What is diversity ?*

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Abstract

This chapter surveys some of the formal approaches that have been proposed for comparing sets of objects on the basis of their diversity. The first part of the chapter discusses a few general principles for making these comparisons. The second part reviews the methods used by biologists to evaluate biodiversity. The third part examines the approaches that view the diversity of a set as resulting from the agregation of the pairwise dissimilarity of its elements. Within these approaches, a distinction is made between those that use a cardinal information on dissimilarity and those based only on a qualitative, or ordinal, notion of dissimilarity. Finally, the chapter also discusses the approach, essentially developped by Nehring and Puppe, that bases diversity appraisal on the valuation of different attributes possessed by the objects.

1 Introduction

Diversity is an issue that is attracting increasing attention in various spheres. Governments of more than 150 countries have ratified the Rio convention of 1992 which requires them to adopt economically costly policies aiming at the "conservation of *biological diversity*" (article 1 of the Rio convention). More recently, UNESCO has approved, in October 2005, the convention on "the promotion and protection of *diversity of cultural expressions*". This convention has been invoked by the representatives of many countries in the negotiations at the World Trade Organization (WTO) in order to remove certain "cultural goods" from the scope of free trade agreements. In

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economics, there is a long standing tradition of research in industrial organization that is concerned with *product diversity* and how it can be promoted by various forms of market competitions (see Dixit and Stiglitz (1977) for a classical analysis). Diversity appears also to be an important aspect of the freedom of choice that individuals may have in different situation (see e.g. Barberà, Bossert, and Pattanaik (2004) and Sugden (1998) for surveys of the literature on the measurement of freedom of choice). More colloquially, one finds a significant concern in popular discussions about the *diversity of opinions* expressed in the medias or in the political arena.

But what is diversity? Would the killing of 500 000 flies of a specific species have the same impact on the reduction of biological diversity than the elimination of the last 6000 remaining tigers on earth ? Is the diversity of opinions expressed in the written press larger in France than in the US ? Is the choice of models of cars offered more diverse at General Motor than at Volkswagen ? In order to put the Rio or the UNESCO convention into force, or to study the impact of concentration of the media industry on the diversity of opinions expressed in the media, it is of some importance to have available clear, but yet accurate, answers to questions like this. I propose in this chapter to critically examine some of the approaches that have been proposed in various disciplines - biology and economics for the most part - for answering such questions.

It could of course be argued, without play-words, that the diversity of contexts to which these questions refer is such that they can not be handled by a single concept. The biodiversity of an ecosystem is not the same thing as the diversity of opinions founded in the medias and the two types of situation require completely different notions.

The discussion of this chapter will certainly give some credential to this view by illustrating, on several occasions, how the relative merit of the various proposed approaches to diversity appraisal are dependant upon the context. Yet the view point that is taken herein is that, different as they are, all problems of diversity appraisal share a common formal structure that will be the main object of analysis. At the most abstract level, the formal structure is very simple. A universe of objects (e.g. living organisms, opinions, car models, cultures, etc.) is given to the diversity appraiser, and various subsets of these objects (ecosystems, newspapers, car retailers, countries) are to be compared on the basis of their diversity. The general question examined is: how can one make these comparisons ?

It should be noticed that this very way of addressing the problem limits somehow the kind of issues handled by the approaches covered herein. An important limitation has to do with *uncertainty*. The consequences on biodiversity (however defined) of many human decisions (deforestations, carbon emissions, etc.) are uncertain. So are the consequences, on cultural diversity (however defined), of allowing WTO to treat "cultural goods" as standard freely tradable goods. Hence a question that *could be* asked, but that *will* *not be* asked herein, is: how should one rank decisions with uncertain consequences in terms of diversity ?

Answering such a question is obviously a more ambitious task than answering the previous question. If we know how to rank all decisions that have uncertain consequences in terms of diversity, then we know how to rank the subclass of these decisions that have certain consequences but the converse is obviously false. Some important contributions to the literature on diversity appraisal, notably those of Weitzman (1992),Weitzman (1993), Weitzman (1998) and Nehring and Puppe (2002), have proposed answers to this broader question that contain therefore also answers to the first question. Yet, in discussing these answers, attention will be limited to aspects that concern the narrow problem at hand. To a large extent, this choice of narrowing the discussion to the issue of "what is diversity", rather than to that of "how to rank decisions that have (possibly uncertain) consequences on diversity" is motivated by a prosaic consideration of time and space. Yet this choice also reflects a sentiment that diversity is a delicate notion that remains poorly understood. For this reason, progress in understanding this notion are more likely to come from a narrow focusing on its very meaning in some ideally simple situations than from a mixing of it with other complex notions such as uncertainty.

In making diversity comparisons of different sets of objects, it is important to be explicit about the exact *information* that is assumed to be available on the objects and to be useful for diversity appraisal. All approaches for evaluating diversity differ in the kind of information that they assume and on the role played by the information. For example, in the context of biological diversity, the elements of the ecosystems are very often assumed to be *living organisms* grouped into *species*. This grouping obviously plays a crucial role in the appraisal of diversity commonly done by biologists. In other theories of diversity appraisal, especially those developed in economics following Weitzman (1992), a primitive notion of *dissimilarity* between objects is assumed to play a role. The precision of the information conveyed by the underlying notion of dissimilarity differs across theories. In papers such as Bossert, Pattanaik, and Xu (2003), VanHees (2004), Weitzman (1992), Weitzman (1993) and Weitzman (1998) this information takes the form of a *cardinally meaningful* numerical distance between the objects. In other approaches, such as that considered in Bervoets and Gravel (2007), Nehring (1997) and Pattanaik and Xu (2000), the information on dissimilarity is assumed instead to be only *qualitative* or *ordinal* in nature. Finally, in the approach developed by Nehring and Puppe (2002) (see also Nehring and Puppe (2003)), it is an information on the valuation of various *attributes* of the objects that plays a key role in appraising diversity.

It could be argued that the relevant information for evaluating diversity, and more generally the evaluation of diversity itself, should depend upon the final objective reached by this evaluation. Why is one interested in evaluating diversity ? Why is diversity important ? The answer to these question is likely to depend upon the context in which the evaluation is performed. The reasons for being interested in biodiversity are certainly different from those that justify an interest in the promotion of cultural diversity, or a concern for product diversity. Should the reasons that motivate an interest for a measurable phenomenon affect the way by which the measurement of this phenomenon is to be performed?

Opinions on this matter differ. In the context of freedom appraisal, some authors, such as Carter (1999) or VanHees (1997), have argued that the *reasons* for being interested in a phenomenon such as freedom of choice - what they call the "*value of freedom*"- should not affect the definition, and therefore the measurement, of the phenomenon. A phenomenon can be defined and appraised irrespective of the role played by the phenomenon in the theory used by the appraiser. On the other hand it is quite possible to hold the converse view that the definition and appraisal of a phenomenon depends *totally* upon the consequences that this phenomenon is expected to have or upon the reasons for being interested in it. After all, the precise measurement of temperature in physics is entirely motivated by the role played by temperature (as measuring the speed of movement in particles) in the physical theory and does not attempt to measure an *a priori* notion of "temperature" that is independent from this role.

While my sympathy tend to go somewhat toward this later view, this survey will not discuss the reasons for developing an interest in diversity appraisal. Here again, my reluctance for examining this complex issue is the result of a mixture of prosaic considerations of time and space as well as a more substantial belief that the reasons for developing a concern for diversity are not clearly understood. Human sciences and even, for this matter, biology do not seem to have developed a theoretical apparatus in which the notion of diversity could play a role as precise as the one played by temperature in physics.

As mentioned above, the proposed survey will focus on the *formal structure* of the various existing proposals for defining diversity. Yet the presentation of this formal structure is intended to be *non-technical*. This means that the formalism will be kept to a minimum and that the properties and results will be stated as heuristically as possible. In particular, I shall not provided detailed proofs of the results even though I will, on occasion, provide indicative comments as to how such a detailed proof can be constructed from various results in the literature.

Specifically, the presentation is organized as follows. In the next section, the formal framework is introduced, and some principles for diversity appraisal, as well as their implications, are examined in the case where no information on the objects - other than their existence- is provided. Section 3 examines the approaches developed in biology where the objects are partitioned into categories (species) and where the diversity of a set is assumed to be measured by some generalized entropy. The approaches that base diversity appraisal on the aggregation of an *a priori* notion of dissimilarity between objects are considered in section 4 while section 5 discusses the multi-attributes approach. Finally, section 6 provides some elements of conclusion.

2 Comparing sets of objects: General principles

2.1 The framework

At the center of any exercise of diversity evaluation is a (typically finite) set X of objects. While the objects in X can be *a priori* anything (living organisms, products, opinions, cultural expressions, etc.), they need to be clearly identified at the outset. This identification is not always made in the literature, and is not always easy to make in practice. For instance, in the case of biodiversity appraisal, there is sometime some ambiguity as to whether the living individual or the species should be treated as the unit of analysis. This choice has obviously some impact for the formal analysis. Suppose we have an ecosystem in which there are 10 000 trees of a species A and 5000 tree of a species B. If the tree is taken as the unit of analysis, this ecosystem has 15 000 elements while if we take species to be the unit of analysis, it has only 2 elements. Furthermore, most indicators used by biologists to evaluate diversity use information on the *relative frequency* of living individuals belonging to each species. Here again, the way by which this information can be handled in the formal analysis will depend upon the unit of analysis chosen. If species is the unit of analysis, then some extra (non set-theoretic) information need to be included (e.g. the set has 2 species, species 1 comes with frequency 2/3 and species 2 with frequency 1/3). If living individual is the unit of analysis, then information on the relative frequency is provided, more parsimoniously, by a partition of the set of individuals into two (species) groups.

Given this clear identification of the objects, the problem of diversity evaluation is that of providing a ranking, denoted $\succeq$, with asymmetric and symmetric factors $\succ$ and $\sim$, of the various *sets of objects* in X. In this setting, $A \succeq B$ means "set A offers at least as much diversity as set B", $A \succ B$ means "set A offers strictly more diversity than set B" and $A \sim B$ means "set A offers the same diversity as set B". Without any reason for excluding *a priori* certain sets of objects, it is typically assumed that every non-empty subsets of X can be considered for diversity appraisal. I denote by $P(X)$ the set of all non-empty subsets of X.

The diversity ranking is always assumed to be *reflexive* (any set is at least as diverse as itself) and *transitive* (A is at least as diverse as C if A is at least as diverse as B which is itself at least as diverse as C). Also, all rankings considered in the literature and used in practice are *complete* in the sense of being capable of comparing any pairs of sets in terms of their diversity. While completeness is quite convenient for applications (funding agencies do not like usually answers like "we don't know whether or not biodiversity of this particular ecosystem has been reduced or increased in the last 15 years"), it is probably the least desirable of these requirements. Yet, no attractive diversity ranking that is incomplete has emerged as yet. A reflexive, transitive and complete ranking is called an *ordering*. I will also often assume that the ranking of sets is *non-trivial* in the sense that it does not consider all sets to be equivalent in terms of diversity.

