problem 2. Which conditions (if any) must be imposed on the numbers a et b in order for the technology represented by the production set $Y = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \leq \ln(a - y_2) - b\}$ to satisfy the conditions of irreversibility, impossibility of free production, possibility of no production, free disposal and convexity?

Answer:

-(i) Production set satisfies irreversibility if $y_1 \leq \ln(a - y_2) - b$ and $-y_1 \leq \ln(a + y_2) - b$ imply $y_1 = y_2 = 0$, or analogously, if $y_2 \leq a - e^{y_1+b}$ and $-y_2 \leq a - e^{-y_1+b}$ imply $y_1 = y_2 = 0$. Clearly, if $a > e^b > 0$, then, considering $y_1 = 0$, it is clear that one can find $y_2 > 0$ such that $y_2 \leq a - e^b$ and $-y_2 < 0 \leq a - e^b$. Hence if the technology is to satisfy irreversibility, we must have $a \leq e^b$. One can show conversely that $a \leq e^b$ is sufficient for irreversibility.

-(ii) This production set will satisfy impossibility of free production if ($y_1 \geq 0$ et $y_2 \geq 0$) $\Rightarrow$ ($y_1 = y_2 = 0$). Suppose $y_1 = 0$. In this case, we have $\ln(a-y_2)-b \geq$

$0 \Leftrightarrow a \geq e^b + y_2$ so that $y_2 > 0$ is compatible with this inequality whenever $a > e^b$. Hence to rule out $y_1 = 0$ and $y_2 > 0$, we need to rule out $a > e^b$ and to assume $e^b \geq a$. Let us make this assumption and let us suppose now that $y_2 = 0$. We then have $y_1 \leq \ln a - b \leq 0$ and, therefore, that $y_1 > 0$ is ruled out. Hence $e^b \geq a$ is a necessary and sufficient condition for the technology to satisfy impossibility of free production.

-(*iii*) Possibility of no production requires $0 \leq \ln a - b$, which is nothing else than requiring $e^b \leq a$. Hence, the only way with which the technology can satisfy both irreversibility and possibility of no production is to have $e^b = a$.

-(*iv*) Free disposal will be satisfied if $y_1 \leq \ln(a - y_2) - b$ and $\widehat{y}_j \leq y_j$ for $j = 1, 2$ implies $\widehat{y}_1 \leq \ln(a - \widehat{y}_2) - b$. But clearly, if y_1 and y_2 are such that $y_1 \leq \ln(a - y_2) - b$, then if $\widehat{y}_2 \leq y_2$, one has $y_1 \leq \ln(a - \widehat{y}_2) - b$ and, similarly, $\widehat{y}_1 \leq y_1$ implies $\widehat{y}_1 \leq \ln(a - y_2) - b$. Hence, no restriction on the parameters a and b are needed for this technology to satisfy free disposal.

(*v*) Convexity will be satisfied if $y_1 \leq \ln(a - y_2) - b$ and $z_1 \leq \ln(a - z_2) - b$ imply that $\lambda y_1 + (1 - \lambda)z_1 \leq \ln(a - \lambda y_2 - (1 - \lambda)z_2) - b$ for all real number λ between 0 and 1. Assume that $y_1 \leq \ln(a-y_2)-b$ and $z_1 \leq \ln(a-z_2)-b$. Clearly, this implies that $\lambda y_1 \leq \lambda \ln(a-y_2)-\lambda b$ and $(1-\lambda)z_1 \leq (1-\lambda)\ln(a-z_2)-(1-\lambda)b$ for every $\lambda \in]0, 1[$ and, therefore, that:

$$\lambda y_1 + (1 - \lambda)z_1 \leq \lambda \ln(a - y_2) + (1 - \lambda)\ln(a - z_2) - b \tag{1}$$

Now the function $\Phi : S(a) \to \mathbb{R}$ defined by

$$\Phi(x) = \ln(a - x)$$

where $S(a) \subset \mathbb{R}$ is the set, which depends upon a, of all real numbers x for which $\ln(a - x)$ exists is a function whose second order derivative Φ" is given by:

$$\Phi''(x) = \frac{-1}{(a - x)^2} \leq 0$$

Hence, by standard calculus arguments, this function is a concave function and is such that:

$$\Phi(\lambda x + (1 - \lambda)y) \geq \lambda\Phi(x) + (1 - \lambda)\Phi(y)$$

Because of this, (1) above implies that

$$\lambda y_1 + (1 - \lambda)z_1 \leq \ln(a - \lambda y_2 - (1 - \lambda)z_2) - b$$

and, therefore, that the technology is convex.