If the diversity ranking $\succeq$ is an ordering, and the number of possible sets is finite, we know that $\succeq$ can be represented by a numerical diversity index $I_d : P(X) \to \mathbb{R}$ in such a way that $I_d(A) \geq I_d(B)$ if and only if $A \succeq B$. The index I_d can therefore be though of as "measuring" the diversity of various sets even though the usual ordinal caveat applies to this measurement: it is unique up to a monotonic transformation.

Which principles could plausibly underlie a ranking (or an ordinal diversity index) of sets on the basis of their diversity? Without further information on the structure of the objects themselves (i.e. their dissimilarities, the attributes that they possess, etc.), the list of these principles is likely to be rather small. Yet it is not empty. It is therefore instructive to go through this list and to examine some of these principles as well as the rankings to which they naturally lead. While most of the principles and rankings discussed in this section have been proposed in other contexts than diversity appraisal, they do have a bearing on the problem at hand when properly reinterpreted.

2.2 Monotonicity

A first well-known category of principles concerns the *monotonicity* of a diversity criterion with respect to the addition of objects: diversity is never reduced when objects are added to a set. This principle can be expressed axiomatically as follows.

Axiom 1 *(monotonicity) For every distinct sets A and B in $P(X)$, $A \supset B \Rightarrow A \succeq B$.*

While this principle looks quite reasonable for diversity appraisal, its general acceptance depends upon the interpretation given to the objects. In the context of biodiversity appraisal, if the elements of the sets are assumed to be species, then this axiom is quite reasonable: Adding a species does not reduce diversity and eliminating a species never increase diversity. On the other hand, if the elements of the set are living individuals, then it is easy to think of plausible conceptions of diversity that violate this principle. For example, it is not uncommon in biology (see section 3 below) to measure the diversity of an ecosystem by the Shannon (1948) entropy defined on the *relative frequencies* of the various species present in the ecosystem. This measure violates monotonicity if the objects of the set are assumed to be living individuals. Suppose in effect a world in which the population of living individuals is equally split between species 1 and 2. The addition to such a world of living individuals of species 1 will increase the ratio of individuals from species 1 over those of species 2 and will therefore reduce the entropy of the ecosystem.

Yet it is fair to say that many criteria proposed to evaluate diversity satisfy monotonicity. It could even be argued that monotonicity is, in fact, a weak requirement, compatible with very coarse rankings such as the trivial one that considers all sets to be equally as diverse. It is therefore not rare to see notions of diversity that satisfy stronger versions of the same idea.

The strongest of all these versions is certainly the following requirement of strong monotonicity according to which the addition of objects in a set *strictly increases* diversity.

Axiom 2 *(strong monotonicity). For every distinct sets A and B, $A \supset B \Longrightarrow A \succ B$.*

This property appears also somewhat natural in the context of biodiversity appraisal if, as in the previous discussion, objects in X are interpreted as species rather than individual organisms. After all it is not absurd to postulate that the elimination of a species from an ecosystem reduces this ecosystem's diversity. A similar kind of consideration seems to hold for other instances of diversity measurement (such as products diversity or, more generally, the diversity of choices offered to a decision maker). Yet, as will be seen below, there are several plausible conceptions of diversity that violate this strong monotonicity principle while satisfying the weak version of it. The reason for which these conceptions of diversity violate this principle is that they base diversity appraisal on a richer information than what is considered here. For example, in section 5, some attention will be devoted to approaches that view the diversity of a set as the sum of the values of the attributes realized by the objects in the set. In this setting, an object contributes to the diversity of a set *only if* it possesses an attribute not possessed by the other objects in the set. Unless all objects possess a unique attribute that is not possessed by the others, it is clear that such an approach will violate strong monotonicity while satisfying weak monotonicity (adding objects never hurt).

Monotonicity properties provide a natural "direction", so to speak, in which it makes sense to talk about diversity improvement, or deterioration. This leads naturally to the question of defining a set of minimal, or zero, diversity. Singletons are natural candidates for this purpose. A car dealer that offers only one model of car is clearly not diverse. Nor is an ecosystem in which only one species is represented, or a newspaper in which only one opinion is reflected. Yet there are as many singletons as there are objects. How should a plausible diversity ranking compare different singletons ?

One answer to this question, proposed by Pattanaik and Xu (1990) in their influential contribution on freedom appraisal, would be to consider them as equally "non-diverse". This answer is contained in the following axiom.

Axiom 3 *(indifference between non-diverse situations). For all distinct objects x and y, $\{x\} \sim \{y\}$.*

This answer is certainly consistent. The principle of indifference between non-diverse situations is actually satisfied by most indices used by biologists to measure biodiversity, irrespective of the fact that the objects in the sets represent species or individual organisms. This axiom is also satisfied by all approaches that view the diversity of a set as resulting from the aggregation of the pairwise dissimilarities of its elements (as there is not much dissimilarity between an object and itself). On the other hand, conceptions of diversity that focus on the attributes of the objects rather than on the objects themselves have no reason to respect this principle. After all should an ecosystem with only protozoans be considered as diverse as one populated only by humans ?

2.3 Independence

Another category of principles that have been invoked for comparing sets of objects concern the extent to which the contribution of objects to diversity should *depend* upon the set in which they are added or subtracted.

> "An angel is more valuable than a stone. It does not follow, however, that two angels are more valuable than one angel and a stone".[1]

Thus spoken Thomas d'Aquina, who seemed to believe that the contribution of an angel to diversity was depending upon the set to which the angel was added. And Thomas d'Aquina was probably right on this. An essential feature, it seems, of diversity appraisal is that the contribution of an object to the diversity of a set depends upon the objects already present in the set. Eliminating a species of fly does not have the same impact on biodiversity in a situation where there are thousands of different species of fly than in a situation where there are only 10. A particular right-wing opinion contribute less to diversity of opinions when it is added to a right-wing journal then when it is added to a left-wing journal.

[1]Thomas D'Aquinas, Summa contra Geniles, III. The quotation is taken from Nehring and Puppe (2002).

Yet the issue of dependency of the contributions of objects with respect to the sets to which they are added or subtracted is delicate. After all, one of the most widely used measure of biodiversity, *species counting*, assumes that the contribution of a species to the diversity of an ecosystem is one no matter what the ecosystem is. It is therefore of some interest to clarify the notion of dependence, or independence, involved.

Following the work of deFinetti (1937), qualitative probability theorists such as Kraft, Pratt, and Seidenberg (1959) (see also Fishburn (1969)) have identified, in a somewhat different context, an axiom that describes quite compactly the property involved. A good starting point for understanding this axiom is trough the notion of a bilateral *transfer of objects* from a set A to a set B. To remain in the realm of biodiversity applied to objects interpreted as species, suppose that a species initially presented in ecosystem B and absent from ecosystem A is removed from B and added to A (for instance all bears from Slovenia are transferred to the French Pyreneans). How should a plausible ranking of sets on the basis of their diversity record such a transfer ? Specifically, is it possible that this transfer increases diversity in one set without reducing that in the other ?

Anyone believing that the contribution of an object to diversity is independent from the set to which the object belongs should be inclined to answer negatively to this question. If the object involved in the transfer contributes to diversity in a way that is independent from the set, the transfer of the object from one set to another is bound to have *opposite* effect in the two sets. There are really only *three* possibilities here. Either the object *contributes positively* to diversity, in which case the transfer increases diversity in the receiving set and reduces that of the donating set, or it *contributes negatively* to diversity, in which case the reverse conclusion holds, or it *does not contribute at all* to diversity, in which case no change in the diversity of the two sets is recorded. Any other possibility would be indicative of a contribution to diversity that depends upon the set (for instance the transfer does not reduce the diversity of the donating set but increases strictly diversity in the receiving set).

One can now take a step further and consider transferring objects across sets in a sequential manner: First transferring an object from a set A to a set B, and then transferring an object from a set C (not necessarily distinct from A or B) to some set D (again not necessarily distinct from A or B) and so on, a finite number of times. Can this finite sequence of transfers increases the diversity in one set without reducing that in at least one other set ? Even though the intuition here is less straightforward than in the elementary case of a transfer of one object between two sets, the answer to this question should be negative for someone who believe in the set-independence of the contribution of objects to diversity. Moving around objects across a given list of sets can not increase strictly the diversity in some sets without reducing that of some other set if the contribution of any object to diversity is set-independent. This general principle can be stated formally as follows.

Axiom 4 *(general independence) For any two lists of k sets* $(A_1, A_2, ..., A_k)$ *and* $(B_1, B_2, ..., B_k)$ *(for* $k = 2, ...$*), if the list* $(A_1, A_2, ..., A_k)$ *is the result of performing a finite sequence of transfers of objects between sets in the list* $(B_1, B_2, ..., B_k)$*, then if* $A_h \succ B_h$ *for some* $h \in \{1, ..., k\}$*, one must have* $B_j \succ A_j$ *for some* $j \in \{1, ..., k\}$*.*

As will be seen shortly, this axiom captures this notion of independence in a very exhaustive way, since it identifies all complete and reflexive diversity rankings that compare sets on the basis the *sum* of the "contributions" to diversity of their elements, with the elements' "contributions" being measured by some function that does not depend upon the sets. It is actually a quite demanding axiom that implies other notions of independence discussed in the literature.

One of them is that considered in Pattanaik and Xu (1990), which asserts that adding or subtracting the same objects to two sets does not affect their ranking.

Axiom 5 *(Pattanaik and Xu's independence) For any sets A, B and C such that* $A \cap C = B \cap C = \emptyset$*,* $A \succeq B \Leftrightarrow A \cup C \succeq B \cup C$

It can easily be seen (see for example Gravel, Laslier, and Trannoy (1998)) that general independence implies Pattanaik and Xu's independence but that the converse does not hold.

But there is another interesting direction in which the notion of independence can be weakened. For the discussion so far has concerned the extent to which the contribution of an object to diversity of a set should depend upon the set. Yet one could also consider a (much weaker) notion of independence that would require only the contribution of an object to the diversity of a set to be independent from the other objects already present in the set. Coming back to the earlier example, suppose that it is recognized that adding new bears in the french Pyreneans (say by transferring them from Slovenia) increases the diversity of the French Pyreneans ecosystem. Should this verdict be maintained if the last remaining 42 couples of eagles who inhabit the French Pyreneans (according to recent accounts) were to be eliminated ? An "independentist" would undoubtedly answer yes to such a question. If the presence of bears increases biodiversity in the French Pyreneans with eagles, it should also increase biodiversity in the French Pyreneans without eagles. But saying this is obviously not to say that the contribution of bears to diversity should be the same in every set. For instance, it is quite possible to hold this statement while believing that the transfer of bears from Slovenia to France has no impact on the diversity in Slovenia (say because there are a sufficient numbers of bears there).

This property of limited independence has been called *contraction consistency* by Nehring and Puppe (1999) and Puppe (1998) and is equivalent, at least for rankings of sets that are weakly monotonic, reflexive and transitive, to an axiom introduced by Kreps (1979) (property 1.5) and analyzed extensively by Nehring (1999). It is stated formally as follows.

Axiom 6 *(contraction consistency) For every sets A and B such that $A \supset B$ and for every object x, if $A \cup \{x\} \succ A$, then $B \cup \{x\} \succ B$.*

This axiom, it should be emphasized, is rather weak. It is obviously satisfied by any ranking that satisfies general independence (and also by rankings satisfying Pattanaik and Xu's independence). But contraction consistency is also trivially satisfied by any strongly monotonic ranking (contraction consistency has obviously no bite if adding objects to sets *always* increase diversity). The key essential feature of contraction consistency is the idea of "*decreasing contribution*" of objects to diversity with respect to the size of the set: an object can not contribute to the diversity in a set if it does not contribute to the diversity of one of its proper subsets.

Yet it is not hard to think of plausible notions of diversity that would violate contraction consistency. This is certainly the case of notions such as those considered in section 4 of this chapter which attach importance to the underlying "dissimilarity" between the objects. To take a simple example, suppose that we are interested in comparing the diversity of meals offered in various restaurants. Consider first a (rather carnivorous) restaurant offering to its clients only chicken and lamb dishes. It would not be absurd to consider that adding to this restaurant a beef dish does not increase significantly the diversity of plates. This could be so because the dissimilarity between beef and chicken or between beef and lamb may be judged lower than the existing dissimilarity between lamb and chicken. Yet it would not be absurd either to consider that adding the same beef dish to a restaurant offering initially chicken, lamb and paneer (an Indian vegetarian delicacy) would increase the diversity of meals offered at this restaurant. Again this could happen simply because the dissimilarity between beef and paneer may appear much larger than any existing dissimilarity between lamb and chicken, paneer and chicken or paneer and lamb (after all, many Indian gourmets consider beef to be a rather extravagant kind of food, especially as compared with paneer).

2.4 Some implications of the principles

The properties just discussed are actually quite exhaustive in terms of the kind of diversity rankings that they allow.

Consider first the property of general independence. As it turns out, this axiom characterizes all reflexive and transitive rankings of sets that result from the comparisons of the sum of "contributions" of their elements, with the contributions measured by numbers pre-assigned to every object of the universe. If the ranking of sets is further required to be monotonic, the contribution of objects to diversity is constrained to be non-negative. If the ranking of sets is required to be strictly monotonic, then the contributions have to be strictly positive.

This result is stated formally as follows.

Theorem 1 *Let $\succeq$ be a reflexive and complete diversity ranking of $P(X)$. Then $\succeq$ satisfies general independence if and only if there exists some function $c : X \longrightarrow \mathbb{R}$ that assigns to each object x its numerical contribution $c(x)$ such that, for every sets A and B, $A \succeq B$ if and only if $\sum_{a\in A} c(a) \geq \sum_{b\in B} c(b)$. Moreover, if $\succeq$ is weakly (resp. strongly) monotonic, $c(x) \geq 0$ (resp. > 0) for every object x.*

Proof. *The main part of the proof consists in showing that an axiom formally equivalent to general independence, and called* additivity *by Fishburn (1970), implies the additive representation. This proof, which rides on the theorem of the alternative, can be found in Kraft, Pratt, and Seidenberg (1959) or Fishburn (1970) The equivalence between the current formulation in terms of sequence of transfers and additivity is proved in Gravel, Laslier, and Trannoy (1998) (lemma 1). Yet the proof in Gravel, Laslier, and Trannoy (1998) uses a slightly different kind of transfer than what has been (informally) discussed here since it allows for transferring objects from singleton to other sets . It is possible to prove a variant of lemma 1 in Gravel, Laslier, and Trannoy (1998) with the transfers discussed herein if some care is taken with the treatment of singleton sets. Details of the construction are omitted.* ■

Let me refer to the rankings characterized by theorem 1 as to *additive rankings*. An important special case of additive rankings is the *cardinality ranking* that compares sets on the basis of their number of elements (for instance the number of species if species are interpreted as being the objects). This ranking has been the object of extensive discussions in the literature on freedom of choice where it received a nice axiomatic characterization by Pattanaik and Xu (1990) (see also Jones and Sugden (1982) and Suppes (1987)). The cardinality ranking differs from other additive rankings by assigning the *same* numerical contribution to diversity to *all* objects. While this "equal treatment" property may look arbitrary - there is little reason after all to consider that a specific species of flies contributes the same way to biodiversity as a chimpanzee - it may equally well be seen as an appropriate assumption to make in the abstract setting considered in this section where no information is assumed to be available on the nature of the objects. Without any reason for treating different objects differently, why distinguishing them ?

This non-distinction between objects is clearly at the heart of the axiom of indifference between non diverse situations. Quite obviously, this axiom suffices to select, in the class of non-trivial, monotonic, complete and reflexive rankings that satisfy generalized independence, the cardinality ranking. Let me state this fact formally as follows.

Proposition 1 *Let $\succeq$ be a reflexive, complete, monotonic and non-trivial diversity ranking of $P(X)$. Then $\succeq$ satisfies general independence and indifference with respect to non-diverse situations if and only if it is the cardinality ranking.*

Proof. *From theorem 1, a reflexive, complete and monotonic diversity ranking satisfies general independence if and only if there exists a function c that assigns a non-negative numerical contribution $c(x)$ to every object x in X such that $A \succeq B$ if and only $\sum_{a \in A} c(a) \geq \sum_{b \in B} c(b)$. For this ranking to satisfy indifference with respect to non-diverse situations, we must have $c(x) = c(y)$ for every objects x and y. As non-triviality precludes the case where $c(x) = c(y) = 0$ for every x and y, we must conclude that $c(x) = c(y) > 0$ for every x and y.* ∎

It is probably worth comparing the characterization of the cardinality ranking provided by this proposition with the widely discussed one of Pattanaik and Xu (1990). Both characterizations share in common the principle of indifference between singletons. Yet Pattanaik and Xu (1990) require the ranking to be transitive (but not necessarily complete) while proposition 1 only requires it to be complete (but not necessarily transitive). Proposition 1 requires the ranking to be non-trivial and monotonic while Pattanaik and Xu (1990) require it to satisfy the "preference for choice over non-choice" principle according to which a set made of two distinct objects should be ranked above any of the two singletons made of one of the two objects. On the other hand, proposition 1 rides on the general independence principle while Pattanaik and Xu (1990) obtains cardinality out of their (much weaker) own independence condition.

This later state of affairs is actually an interesting logical fact. Without indifference between non-diverse situations, the class of strictly monotonic rankings satisfying Pattanaik and Xu's independence is larger than the additive class (as shown in Kraft, Pratt, and Seidenberg (1959), there are non-additive strictly monotonic rankings who satisfy Pattanaik and Xu's independence). The fact that the addition of indifference between non-diverse situations is capable of shrinking the (large) class of strictly monotonic rankings satisfying Pattanaik and Xu's independence up to the unique cardinality ranking is somewhat surprising.

Any additive ranking has the property that it can be thought of as measuring the diversity of a set by summing the numerical contributions of all its elements. Hence such a ranking can be viewed as resulting from a two-step procedure:

1) Measuring the numerical contribution of every object in the universe by a numerical function c and

2) Comparing sets on the basis of the sum of the contributions of their elements as measured by the function c.

A disputable feature of this approach is that the contribution made by an object to the diversity is borne by the object itself and not by some more fundamental *attributes* that the object may have. Yet there are many contexts in which it is the attributes of the objects, rather than the objects themselves, who are contributing to diversity. Industrial organization, and the characteristics-based or hedonic approach to product differentiation developed by Lancaster (1966) and Rosen (1974), has made economists aware that differentiated products such as cars, televisions, etc. are best seen as vectors in a space of more fundamental characteristics.

If one adopts the characteristics-based view of products differentiation, it becomes clear that the appraisal of diversity in, say, different car retailers should depend upon the characteristics possessed by the cars and not so much upon the car themselves. In recent years, Nehring and Puppe (2002) (see also Nehring and Puppe (2003)) have forcefully defended the view that attributes, rather than objects, should be the ultimate contributors to diversity.

Of course the abstract framework considered in this section does not provide information on the attributes possessed by the objects. A natural, if not formal, way to define attributes in this setting would be to view them as *sets of objects*. Each such set would be interpreted to be the list of all those objects possessing the attribute considered. A surprising consequence of the principle of contraction consistency is that, along with monotonicity, it characterizes all rankings that *can be thought of* as comparing sets on the basis of the *sum* of the *values of the attributes* that are *realized* in the sets for some family of attributes, and for some function that values the contribution of the attributes. Put differently, when imposed on an ordering, the properties of monotonicity and contraction consistency imply the existence of a family of attributes and a numerical function that values these attributes in such a way that the ranking of sets can be viewed as resulting from the comparisons of the sum of the values of the attributes that are realized in the sets.

This result, first proved by Kreps (1979), and reinterpreted, in terms of attribute weighting, by Nehring (1999) and Nehring and Puppe (2002) on the basis of results developed in Chateauneuf and Jaffray (1989), is stated formally as follows.