Problem 3: True or false ? If a technology is represented by a production function $f : \mathbb{R}^l_+ \to \mathbb{R}$ that is quasi-concave et that satisfies $f(0^l) = 0$, then this technology can not have increasing returns to scale. (We recall that a production fonction is quasi-concave if, for any level of production y, the set $V(y) = \{x \in \mathbb{R}^l_+ : f(x) \geq y\}$ is convex).

Answer: False. Consider the following technology (inputs are represented by positive numbers): h

$$Y = \{(y, x_1, ..., x_l) \in \mathbb{R}^{l+1}_+ \mid y \leq x_1^2 x_2^2 ... x_n^2\}$$

A typical $V(y)$ set associated to this technology is defined by:

$$V(y) = \{(x_1, ..., x_l) \in \mathbb{R}^l_+ \mid y \leq x_1^2 x_2^2 ... x_n^2\}$$

One can verify easily that this set is convex (draw a picture in $\mathbb{R}^2_+$). Yet this technology admits increasing returns to scale because if, say, starting from the factors combination $(x_1, ..., x_l) \in \mathbb{R}^l_+$ enabling the firm to produce at most y unités d'output we double the level of use of all factors, we will be capable of producing at most $(2x_1)^2(2x_2)^2...(2x_n)^2 = 4y$ units of output.

Problem 4 For every of the following production functions, (where $a, b \in \mathbb{R}_{++}$), indicates if the technology that it represents satisfies the 6 assumptions that we have seen in class (except irreversibility).

(i) $f(x_1, x_2) = e^{\min(ax_1, bx_2)}$

Answer: The technology represented by this production function is not convex (for instance, the production activities $(1, 0, 0)$ et $(e, \frac{1}{a}, \frac{1}{b})$ are technologically feasible because they satisfy, respectively, $1 \leq e^{\min(0,0)}$ and $e \leq e^{\min(1,1)}$ but the convex combination $(\frac{e+1}{2}, \frac{1}{2a}, \frac{1}{2b})$ of these two production activies is not (because $\frac{e+1}{2} > e^{\frac{1}{2}} = e^{\min(\frac{a}{2a}, \frac{b}{2b})}$. This technology violates also the impossibility of free production (because producing $1 = e^{\min(0,0)}$ unit of output is possible without hiring any input. The technology satisfies all other conditions.

(ii) $f(x_1, x_2) = ax_1 + x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} + bx_2$

Answer: The technology described by this production function satisfies all conditions seen in class (one verifies that the function is concave by noticing that $f_{ii} = \frac{-x_j^{\frac{1}{2}}}{4x_i^{\frac{3}{2}}} \leq 0$ for $i = 1, 2$ and $j \neq i$ and $f_{11} f_{22} - f_{12}^2 = \frac{1}{16x_i x_2} - \frac{1}{16x_i x_2} \geq 0$; A technology represented by a concave production function is convex.)

(iii) $f(x_1, x_2) = ax_1 - x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} + bx_2$

Answer: *(iii)* The technology described by this production function violates the free disposal assumption because the production function is not increasing in all its argument (using calculus arguments, which are appropriate here because the function is differentiable, its partial derivatives are not always positive). For example:

$$\frac{\partial f(x_1, x_2)}{\partial x_1} = a - \frac{1}{2}(\frac{x_2}{x_1})^{\frac{1}{2}} < 0 \text{ if } \frac{x_2}{x_1} > 4a^2$$

This technology is also not convex. Indeed, if one considers the factors combinations $x = (\frac{b}{a}, 0)$ and $z = (0, 1)$, one can see that they both enable the firm to produce at most b units of output since $f(\frac{b}{a}, 0) = f(0, 1) = b$. Consider now the (convex) combination $(\frac{b}{2a}, \frac{1}{2}) = \frac{1}{2}x + \frac{1}{2}z$. This combination enables the firm to produce:

$$f(\frac{b}{2a}, \frac{1}{2}) = b - \frac{1}{2}(\frac{b}{a})^{\frac{1}{2}} = b^{\frac{1}{2}}(b^{\frac{1}{2}} - \frac{1}{2a^{\frac{1}{2}}}) < b$$

in violation of convexity.