Theorem 2 *An ordering $\succeq$ on all sets in $P(X)$ satisfies weak monotonicity and contraction consistency if and only if there exists an attribute valuation function λ that assigns a non-negative value $\lambda(S)$ to any subset S of X ($\lambda(S)$ is zero if S does not correspond to an attribute or, equivalently in this theory, if it corresponds to an attribute that is not valued) such that, for every two sets A and B, $A \succeq B$ if and only if $\sum_{C:C\cap A\neq\emptyset} \lambda(C) \geq \sum_{D:D\cap B\neq\emptyset} \lambda(D)$.*

Proof. *As noticed above, a complete, reflexive and transitive diversity ranking $\succeq$ can be numerically represented by a diversity indicator $I_d : P(X) \to R$. There is actually a wide range of such diversity indicators as any increasing transformation of a diversity indicator is also a diversity indicator. As any set function, a diversity indicator has a **conjugate Möbius inverse** (see e.g. Nehring (1999), section 3 and Nehring and Puppe (2002), fact 2.1) $\lambda : P(X) \to R$ defined, for every B, by:*

$$\lambda(B) = \sum_{S\subset B} (-1)^{\#(B\backslash S)+1} I_d(X\backslash S) \tag{1}$$

and satisfying, for every set A, the equality:

$$I_d(A) = \sum_{S:S\cap A\neq\emptyset} \lambda(S) \tag{2}$$

*In view of (2), the only thing that needs to be shown is the existence of a diversity indicator which admits a non-negative conjugate Möbius inverse for every set other than X. This non-negative conjugate Möbius inverse can be interpreted as a method for valuing the attributes (any set receiving a positive value is considered as an attribute). We don't care much about the valuation of the attribute "being an object of the universe" because it is an attribute that is realized in every non-empty set and that can not serve therefore in distinguishing sets. As shown by Chateauneuf and Jaffray (1989) a necessary and sufficient for a set function I_d to have a non-negative conjugate Möbius inverse for all sets other than X is to be weakly monotonic and **totally submodular** (see e.g. Chateauneuf and Jaffray (1989) for a definition of this property). As a numerical representation of a monotonic ranking of sets, any diversity indicator is obviously monotonic. Hence the only thing that needs to be shown is that, among all diversity indicators that numerically represent $\succeq$, there is at least one that is totally submodular. Nehring (1999) has shown that, for a monotonic ranking, the existence of a totally modular numerical representation of it amounts to requiring the ranking to satisfy the following condition, for every sets A, B and C:*

$$A \succeq A \cup B \Rightarrow A \cup C \succeq A \cup B \cup C \tag{3}$$

Now, it is immediate to verify that, for a reflexive, transitive and monotonic ranking, this condition, first formulated by Kreps (1979), is equivalent to contraction consistency. ■

The fact that one can think of an ordering of sets as resulting from the additive valuations of "attributes" is, in itself, not terribly informative. For any numerical function having as domain all non-empty subsets of a finite set can be thought of, by applying Möbius inversion, as a sum of valuations of attributes, with some attribute negatively valued and others, positively. The attribute valuation function provided by (1) is constructed in a mechanical manner from the numerical representation of the diversity ranking itself. For this reason, it depends crucially upon this numerical representation.

A more substantial information is obtained from the fact that, if the ordering satisfies contraction consistency and weak monotonicity, there exists at least one numerical representation of the ranking for which Möbius inversion provides a non-negative valuation of all attributes other than the universe. Yet, it is important, here again, to notice the dependency of the valuations of the attribute with respect to the particular diversity index used to represent the ranking. Different families of attributes and different valuations of them will obtain as one changes from one numerical representation of the same diversity ranking to another. This is problematic for the interpretation of this theorem as providing an argument in favour of a multi-attributes approach to diversity measurement. For it is somewhat bizarre for the attributes that are considered relevant for appraising diversity as per a given ranking be dependent upon the numerical representation of the ranking. It is somewhat bizarre that the attribute of, say, being a mammal, be valued positively and largely for one representation of the ranking while being given no weight at all for other representations of the same ranking. If it is believed that a diversity ranking of sets can be thought of as aggregating the values of the attributes realized in the sets, the concerned attributes and their relative importance should be independent from the particular numerical representation of the ranking.

An example may perhaps be useful for understanding this point. Suppose one is interested in ranking subsets of the set $\{lizard, shark, whale\}$ using the following ordering $\succeq$:

$$\begin{aligned} \{lizard, shark, whale\} &\succ \{shark, whale\} \sim \{lizard, whale\} \succ \\ \{whale\} &\succ \{lizard, shark\} \succ \{shark\} \sim \{lizard\} \end{aligned} \quad (4)$$

It is easy to verify that this ordering satisfies contraction consistency and weak monotonicity. Yet the valuation of attributes revealed by this ranking is far from clear. If we except the trivial attribute "being an object in the universe" (represented by X), the possible attributes here are "fish", "reptile" and "mammal", represented respectively by the singletons $\{shark\}$, $\{lizard\}$ and $\{whale\}$ as well as "ocean-living" ($\{shark, whale\}$), "air-breathing" ($\{lizard, whale\}$) and "cold-blooded" ($\{lizard, shark\}$). The problem is that the diversity ranking provided by (4) does not tell us much about the valuation of these various attributes. The only obvious thing that can be said from looking at the ranking is that a positive value is attached to singleton attributes (e.g. "fish", "mammal" and "reptile") (because the triplet X is ranked strictly above any ecosystem consisting of pairs of animals). The valuation of the other attributes will depend crucially upon the particular diversity index used.

For instance, if one represents the ranking by the diversity index I_d defined by:

$$\begin{aligned}
I_d(\{lizard, shark, whale\}) &= 4 \\
I_d(\{shark, whale\}) &= I_d(\{lizard, whale\}) = 3 \\
I_d(\{whale\}) &= 2 \\
I_d(\{lizard, shark\}) &= 1 \\
I_d(\{lizard\}) &= I_d(\{shark\}) = 0
\end{aligned}$$

then, applying the formula (1), one obtains the following definition of the attribute valuation function λ:

$$\begin{aligned}
\lambda\{lizard\} &= \lambda\{shark\} = 1 \\
\lambda(\{lizard, shark\}) &= \lambda(\{lizard, whale\}) = \lambda(\{whale, shark\}) = 0 \\
\lambda(\{whale\}) &= 3 \\
\lambda(X) &= -1
\end{aligned}$$

According to this definition, the only attributes that are valued positively are "mammal", "fish" and "reptile", with mammal being given a larger value than fish and reptile and reptile and fish being equally valued. No values is attached to the other attributes.

Now a somewhat different story is obtained if the following numerical representation $\widehat{I_d}$ of the same ranking is used instead.

$$\begin{aligned}
\widehat{I_d}(\{lizard, shark, whale\}) &= 5 \\
\widehat{I_d}(\{shark, whale\}) &= \widehat{I_d}(\{lizard, whale\}) = 4 \\
\widehat{I_d}(\{whale\}) &= 1 \\
\widehat{I_d}(\{lizard, shark\}) &= 0 \\
\widehat{I_d}(\{lizard\}) &= \widehat{I_d}(\{shark\}) = -1
\end{aligned}$$

For using again the formula (1), this diversity index provides us with the attribute valuation function $\widehat{\lambda}$ defined by:

$$\begin{aligned}
\widehat{\lambda}(\{lizard, whale\}) &= \widehat{\lambda}(\{shark, whale\}) = \widehat{\lambda}(\{lizard\}) = \widehat{\lambda}(\{shark\}) = 1 \\
\widehat{\lambda}(\{lizard, shark\}) &= 2 \\
\widehat{\lambda}(\{whale\}) &= 4 \\
\widehat{\lambda}(X) &= -4
\end{aligned}$$

which values positively all logically conceivable attribute (except of course the trivial attribute X). Notice also how different is the ordering of attributes in terms of the way they are valued. The fact of being cold blooded is valued more here than the fact of being a reptile or a fish while the converse ranking of the attributes was holding with the index I_d.

We have examined, in this section, alternative principles that a ranking of sets of object could satisfy in order to serve as a plausible definition of diversity in a framework where no information is available on the nature of the objects. We have also summarized the main implications of these principles in terms of the rankings of objects that they allow. Two broad types of rankings have been identified in this perspective:

1) additive *object-based* rankings, which can be thought of as resulting from a comparisons of the sum of the contributions of objects to diversity, and

2) additive, *attribute-based* rankings, which can be thought of as resulting from the comparisons of the sum of the contributions of *objects' attribute* to diversity.

Additive object-based rankings are tightly characterized by the principle of general independence in the sense that a *unique* ranking of the objects on the basis of their contributions to diversity is obtained from the ranking of sets applied to singletons. On the other hand, the principle according to which each object contributes to diversity independently of the set to which it is added or subtracted does not seem to be well-adapted to diversity appraisal. Additive attribute-based rankings are not tightly characterized by the principles of weak monotonicity and contraction consistency since several rankings of attributes on the basis of their contribution to diversity may come out of the same ranking of sets of objects. Yet additive attribute-based rankings are probably more suitable than their object-based counterpart to serve as plausible criteria for appraising diversity provided that some extra information on the attribute valuations is added to the model.

3 Diversity as generalized entropy

Biologists have been probably the first scientists to develop an interest in diversity appraisal. As typically conducted in biology, diversity appraisal interprets the objects as being species, and view ecosystems as vectors of *species abundance*, each component of the vector being interpreted as the number of individuals belonging to the corresponding species in the ecosystem.[2] The extra structure brought about by the grouping of living individ-

[2] As mentionned earlier, an alternative would be to view objects as being individuals, partitionned into a number of different species. Everything that will be said in this section could be also said in this alternative framework, albeit at some extra cost in terms of notation.

uals into species is obviously natural in biology, even though there is some disagreement among biologists as to the precise definition of what constitutes a "species". A widely used criterion defines a species as a set of actually or potentially interbreeding individuals. This definition obviously only applies to organisms that reproduce sexually. Yet the reproduction criterion is important because it is through reproduction that individual organisms exchange genes and evolve over time.

In this setting, if there are m species, an ecosystem is described by a vector $x = (x_1, ..., x_m)$ where $x_k \geq 0$ denotes the number of individuals in the ecosystem who belong to species k. I shall assume in the discussion that x belongs to $\mathbb{R}^m_+$. This means that there can be any non-negative number of individuals in any species and, therefore, that the underlying number of individuals is uncountably infinite. It is also worth noticing that this way of doing attaches no importance whatsoever to the diversity between individuals of the same species.

It is not rare in biology to evaluate diversity on varying sets of species. For instance, one may be interested in tree diversity, in planktonic diversity, in mammal diversity, and so on. Biologists therefore attach some importance to the fact that the diversity ranking be defined for all specifications of the set of species. I shall denote accordingly by $\succeq_m$ the ranking of ecosystems based on m species (for any $m = 1, 2, ...$). For any vector x in $\mathbb{R}^m_+$, let $\mu(x) = (\sum_{k=1}^{m} x_i)/m$ denote the *average abundance* of the ecosystem, interpreted to be the average number of living individuals that can be found in a species. For any vector x in $\mathbb{R}^m_+$, I also denote by $\overline{x}$ the *ideally diverse ecosystem* corresponding to x that is defined by $\overline{x}_k = \mu(x)$ for every k.

By far the most widely used class of diversity-based rankings of ecosystems in biology is that associated with the general entropy measure of Renyi (1961) defined over the distributions of *relative frequencies* of species in various sets. This class is parameterized by a real number r. For r different from 1, the ranking $\succeq^r_m$ of m-species ecosystems generated by this class is defined by:

$$x \succeq^r_m y \Longleftrightarrow (\sum_{k=1}^{m} (\frac{x_k}{m\mu(x)})^r)^{\frac{1}{1-r}} \geq (\sum_{k=1}^{m} (\frac{y_k}{m\mu(y)})^r)^{\frac{1}{1-r}} \tag{5}$$

if $r \geq 0$ and by

$$x \succeq^r_m y \Longleftrightarrow -(\sum_{k=1}^{m} (\frac{x_k}{m\mu(x)})^r)^{\frac{1}{1-r}} \geq -(\sum_{k=1}^{m} (\frac{y_k}{m\mu(y)})^r)^{\frac{1}{1-r}}$$

if $r < 0$[3] while for $r = 1$, the ranking is defined by:[4]:

$$A \succeq_m^1 B \Longleftrightarrow -\sum_{k=1}^{m} \frac{x_k}{m\mu(x)} \ln \frac{x_k}{m\mu(x)}) \geq -\sum_{k=1}^{m} \frac{y_k}{m\mu(y)} \ln \frac{y_k}{m\mu(y)} \tag{6}$$

using the convention that $0 \ln 0 = 0$. Posing by convention also that $0^0 = 0$, the case $r = 0$ corresponds to the widely-used criterion of species counting discussed in the previous section. The (limiting) case $r = 1$ corresponds to the well-known Shannon (1948)'s entropy index first proposed, in the context of biodiversity measurement, by Good (1953). Other interesting cases are $r = 2$, where the ranking is induced by the comparison of the Simpson (1949) index (also known in economics as the Herfindahl index of concentration), and r approaching infinity, where the ranking is obtained by comparing the (inverse of) the relative frequency of the *most abundant species* in the ecosystem. This later index is known as the Berger and Parker (1970) index of diversity.

All rankings in this family consider that an ecosystem in which all existing species are represented with the same abundance is the most diverse of all. Increasing the value of r amounts to increasing the weight given to the more abundant species in the ecosystem, with the limiting case given by the Berger and Parker (1970) index where the weight given to any species other than the most abundant one is zero. The justification given to these rankings of ecosystems by biologists (see for instance Magurran (1998)) is a mixture of prosaic considerations of ease of calculations, and of intuitive probabilistic ideas about the appeal of having ecosystems that are "unpredictable" in terms of the probability of success of the various species. For instance the Simpson index can be interpreted as measuring the (reciprocal of) the probability for two living individuals drawn independently at random from the ecosystem to belong to the same species. The lower the probability, the more diverse is the ecosystem. Analogously, Shannon's entropy is usually viewed in thermodynamics as a measure of "disorder" of the system. An ecosystem that is maximally "disordered" is this sense - that is an ecosystem in which it is very hard *a priori* to predict which species will win the race for survival - is maximally diverse.

The properties that characterize the generalized entropy family of rankings have been identified, in the somewhat different context of income inequality measurement, by Shorrocks (1984) (see also Shorrocks (1980)). It is of interest to recall here these properties because they identify clearly the limitations of generalized entropy as a method for evaluating diversity.

[3]Biologists rarely, if at all, consider negative values of r.

[4]Applying de L'Hospital's rule, it is immediate to see that, if $p_1, ..., p_m$ are any m non-negative numbers summing to 1, the limit of the expression $\frac{1}{1-r} \ln(\sum_{i=1}^{l} p_i^r)$ as r tends to 1 is $-\sum_{i=1}^{l} p_i \ln p_i$.

The first one is an axiom of symmetry. It says, in a rather disputable fashion for diversity appraisal, that "species does not matter" and that the only things that are relevant for diversity is the ordering of the species in terms of their abundance. The fact that the more abundant species is chimpanzee or mosquitoes has no bearing whatsoever on our evaluation of biodiversity. This axiom is stated as follows.

Axiom 7 *(symmetry between species). For every number m of species, for all ecosystems x and y in $\mathbb{R}_+^m$, if x is a permutation of y, then $x \sim^m y$.*

The second axiom is also somewhat problematic for diversity appraisal. For it says, in contradiction with the principle of weak monotonicity, that it is only the *relative* abundance that matters and that increasing uniformly the population size of each species in an ecosystem has no effect whatsoever on diversity. A world in which each species has 10 individuals is just as diverse as a world in which each species has 1 million individuals.

Axiom 8 *(scale invariance). For every number m of species, and for any ecosystem x in $\mathbb{R}_+^m$, $tx \sim^m x$ for every real number t.*

The third axiom captures the "equalizing" feature of diversity as appraised by generalized entropy rankings. It says that reducing disparities in abundance in an ecosystem is, *ceteris paribus*, a good thing for diversity. Specifically, any change that would lead to the elimination of a certain number of individuals in an abundant species and to an increase in the same number of individuals in a less abundant species would be worth doing from the view point of diversity if the change does not affect the ranking of the two species in terms of their abundance.

Axiom 9 *(abundance equalization). For every number m of species, and for all ecosystems x and y in $\mathbb{R}_+^m$, if there exists two species h and k such that $x_i = y_i$ for all i other than h and k and for which one has*

$$x_h = y_h + \Delta < y_k - \Delta = x_k$$

for some strictly positive real number Δ, then, $A \succ B$.

The last substantive axiom that characterizes generalized entropy rankings reflects the concern for convenience in applications alluded to above. It is common for biodiversity appraisal to be made separately and sequentially on various subsets of species. For example, in examining the biodiversity of a particular forest, some biologists will focus on the different species of trees, others will be interested in species of fungus, while other will investigate mammals and birds. Yet it is of practical importance that these various appraisals of diversity be made in a consistent fashion with each other. Suppose we first appraise the diversity of trees in a forest, and we then appraise the diversity of fungus in this forest. How can one obtain, from these two separate evaluations, an overall appraisal of the diversity of this forest in terms of both fungus and trees ? It would be nice if, for instance, the overall evaluation could be obtained in a simple fashion from the two "sub evaluations" made in isolation, rather than requiring a new appraisal of diversity in the large population made of both fungus and trees. A simple and convenient way of proceeding would be to write the overall diversity as the sum of the diversity of fungus, the diversity of trees and the diversity of the "aggregate" ecosystem made only of the *average abundance* of the species of fungus and the average abundance of the species of trees. Using the terminology of Shorrocks (1984), I call "decomposable" a diversity indicator that satisfies this property.

In the ordinal setting considered herein, this property is expressed as the requirement that, among all numerical indicators that represent the diversity ranking, there is at least one that is decomposable.

Axiom 10 *(decomposable representability). There exist functions I_d^m from $\mathbb{R}_+^m$ to $\mathbb{R}$ representing numerically the ranking $\succsim_m$ for every m such that, for any $x \in \mathbb{R}_+^{m_1}$ and $y \in \mathbb{R}_+^{m_2}$, one has:*

$$I_d^{m_1+m_2}(x,y) = I_d^{m_1}(x) + I_d^{m_2}(y) + I_d^{m_1+m_2}(\overline{x},\overline{y})$$

for every number of species m_1 and m_2, where the vector (x,y) is defined by $(x,y) = (x_1,...,x_{m_1},y_1,...,y_{m_2})$.

The attractiveness of this property is, of course, a pure matter of computational convenience. It is convenient, in evaluating the diversity of fungus and trees, to rely only on the (presumably) known diversity of fungus, the (presumably) known diversity of trees and the easy-to-calculate diversity of the simple ecosystem in which all fungus species have their average abundance and all tree-species have their average abundance. But beside this convenience, there is nothing intrinsically appealing in this property.

A powerful result, due to Shorrocks (1984), shows that any *continuous* ordering[5] of ecosystems in $\mathbb{R}_+^m$, for any m, satisfying decomposable representability, abundance equalization, scale invariance, and symmetry between species can be thought as resulting from the comparison of generalized entropy, as calculated by (5) or (6), for some value of r, with the exception of $r = 0$. This result is expressed in the following theorem.

Theorem 3 *An infinite list $\succeq_m$, for $m = 1,...,$ of continuous orderings of ecosystems (each of which involving m species) satisfies symmetry between*

[5] Continuity of an ordering on $\mathbb{R}_+^m$ is defined in the standard way (see for instance Debreu (1954)) by the requirement that the no-better than and the no-worse than sets be closed in $\mathbb{R}_+^m$.

species, scale invariance, abundance equalization and decomposable representability if and only there exists a real number r different than zero such that, for any m, and for every two ecosystems x and y, one has:

$$x \succeq_m y \Leftrightarrow (\sum_{k=1}^{m}(\frac{x_k}{m\mu(x)})^r)^{\frac{1}{1-r}} \geq (\sum_{k=1}^{m}(\frac{y_k}{m\mu(y)})^r)^{\frac{1}{1-r}}$$

if r is different from 1 and

$$x \succeq_m y \Leftrightarrow \sum_{k=1}^{m} \frac{x_k}{m\mu(x)} \ln \frac{x_k}{m\mu(x)} \geq -\sum_{k=1}^{m} \frac{y_k}{m\mu(y)} \ln \frac{y_k}{m\mu(y)}$$

otherwise.

Proof. *By Debreu (1954) theorem, any continuous ordering $\succeq_m$ on $\mathbb{R}_+^m$ can be represented numerically by a continuous function $I : \mathbb{R}_+^m \to \mathbb{R}$. Given symmetry and abundance equalization, this function satisfies $I(\overline{x}) \geq I(x)$ for all x in $\mathbb{R}_+^m$. Hence this function is just like (the inverse of) an inequality index as defined in Shorrocks (1984). Given decomposable representability, it satisfies all axioms of a diversity index considered in Shorrocks (1984). For this reason, the analysis of Shorrocks (1984) applies and, by virtue of the theorem 5 of his paper, there exists a real number r and a numerical representation I_d^m of the ordering $\succeq_m$ such that:*

$$I_d^m(x) = \begin{array}{c} \frac{1}{m}\frac{1}{r(1-r)} \sum_{k=1}^{m} \left(\frac{x_k}{\mu(x)}\right)^r \text{ if } r \neq 0,1 \\ \frac{-1}{m} \sum_{k=1}^{m} \frac{x_k}{\mu(x)} \ln\left(\frac{x_k}{\mu(x)}\right) \text{ if } r = 1 \\ \frac{-1}{m} \sum_{k=1}^{m} \ln\left(\frac{\mu(x)}{x_k}\right) \text{ if } r = 0 \end{array}$$

Yet, it is clear that, for $r < 0$, the ranking of vectors induced by the function:

$$\frac{1}{m}\frac{1}{r(1-r)} \sum_{k=1}^{m} \left(\frac{x_k}{\mu(x)}\right)^r \tag{7}$$

is equivalent to the ranking induced by the function:

$$-[\sum_{k=1}^{m} \left(\frac{x_k}{\mu(x)}\right)^r]^{\frac{1}{1-r}}$$

For r strictly between 0 and 1 and for r strictly greater than 1, the ranking induced by the function (7) is equivalent to that induced by the function defined by (5). The equivalence is of course trivial for $r = 1$. ■

This theorem, which does not provide any characterization of species counting (the case $r = 0$), clearly identifies the principles that justify generalized entropy as a sensible method for appraising diversity. The most problematic of these principles is clearly symmetry across species, which neglects the varying dissimilarities that may exists between species. Yet some other principles are also open to discussion.