(iv) $f(x_1, x_2) = \min(1 - a/x_1, 1 - b/x_2)$

Answer: The technology represented by the production function $f(x_1, x_2) = \min(1 - a/x_1, 1 - b/x_2)$ violates the possibility of no-production assumption ($a > 0$ unit of input 1 and $b > 0$ units of input 2 are required for producing 0 output). Moreover, it is impossible to operate the technology without using some quantity (however small) of the two inputs. All other assumptions are verified (in particular convexity). Here is the proof of this state of affairs. Consider any pair (y, x_1, x_2) and (y', x'_1, x'_2) of technically feasible productive activities. Let us show that $(ty + (1-t)y', tx_1 + (1-t)x'_1, tx_2 + (1-t)x'_2)$ is also technically feasible for every real number $t \in [0, 1]$. The productive activity (y, x_1, x_2) is technically feasible if and only if:

$$y \leq \min(1 - a/x_1, 1 - b/x_2)$$

Similarly (y', x'_1, x'_2) is technically feasible if and only if:

$$y' \leq \min(1 - a/x'_1, 1 - b/x'_2)$$

Of course, if $t \in]0, 1]$

$$ty \leq t \min(1 - a/x_1, 1 - b/x_2)$$

and

$$(1-t)y' \leq (1-t) \min(1 - a/x'_1, 1 - b/x'_2)$$

Adding these two inequalities yields:

$$ty + (1-t)y' \leq t \min(1 - a/x_1, 1 - b/x_2) + (1-t) \min(1 - a/x'_1, 1 - b/x'_2) \tag{A}$$

One can see that, for every positive real numbers α and β, $1 - a/\alpha \leq 1 - b/\beta \Leftrightarrow \beta \geq \frac{b}{a}\alpha$. Defining $\Phi_1(x) = 1 - a/x_1$ and $\Phi_2(x) = 1 - b/x$, one has that $\min(\Phi_1(x_1), \Phi_2(x_i)) = \Phi_i(x_i) \Leftrightarrow \min(\Phi_1(tx_1), \Phi_2(tx_i)) = \Phi_i(tx_i)$ for all number t (for $i = 1, 2$). Moreover, one verifies that the functions Φ_i ($i = 1, 2$) are (weakly) concave since their second order derivatives are zero. We therefore have:

$$t \min(1 - a/x_1, 1 - b/x_2) + (1-t) \min(1 - a/x'_1, 1 - b/x'_2) \leq \min(\Phi_1(tx_1 + (1-t)x'_1), \Phi_2(tx_2 + (1-t)x'_2) \tag{B}$$

Combining inequalities (A) and (B) lead us to the conclusion that the productive activity $(ty + (1-t)y', tx_1 + (1-t)x'_1, tx_2 + (1-t)x'_2)$ is technically feasible, as required.

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Problem 5 Show that if a technology is represented by a production function $f : R^l_+ \to R$ is additive in the sense that $f(x + z) = f(x) + f(z)$ for every x, $z \in R^l_+$, the technology will be convex if inputs are perfectly divisible.

Answer: By contradiction, assume that $(\mathbf{x}, y)$, $(\mathbf{x}', y')$ and t (where $\mathbf{x}$, $\mathbf{x}' \in \mathbb{R}^l_+$, y, $y' \in \mathbb{R}_+$ and $t \in [0, 1]$) are such that $y \leq f(\mathbf{x})$ and $y' \leq f(\mathbf{x}')$ but, contrary to what is required by convexity, are also such that:

$$ty + (1-t)y' > f(t\mathbf{x} + (1-t)\mathbf{x}') \tag{2}$$

By additivity, inequality (2) implies that:

$$ty + (1-t)y' > f(t\mathbf{x}) + f((1-t)\mathbf{x}')$$

which, when combined with $y \leq f(\mathbf{x})$ et $y' \leq f(\mathbf{x}')$, implies either one of the following two violations of the divisibility assumption:

$$ty > f(t\mathbf{x})$$

or

$$(1-t)y' > f((1-t)\mathbf{x}')$$

QED.

Problem 6: For every production, find the profit function that corresponds to it.

(a) $f(x) = \ln x$ si $x \geq 1$
$= 0$ otherwise.