Scale invariance is one of them. Even if one accepts the view that equalizing the probabilities of survival across species is the best way to promote biodiversity, it is not clear that this probability of survival does not depend to some extent upon the scale of abundance of the various species. On the one hand, it is widely acknowledged that a minimal threshold of abundance within each species is required to ensure species survival. For instance, species have a much lower chance of survivals when they all contain 10 specimen than when they contain 100 000 of them. On the other hand, there seems also to be some upper bound on the abundance that is sustainable given the limited physical resources that planet Earth has. For instance, the survival probability of mammals would certainly be challenged if the population of each species of mammal were to raise to 10 billions.

The axiom of abundance equalization can also be questioned as a fundamental principle of diversity appraisal. The axiom derives its appeal from the evolutionary context of biology where the abundance of objects (living organisms) belonging to a particular category (species) appears to be related to the probability of survival of the category. Another evolutionary context that satisfies this feature is linguistic diversity where the number of users of a particular language can be thought as a good indicator of the probability of survival of this language. In all such evolutionary contexts, equalizing the probabilities of survival of the different categories can be seen as a natural thing to do in an *ex ante* situation, provided of course that the different categories can be thought as symmetric. Yet such equalization would not be natural in non-evolutionary contexts. Consider for instance the case of products diversity. Even though a grouping of products - cars say - into categories - such as luxury cars, sport cars, family cars, etc.- could be imagined that would play a similar role as the biological grouping of living organisms in species, there would not be much justification for imposing the axiom of abundance equalization in this context. Why would a car retailer offering an equal quantity of sport cars and family cars be considered more diverse than a retailer who,following the demand trend for the two categories of car, would offer them in different proportions ?

As the symmetrical treatment of the categories provided by generalized entropy is the main weakness of this family of methods for appraising diversity, it is important to move away from this assumption. An attractive direction for making this move is to introduce explicitly some information pertaining to the underlying varying dissimilarity between the objects.

4 Diversity as aggregate dissimilarity

From a formal point view, partitioning objects into various categories as done in the previous section can be seen as a (very) particular method for defining a notion of proximity, or similarity, between objects. Two objects belonging to the same category are assumed to be similar and two objects belonging to different categories are assumed to be "dissimilar". Of course the notion of similarity that underlies this way of doing is rather crude: two objects can either be similar, or dissimilar. They can not have intermediate levels of dissimilarity. It is to some extent this rigidity of the notion of similarity that underlies the grouping of objects into categories which limits the interest of generalized entropy as a plausible method for appraising diversity. Would not it be possible to appraise diversity on the basis of a notion of similarity between objects that could be finer than the crude one considered thus far ?

4.1 Cardinal notions of dissimilarity

In mathematics, proximity or similarity judgements are usually expressed by means of a *distance*, or metric, function. In the present context, a distance is a function $d : X \times X \to \mathbb{R}_+$ that associates, to every pair (x, y) of objects in X, a (non-negative) number $d(x, y)$ interpreted as the distance that separates x from y. To motivate this interpretation, it is common to require d to be symmetric (the distance is the same in going from y to x than in going from x to y) and to satisfy the normalization property that $d(x, x) = 0$ for any object x (the distance between any object and itself is zero). It is also quite common to require a distance to satisfy the property that $d(x, y) > 0$ for every distinct objects x and y as well as the so-called "triangle inequality" according to which $d(x, z) \leq d(x, y) + d(y, z)$ (the distance of a "direct" trip between z and x can never exceed that of an "indirect" trip). Yet this triangle inequality has no particular appeal here. One could very well accept for instance that the dissimilarity, in terms of modes of transportation, between a bicycle and a car be greater than the sum of dissimilarities between a bicycle and a motorcycle and between a motorcycle and a car.

Taking as given such a distance function, Weitzman (1992) (see also Weitzman (1993) and Weitzman (1998)) has proposed a recursive method for evaluating the diversity of a set based of a specific aggregation of the dissimilarities between the pairs of objects that it contains. The general idea of Weizman's method is the following. Given any set, one first calculates, for each object, the *increasingly ordered vector* of all distances that separate the object from the other objects in the set. This provide us with as many such ordered vectors as there are objects in the set, each vector having itself also as many components as there are objects in the set. One then compares these ordered vectors by the lexicographic criterion and selects the element in the set which corresponds to the vector which ranks last according to this criterion. There might be several such elements, in which case they will all have the same ordered vector of distances. It this is the case, any one of these elements can be selected. Once this element is selected, one records the *smallest non-zero distance* in its vector and remove the element from the set and redo the procedure with the remaining elements in the set. At the end, one is left with a unique element in the set and one has obtained a finite list of smallest non-zero distance. Weitzman (1992) has proposed to compare sets on the basis of the *sum* of these smallest non-zero minimal distances, sets with larger sums being considered more diverse than those with smaller sums.

Weitzman himself has never provided axiomatic characterizations of his criterion, even though he has discussed some of its properties. The criterion does not seem completely unreasonable, be it simply because it provide the first - to my knowledge - definition of diversity as an aggregation of dissimilarities. Yet the procedure proposed is rather specific. It combines, in a rather complex way, a *lexicographic* procedure for selecting sequentially the objects associated to the minimal distance and an *additive* procedure of summing these minimal distances. One would therefore like to get a better sense of its plausibility by knowing the elementary axioms that justify this way of doing, and only this one.

The only existing - to my knowledge - axiomatic characterization of Weitzman procedure is due to Bossert, Pattanaik, and Xu (2003). The characterization uses three axioms and requires the set X and the distance d to allow for a sufficient richness in the ranking of alternative pairs of objects. Specifically Bossert, Pattanaik, and Xu (2003) assumes the set X and the distance function d to be such that, for any two non-negative real numbers s and t satisfying $s \leq t$, there exists three objects x, y and z such that $s = d(x, y) \leq d(x, z) \leq d(y, z) = t$. This assumption obviously requires the set X to contain uncountably many objects.

The first axiom considered by Bossert, Pattanaik, and Xu (2003) is hardly disputable in the current context. It says that the ranking of sets consisting of one or two objects should coincide with the ranking of the corresponding pairs of elements on the basis of their distance. Formally:

Axiom 11 *(dissimilarity monotonicity). For any (not necessarily distinct) objects w, x, y and z in X, $\{w, z\} \succeq \{x, y\}$ if and only if $d(w, z) \geq d(x, y)$.*

Since the statement of the axiom does not require the objects w, x, y and z to be distinct, and since the distance function satisfies $d(x, y) > d(x, x) = 0$ for every distinct x and y, the axiom implies both the principle of indifference between non-diverse situations as well as Pattanaik and Xu (1990)'s axiom of strict preference for choice over non-choice.

The second axiom is certainly more disputable. It captures the idea that the diversity ranking of two sets should be independent from the addition or the deletion of objects when the distances that separate the objects from the other objects in the set satisfy certain conditions. Specifically, the independence of the ranking is assumed to hold when:

1) the smallest distance that separates the added or subtracted object from the other objects in the set is smaller than the smallest distance between the other objects themselves and

2) the minimal distance between the object and the other objects in the set is the same in the two sets.

This axiom is expressed formally as follows.

Axiom 12 *(minimal distance independence). For any two sets A and B, and for any two objects x and y that do not belong to A and to B respectively, if the smallest distance between x and the other objects in A is smaller than the smallest distance between all distinct objects in A, if the same is true for y with respect to B and if the smallest distance between x and the other objects in A is the same than the smallest distance between y and the other objects in B, then $A \succeq B$ if and only if $A \cup \{x\} \succeq B \cup \{y\}$.*

Hence, the idea captured by this axiom is that the ranking of two sets should not depend upon the object that produces the minimal distance in the set when the minimal distance thus produced is the same. As much of the independence notions discussed earlier, this axiom has a strong additive feature that contribute for no small part to the clearly additive feature of Weitzman's method for evaluating diversity.

The last axiom considered by Bossert, Pattanaik, and Xu (2003) is certainly the more opaque of the three. It applies to pairs of objects such as $\{a, c\}$ and $\{x, y\}$ who have the property that the distance between x and y is greater than the distance between a and b. In this context, consider adding an object b to the set $\{a, c\}$ that is "in between" a and c in the sense that the distance that separates b from c or from a is lower than the distance between a and c. Such an object b would be called by Weitzman (1992) a *link* of $\{a, c\}$ to $\{x, y\}$ if was such that $d(x, y) = d(a, c) + \min(d(a, b), d(b, c))$. That is b is a link of $\{a, c\}$ to $\{x, y\}$ if the minimal distance brought about by the addition of b to a and c is exactly equal to the *difference in distance* between x and y on the one hand and a and c in the other. The third axiom of link indifference introduced by Bossert, Pattanaik, and Xu (2003) would require the set $\{x, y\}$ to be diversity-wise equivalent to the set $\{a, b, c\}$.

Axiom 13 *(link indifference). For all objects a, b, c, x and y in X such that $x \neq y$ and a, b and c are pairwise distinct, if $\{x, y\} \succ \{a, c\}$ and if b is such that $\max\{d(a, b), d(b, c)\} \leq d(a, c)$ and $d(a, c) + \min\{d(a, b), d(b, c)\} = d(x, y)$, then $\{x, y\} \sim \{a, b, c\}$.*

The opacity of this axiom is impressive. And there is no reason really, either practical or philosophical, to adhere to it. Supposed we are interested in appraising the diversity of modes of transportation and that we are told that the distance between riding a bike (a) and riding a motorcycle (c) is 6, that the distance between walking (x) and driving a car (y) is 10. Suppose that we know that the distance between riding a motorcycle and skate boarding (b) is 6 and that the distance between skateboarding and riding a bike is 4. Why should we deduce from these information than the set $\{x, y\}$ offers exactly the same diversity as the the set $\{a, b, c\}$?

Yet this axiom of link indifference is inextricably connected to Weitzman's method for evaluating diversity. For Bossert, Pattanaik, and Xu (2003) have shown the following.

Theorem 4 *If X and d have the richness defined above, then $\succeq$ is an ordering defined over all finite subsets of X satisfying dissimilarity monotonicity, minimal distance independence and link indifference if and only if it ranks sets by Weitzman procedure.*

The lack of appeal of the axiom of link indifference, to mention just this one, sheds serious doubts on the appeal of Weitzman procedure as a sensible method for making diversity appraisal. We do not have, at the present moment, any reason for appraising diversity in the particular way proposed by Weitzman.

Moreover Weitzman's approach suffers from a more fundamental interpretative difficulty. It assumes the existence of a *cardinally meaningful* distance function.[6] That is, it requires from the diversity appraiser an *a priori* capacity to formulate statements such as "the dissimilarity between walking and driving a car is twice that between riding a bicycle and riding a motorcycle". Such a capacity is clearly assumed in the very formulation of the axiom of link indifference. Yet it is far from obvious that this capacity exists, even for an appraiser working in the very developed and sophisticated context of biological diversity. While it seems reasonable to believe that many biologists would agree, say on the basis of a taxonomic or phylogenetic criterion, to say that a bee and a wasp are more similar than a chimpanzee and a protozoan, it is much doubtful to think that they would agree on saying that the dissimilarity between a chimpanzee and a protozoan is exactly 10 times that between a bee and a wasp.

Would not it be possible to view diversity as an aggregation of the underlying dissimilarities between the objects without going as far as requiring these dissimilarities to be measured cardinally by a distance function ?

[6] Another paper that adresses the issue of diversity measurement in terms of a cardinal notion of dissimilarity is VanHees (2004).

4.2 Ordinal notions of dissimilarity

An obvious alternative to a cardinal measure of dissimilarity is an *ordinal*, or qualitative, one which enables the formulation of statements like "w is more dissimilar from z than x is from y" but which does not enable a further quantification of these statements. In particular, statements like "the dissimilarity between w and z is twice that between x and y" have *no* meaning in an ordinal theory of dissimilarity.

A simple ordinal notion of dissimilarity is the crude one implicitly assumed in the partitioning of living organisms in species performed in biology: Two objects are either similar (say if they belong to the same species) or dissimilar (if they belong to different species), with no other intermediate possibilities.

Assuming the existence of such a dichotomous ordinal notion of dissimilarity, Pattanaik and Xu (2000) have proposed an axiomatic characterization, that shall not be detailed here, of a particular criterion for comparing sets on the basis of their diversity. According to their criterion, the diversity of a set can be defined by the number of elements in the *smallest partition* of the set into subsets of similar objects. Applied to the biodiversity context, where the objects are living organisms, and where living organisms are similar if and only if they belong to the same species, Pattanaik and Xu (2000) criterion amounts to evaluating diversity by counting the number of species.

While the species counting criterion has already been characterized in section 2 in a framework where the objects where interpreted as species (and not as objects), the characterization proposed by Pattanaik and Xu (2000) is significantly different. This reemphasizes the importance of specifying properly the nature of the objects that are involved in appraising diversity. If objects are interpreted as species right from the beginning, the characterization of the criterion of species counting is obtained quite straightforwardly from the principles of general independence and indifference between non-diverse situations. If objects are interpreted as living individuals, and are grouped into species on a basis of a dichotomous notion of similarity (living individuals are similar if and only if they belong to the same species), then the principles that characterize the criterion of species counting are quite different.

If one wants to introduce a finer, but yet ordinal, notion of dissimilarity into the picture, it is important to go beyond the dichotomous notion of dissimilarity considered in Pattanaik and Xu (2000).

A general way to describe an ordinal notion of dissimilarity is through the formalism of a *quaternary* relation Q on X which can, alternatively, be viewed as a binary relation on the Cartesian product $X \times X$. In this light, the statement $(w,\ z)\ Q\ (x,\ y)$ is interpreted as meaning "w is at least as dissimilar from z than x is from y". Asymmetric ("strictly more dissimilar than") and symmetric "equally similar" factors of Q can also be defined in the usual fashion and are denoted Q_A and Q_S respectively. To motivate further this interpretation, it is natural to assume that, for every distinct objects x and $y \in X$, both $(x,\ y)\ Q_A\ (x,x)$ and $(x,x)\ Q_S\ (y,y)$ hold (that is, two distinct objects are always strictly more dissimilar than any of the two objects in isolation, and pairs of identical objects are just equally similar). We assume also that Q is *symmetric* in the sense that $(x,y)\ Q\ (y,x)$ holds for every objects x and y and, as a binary relation on $X \times X$, is reflexive, complete and transitive. All these properties of Q would clearly hold true if, like VanHees (2004) or Weitzman (1992), we would accept to go as far as measuring the dissimilarity by a (cardinally significant) distance function (as distances are conventionally assumed to be symmetric and to satisfy $d(x,y) > d(x,x) = 0$ for all distinct x and y). Conversely, any Q satisfying these properties can be represented numerically by a distance function, which can even be chosen in such a way that it satisfies the triangle inequality.[7] Obviously no cardinal significance is attached to a distance function representing a particular quaternary relation as any monotonic transformation of such a distance function is also a numerical representation of the quaternary relation. In particular, a property such as link indifference, which requires that meaning be attached to distance summations, can not be formulated here because the value of a sum is not preserved if a monotonic transformation is applied to the terms of the sum.

What properties can a ranking of sets of objects be required to satisfy in such a setting ?

A principle that is difficult to avoid here is dissimilarity monotonicity, according to which the information on the underlying dissimilarities between the objects is the only one that should be used for ranking singletons and pairs. This principle has been formulated above in terms of a distance function. Yet it is clearly ordinal in nature in the sense that if holds for a particular distance function, it holds also for all monotonic transformation of this distance function. Another highly plausible principle is weak monotonicity.

A third principle has been proposed by Bervoets and Gravel (2007). This principle, to be referred to as *robustness of domination*, requires roughly the domination of a set by another to be robust to the addition, in the dominated set, of options when the options added are themselves dominated in terms of diversity. Specifically, the principle requires that if adding *separately* objects taken from some sets C and D to a set B is insufficient to reverse the domination of the set B by some other set A, then adding *jointly* the objects in C and D to B should also be insufficient to reverse the domination if the added objects are themselves considered less diverse than those contained

[7] See Suppes, Krantz, Luce, and Tversky (1989), p. 162, theorem 1 for a proof of this somewhat surprising fact that any quaternary relation satisfying the aforementionned properties can be represented numerically by a distance function satisfying the triangle inequality.

in A. I formulate this robustness of domination principle as follows.

Axiom 14 *(robustness of domination) For every sets of objects A, B, C and D such that $B \cap C = B \cap D = C \cap D = \emptyset$, if $A \succeq (B \cup C)$, $A \succeq (B \cup D)$ and $A \succeq C \cup D$ then $A \succeq (B \cup C \cup D)$ and if $A \succ (B \cup C)$, $A \succ B \cup D$ and $A \succ (C \cup D)$, then $A \succ (B \cup C \cup D)$.*

A ranking of sets that satisfies the principles of dissimilarity monotonicity, weak monotonicity and robustness of domination is that which compares sets on the basis of their two most dissimilar objects: set A is at least as diverse as set B if the two most dissimilar objects in A are at least as dissimilar as the two most dissimilar objects in B. Let me refer to this ranking as to the "maxi-max" criterion. Such a ranking is certainly extreme in that it only focuses on the two most dissimilar objects to appraise the diversity of a set. For example, in a biological context, with the objects interpreted as living organisms, and using the crude ordinal notion of dissimilarity according to which two living organisms are either similar (if they belong to the same species) or dissimilar (if they belong to different species), the maxi-max ranking would consider all ecosystems in which at least two species are represented to be equally diverse, and will consider any such ecosystem to be more diverse than an ecosystem with only one species represented, irrespective of the number of individuals in each species. Obviously finer rankings could be obtained from the maxi-max criterion if finer notions of dissimilarity were used. Yet it is clear that the maxi-max ranking is rather coarse in that it refuses to recognize any contribution to diversity brought about by pairs of objects that are not maximally dissimilar.

The maxi-max ranking, it should be mentioned, violates all independence principles discussed in section 2, including of course contraction consistency. This can be seen easily from the restaurant example considered earlier. Suppose, plausibly, that the notion of meals dissimilarities that applies to this example is given by:

$$(beef, paneer)\ Q_A\ (lamb, paneer)\ Q_A\ (chicken, paneer)\ Q_A$$
$$(lamb, chicken)\ Q_S\ (beef, chicken)\ Q_A\ (lamb, beef)$$

In that case, the maxi-max criterion would consider that adding a beef meal to a restaurant offering lamb and chicken dishes does not enlarge diversity because it does not affect the dissimilarity of the two most dissimilar meals (that would still be lamb and chicken, or the equally dissimilar beef and chicken). On the other hand, and contrary to the requirement of contraction consistency, adding beef to a restaurant offering chicken, lamb and paneer dishes enlarges diversity as appraised by the maxi-max criterion because such addition increases the dissimilarity of the two most dissimilar dishes (lamb and paneer before the introduction of beef, beef and paneer after the introduction of beef).

Are there other rankings of sets than the maxi-max that satisfy the three principles of dissimilarity monotonicity, weak monotonicity and robustness of domination ? The answer to this question is, unfortunately, negative as the following theorem has been proved in Bervoets and Gravel (2007).

Theorem 5 *A transitive and reflexive ranking* $\succeq$ *of all non-empty subsets of a finite set* X *satisfies weak monotonicity, robustness of domination and dissimilarity monotonicity with respect to some ordinal notion of similarity* Q *if and only if* $\succeq$ *is the maxi-max criterion.*

A somewhat natural direction for extending the maxi-max criterion is the *lexicographic* one. Such an extension, called the lexi-max criterion in Bervoets and Gravel (2007), would rank sets by looking first, like the maxi-max criterion, at their two most dissimilar options. Yet, and contrary to the maxi-max criterion, in case of equal dissimilarity between the two most dissimilar options in the two sets, the lexi-max criterion would move on to the comparisons of the second most dissimilar two objects in the sets and, if a tie is also obtained there, to the third most dissimilar objects and so on. The lexi-max criterion would sequentially perform the dissimilarity comparisons from the most dissimilar to the least dissimilar pair of options until a strict ranking is obtained or, if the two sets have the same cardinality and the same dissimilarity between all their pairs of distinct objects ordered from the most dissimilar to the least dissimilar, by considering the sets as equally diverse. This lexi-max criterion satisfies strong monotonicity (if the underlying quaternary relation considers two distinct objects to be strictly more dissimilar than one object and itself) and dissimilarity monotonicity but violates robustness of domination. Axioms that characterize the lexi-max criterion have been identified in **?** and will not be commented on here.

The gain in discriminatory power brought about by the lexi-max criterion, as compared to the maxi-max one, is probably best seen in the simple case of the crude ordinal notion of dissimilarity underlying the partitioning of the living organisms in species. While the maxi-max ranking would consider in this case all ecosystems with at least two species to be equally diverse, the lexi-max would value both the addition of individuals within existing species and the addition of individuals from other non-existing species (if they are at least two species represented). Again, more interesting ranking of sets could be obtained if the lexi-max criterion was based on a finer notion of dissimilarity.

If the gain of sensitivity with respect to less than maximally dissimilar options of a set obtained in moving from the maxi-max to the lexi-max criterion is not negligible, it is not great either. In fact, one can say that both the maxi-max *and* the lexi-max criteria give a "veto power" to the two most dissimilar options in a set. This veto power is somewhat extravagant, since it prevents any possibility for a large number of significant increases in the dissimilarities of not maximally dissimilar objects to beat an arbitrarily small reduction in the dissimilarity of the two most dissimilar objects. Would it not be possible to construct ranking of sets based on ordinal notion of dissimilarity that allow for smoother trade off between the various pairwise dissimilarities of the objects?