Answer:

$$\pi(w,p) = \max_{x} \; p \ln x - wx$$

The firm can secure itself with a zero profit by producing nothing and hiring nothing. If it hires a positive quantity x^* of the production factor, this quantity will satisfy the first order condition:

$$\frac{p}{x^*} = w \Leftrightarrow x^* = \frac{p}{w}$$

We therefore have $\pi(w) = p(\ln p - \ln w - 1)$ if $p(\ln p - \ln w - 1) \geq 0 \Leftrightarrow w \leq \frac{p}{e}$ and $\pi(w) = 0$ otherwise.

(b) $f(x_1, x_2) = 100 x_1^{\frac{1}{2}} x_2^{\frac{1}{4}}$

Answer:

$$\begin{aligned} \pi(w_1, w_2, p) &= \max_{x_1, x_2} = p100 x_1^{\frac{1}{2}} x_2^{\frac{1}{4}} - w_1 x_1 - w_2 x_2 \\ x_i &\geq 0 \; i = 1, 2 \end{aligned}$$

The first order conditions that must necessarily satisfy the interior solutions x_1^*, x_2^* to this program are:

$$50p \frac{x_2^{*\frac{1}{4}}}{x_1^{*\frac{1}{2}}} - w_1 = 0 \Leftrightarrow x_1^* = 2500p^2 \frac{x_2^{*\frac{1}{2}}}{w_1^2} \tag{foc1}$$

and

$$25p \frac{x_1^{*\frac{1}{2}}}{x_2^{*\frac{3}{4}}} - w_2 = 0 \tag{foc2}$$

Substituting (foc1) in (foc2), one obtains:

$$25p\frac{50p{x_2^*}^{\frac{1}{4}}}{w_1{x_2^*}^{\frac{3}{4}}} = w_2 \Leftrightarrow x_2^* = \frac{p^4 1250^2}{w_1^2 w_2^2}$$

and, substituting this expression back into (foc1), we obtain:

$$x_1^* = \frac{2p^4 1250^2}{w_1^3 w_2}$$

We therefore have:

$$\begin{aligned} \pi(w_1, w_2, p) &= p100(p^4\frac{2500 \times 1250}{w_1^3 w_2})^{\frac{1}{2}}(\frac{p^4 1250^2}{w_1^2 w_2^2})^{\frac{1}{4}} - w_1 p^4\frac{2500 \times 1250}{w_1^3 w_2} - w_2\frac{p^4 1250^2}{w_1^2 w_2^2} \\ &= 76p^4\frac{250^2}{w_1^2 w_2} \end{aligned}$$

(c) $f(x_1, x_2) = (ax_1 + bx_2)^{\frac{1}{2}}$ (for a, $b \in \mathbb{R}_{++}$)

Answer: This technology is based on a perfect substitutability between the two production factors (at a constant marginal rate of substitution of $\frac{a}{b}$). The hiring policy of this firm will therefore be very simple. The firm will hire only factor 2 if $\frac{w_1}{w_2} > \frac{a}{b}$ (case 1), will hire only factor 1 if $\frac{w_1}{w_2} < \frac{a}{b}$ (case 2) and will be indifferent between any combination of the two factors if $\frac{w_1}{w_2} = \frac{a}{b}$. In the case i ($i = 1, 2$), the firm's hiring decision solves:

$$\max_{x_i} p\alpha_i x_i^{\frac{1}{2}} - w_i x_i$$

where $\alpha_1 = a^{\frac{1}{2}}$ and $\alpha_2 = b^{\frac{1}{2}}$. The first order condition satisfied necessarily by an interior solution to this program enables us to obtain:

$$x_i^* = \frac{p^2\alpha_i^2}{4w_i^2}$$

The profit function of the firm therefore writes:

$$\pi(w_1, w_2, p) = \frac{p^2}{4}\max(\frac{a}{w_1}, \frac{b}{w_2})$$

(d) $f(x_1, x_2) = (\min(x_1, x_2))^a$. Which property must satisfy the number a in order for the profit function to be well-defined ?

Answer: This technology exhibits a perfect complementarity between the two production factors so that any optimal (from the view point of the firm) employment level of these two production factors must satisfy $x_1^* = x_2^* = x^*$

(why using one factor in a greater quantity than the other if it is costly to do so and it does not increase output ?).. Le problem solved by the firm is therefore:

$$\max_x px^a - (w_1 + w_2)x$$

This problem is well-defined only if $0 \leq a \leq 1$ (non-increasing returns to scale). Of course, if $a = 1$, returns to scale are constant so that maximal profits must be zero for all levels of input. This later situation will arise with positive production only if $p = w_1 + w_2$. In the case where $a < 1$, one can find the optimal level of employment of either one input by the first order condition:

$$x^* = (\frac{ap}{w_1 + w_2})^{\frac{1}{1-a}}$$

from which we obtain:

$$\begin{aligned} \pi(w_1, w_2, p) &= p(\frac{ap}{w_1 + w_2})^{\frac{a}{1-a}} - (w_1 + w_2)(\frac{ap}{w_1 + w_2})^{\frac{1}{1-a}} \\ &= (\frac{(ap)}{(w_1 + w_2)^a})^{\frac{1}{1-a}}(\frac{1}{a} - 1) \end{aligned}$$

Problem 7: What does the producer's surplus, defined as the integral under the supply curve of the firm calculated between two arbitrary prices of output, measure ?

Answer: From Hotelling lemma, we know that:

$$y(\overline{\mathbf{w}}, \overline{p}) \equiv \frac{\partial \pi(\overline{w}, \overline{p})}{\partial p}$$

where π is the profit function and $y(.)$ is the firm's supply function, that associates to every prices configuration (for inputs and output) the level of output that a competitive profit maximizing firm would like to choose at these prices. The producer surplus $s(p_0, p_1)$ between two output price levels p_0 and p_1 is given by:

$$s(p_0, p_1) = \int_{p_0}^{p_1} y(\overline{\mathbf{w}}, \overline{p})dp = \pi(\overline{w}, p_1) - \pi(\overline{w}, p_0)$$

This producer's surplus therefore measures exactly the change in the firm's profits brought about by the price change.

Problem 8: A stastician has collected the following informations on the behavior of a firm producing one output with two inputs.

	1st trimester	2nd trimester	3nd trimestre
output price	2	3	2,5
price of input 1	1	0,5	2
price of input 2	1	3	2
quantity of output	100	100	90
quantity of input 1	75	75	50
quantity of input 2	75	55	50

Can this behavior be coming from a firm that maximizes its profit in a perfectly competitive environment ? (Justify).

Answer: In order for this behavior to result from a profit maximizing firm evolving in a perfectly competitive environment, it must be the case that, at every prices configuration, the productive activity choosen by the firm at that price configuration yields (weakly) higher profits than any other technically feasible productive activity. While we don't know all the technical possibilities of the firm, we know that any productive activity that has been choosen by the firm is technically feasible. As it happens, this firm can not be maximizing profit. In effect, the profit made by this firm at the first semester is: $2\times100-1\times75-1\times75 = 50$

Yet, by choosing in the first semester the productive activity choosen in the second, the firm would have made in the first semester a profit of: $2 \times 100 - 1 \times 75 - 1 \times 55 = 70$. Hence the choice made by this firm in the first semester is not profit maximizing.

Problem 9 An econometrician has estimated the following output supply and factor demands (respectively) of a firm evolving in a perfectly competitive environment and producing one output (y) using two inputs (x_1 and x_2)..

$$\begin{aligned} y^*(p, w_1, w_2) &= p[\ln(p - w_1) - \ln w_1 + \frac{p}{2w_2}] \\ x_1^*(p, w_1, w_2) &= \frac{p - w_1}{w_1} \\ x_2^*(p, w_1, w_2) &= \frac{p^2}{4w_2^2} \end{aligned}$$

where p is the output price and w_1 and w_2 are, respectively, the prices of inputs 1 and 2. Can these demand and supply functions be resulting from a profit maximizing firm ? Justify.

Answer: No! In order to result from profit maximization, demand and supply functions must be homogenous in degree 0 with respect to all prices. Yet the supply function y^* of this example is not homogenous of degree 0 with respect to all prices. Suppose for instance that $p = 2 > w_1 = w_2 = 1$. The firm supply is then $y^*(2, 1, 1) = 2[\ln(2 - 1) - \ln 1 + \frac{2}{2}] = 2$. Suppose now that prices are all doubled. The firm supply at prices $(p, w_1, w_2) = (4, 2, 2)$ is now $y^*(4, 2, 2) = 4[\ln(4 - 2) - \ln 2 + \frac{4}{4}] = 4$. Hence the firm's choice of output is sensitive to the numéraire used to measure prices. It can therefore not result from profit maximization.