It would be nice to have a positive answer to this question. Yet such an answer is not, for the moment, available. An interesting ranking of sets that allows for trade-offs between the varying dissimilarities of the objects is one which *could be thought* of as defining the diversity of a set by the sum of the distances between all pairs of its elements, for some distance function representing numerically the underlying dissimilarity quaternary relation. Identifying the properties satisfied by such a ranking is clearly a worthwhile objective for future research in the area. While any such ranking will obviously satisfy dissimilarity monotonicity and weak monotonicity, it will also have to satisfy other properties. For instance, assume that we are interested in ranking sets of transportation modes taken from the universe $\{bicycle, car, motorcycle, scooter, walk\}$ and that the underlying dissimilarity ranking of pairs of modes of transportation is (from the more to the less dissimilar):

$$\begin{array}{l}(car, walk)\\(motorcycle, walk)\\(bicycle, car)\\(bicycle, motorcycle)\\(scooter, walk)\\(bicycle, scooter)\\(bicycle, walk)\\(car, scooter)\\(car, motorcycle)\\(motorcycle, scooter)\end{array}$$

Consider now a diversity ranking $\succeq$ of sets that generates the following comparisons between sets of modes of transportation:

$$\{motorcycle, walk\} \succ \{bicycle, car, motorcycle\} \tag{8}$$
$$\{car, motorcycle, scooter\} \succ \{car, walk\} \tag{9}$$

as well as the obvious rankings of pairs of mode of transportation induced by the comparisons of their dissimilarity:

$$\{bicycle, car\} \succ \{motorcycle, scooter\} \tag{10}$$
$$\{bicycle, motorcycle\} \succ \{car, scooter\} \tag{11}$$
$$\{car, walk\} \succ \{motorcycle, walk\} \tag{12}$$

It is easy to see that this ranking is perfectly compatible with weak monotonicity and dissimilarity monotonicity. Yet such a ranking, which violates robustness of domination, can *not* be thought of as resulting from the comparisons of sums of distances representing numerically the underlying ordinal notion dissimilarity. For, if the ranking could be thought of as resulting from such sums, there would be a distance function $d : X \times X \to \mathbb{R}_+$ representing numerically the quaternary relation such that the statements (8) to (12) would write:

$$d(motorcycle, walk) > d(bicycle, car) + d(bicycle, motorcycle) + d(car, motorcycle) \tag{13}$$

$$d(car, motorcycle) + d(car, scooter) + d(motorcycle, scooter) > d(car, walk) \tag{14}$$

$$d(bicycle, car) > d(motorcycle, scooter) \tag{15}$$

$$d(bicycle, motorcycle) > d(car, scooter) \tag{16}$$

$$d(car, walk) > d(motorcycle, walk) \tag{17}$$

Yet summing inequalities (13) to (16) and simplifying leads to:

$$d(motorcycle, walk) > d(car, walk)$$

in contradiction with (17). Identifying precisely the properties satisfied by a ranking of sets that can be though of as resulting from a summation of distance is therefore an open, if not difficult, problem.

While the formalism of quaternary relations appears natural for describing ordinal information on dissimilarity, it is by no means the only conceivable one. The formalism of quaternary relations is natural because it represents the "ordinal" side of the formalism of cardinally meaningful distance functions. Any distance function generates a quaternary relation and, conversely, any quaternary relation satisfying the properties discussed above can be represented numerically by a distance function. Moving from distance functions to quaternary relations is therefore a minimal way of weakening the precision of the information assumed to be available on the underlying dissimilarities between the objects.

It is however possible to weaken the precision of this information to a more significant extent. For instance, Nehring (1997) has explored the possibility of describing the qualitative information on the object dissimilarities in terms of a *ternary* relation. A ternary relation enables one to express statements of "in betweenness" between one object and two other objects. For instance, with a ternary relation, it is possible to say things such as "y is closer to x than z is" but it is not possible to say things like "y is closer to x than z is to w" (unless of course $w = x$). Hence the information on dissimilarity expressed by a ternary relation is significantly less precise than that expressed by a quaternary relation. Are there contexts in which the available information on the objects dissimilarity can take the form of a ternary relation but not of a quaternary relation ? One such context, considered by Nehring (1997), is when objects are assumed to have various attributes. In such a framework, the attributes of the objects naturally define a ternary notion of similarity between objects in the following way: y is closer to x than to z is if all attributes possessed by both x and z are also possessed by y. Yet such a framework does not generate naturally a notion of similarity that takes the form of a quaternary relation. Nehring (1997) has proved interesting results concerning the duality that exists between a ternary relation satisfying certain properties and a family of subsets of X, interpreted as attributes.

The duality is clear at an abstract level. For any statement of the type "y is closer to x than to z", I can define a family of sets (attributes) such that that every time a set in this family contains x and z, it also contains y. Applied, with some care, to all triples of objects over which the ternary relation tells something, this leads to the definition of a family of subsets which can be interpreted as a "multi-attributes" representation of the ternary relation. Conversely, a family of subsets (attributes) of X induces a ternary relation by considering that y is closer to x than to z if any set in the family containing x and z also contains y. Results of Nehring (1997) state that some properties of the ternary relation, such as reflexivity, symmetry and transitivity could be translated into properties of the family of attributes and vice-versa.

Yet it is fair to say that the interest of expressing judgements about objects dissimilarity in the form of a ternary, rather than a quaternary, relation is *derived* from some underlying attribute structure, rather than being *intrinsic*. Without assuming an attribute structure, it may seem rather unusual to express judgements of dissimilarity between objects *only* in the terms of statements involving triples of objects. Why should I be able to compare the similarity between a trout and a salmon with that between a salmon and a shark but unable to compare the dissimilarity between a trout and a salmon to that between a shark and a whale?

5 Diversity as the valuation of realized attributes

In recent years, Nehring and Puppe (2002) (see also Nehring and Puppe (2003)) have forcefully argued in favour of defining the diversity of a set as the sum of the values of the attributes realized by the objects in the set. As discussed earlier, this approach rides on the categorization of the objects into a certain number of attributes (being a mammal, being an ocean living animal, being an air breathing animal, having a specific ancestor, etc.). These attributes are naturally modeled as sets of objects having the considered attributes. Hence, an attributes structure is a collection, $\mathcal{A}$ say, of subsets of X. As discussed in section 2, Nehring and Puppe have proposed to rank every sets A and B by:

$$A \succeq B \Leftrightarrow \sum_{C \in \mathcal{A}: A \cap C \neq \emptyset} \lambda(C) \geq \sum_{D \in \mathcal{A}: B \cap D \neq \emptyset} \lambda(D) \tag{18}$$

for some family $\mathcal{A}$ of attributes and some non-negative valuation function $\lambda : P(X) \to \mathbb{R}_+$ of the attributes. By theorem 2 of section 2, any ordering satisfying contraction consistency and weak monotonicity can be viewed as being generated, in the sense of (18), by a valuation function λ assigning positive values to attributes in some family.[8] Yet, as also discussed in section 2, this result can not really be seen as a justification in favour of measuring diversity by formula (18). For there are typically *different* families of relevant attributes and *different* attribute valuation functions that can rationalize, in the sense of (18) a *given ordering* of sets satisfying contraction consistency and weak monotonicity. When confronted with this wealth of attributes structures and valuation functions, which one should the diversity appraiser choose ? Much more information is therefore required on the diversity ordering of sets to define the family of attributes and the valuation function underlying (18) uniquely.

Leaving this issue aside, an important contribution of Nehring and Puppe (2002) was to examine the implications, for the form of the diversity indicator corresponding to (18), of various assumptions that can be made about the structure of the family $\mathcal{A}$ of attributes. Quite clearly, if this family is very large (for instance any set of objects is a positively valued attribute), the multi-attributes approach to diversity evaluation is likely to be rather cumbersome to apply. At the other extreme, if the family of attributes is only made of singletons (each object is itself an attribute and does not have any other attribute than itself), the ranking induced by (18) becomes a member of the additive family of rankings discussed in section 2 (with each object being valued by a non-negative number).

Most of the properties of λ considered in Nehring and Puppe (2002) are expressed in terms of the (pseudo) distance function δ induced by the difference in diversity, as measured by the diversity indicator corresponding to (18), between pairs and singleton. That is, for a given family $\mathcal{A}$ of attributes and a value function λ generating (18), define:

$$I_d(S) = \sum_{A \in \mathcal{A}: A \cap S \neq \emptyset} \lambda(A) \tag{19}$$

[8] In Nehring and Puppe (2002), the axiomatic justification given to formula (18) is a bit different because it is formulated within a framework in which one is interested in ranking *lotteries* whose consequences are sets of objects rather than *sets of objects*. As argued in introduction, I have decided in this survey to stick to the basic question of "what is diversity ?" rather than to that of "how to rank decisions with uncertain consequences on diversity ?" Theorem 2 of section 2 is the analogue of theorem 2.1 in Nehring and Puppe (2002) in a framework without uncertainty.

for every set S and:

$$\delta(x,y) = I_d(\{x,y\} - I_d(\{y\} = \sum_{\{C\in\mathcal{A}:x\in C \text{ and } y\notin C\}} \lambda(C)$$

for every two objects x and y. Hence, when appraised by the function δ, the distance of x from y is defined as the gain of diversity brought about by the addition of x to y. Because of (18), this distance can be defined as the sum of the values of all attribute that are possessed by x and not possessed by y. While the function δ just defined is non-negative and satisfies $\delta(x,x) = 0$, it is not necessarily symmetric. There is indeed no reason for the gain of dissimilarity brought about by adding x to y to be the same from the gain of dissimilarity brought about from adding y to x. Hence δ is not really a distance function and is sometimes referred to as a pseudo-distance.[9] The pseudo-distance δ will be a distance, and therefore symmetric, if and only if $I_d(\{x\}) = I_d(\{y\})$ for all objects x and y. This case is *a priori* unlikely in a multi-attributes framework. Why would the sum of the values of all attributes possessed by an object be the same for all objects ?

A structure of attributes that plays a role in biology is that described by Nehring and Puppe (2002) under the heading of *taxonomic hierarchy*. This structure is what underlies the grouping of individual organisms into species, themselves grouped into genus, which can be grouped into families and so on. This sequential grouping of attributes imposes a *hierarchical* structure on the family of attributes in the sense that if two sets of attributes have a non-empty intersection, then one set must be a subset of the other (for example the set of chimpanzees is a subset of the set of primate). Nehring and Puppe (2002) have shown (theorem 3.1) that requiring the family $\mathcal{A}$ of attributes to have a hierarchical structure is equivalent, within the framework of diversity appraisal provided by (18), to requiring the diversity indicator (19) to satisfy, for every set S, and every object x, the condition:

$$I_d(S \cup \{x\}) - I_d(S) = \min_{y\in S} \delta(x,y) \tag{20}$$

Hence, hierarchical families of attribute have the property that the diversity of sets, provided that it is appraised by the formula (18), can be seen as a resulting from a sequential aggregation of the (pseudo) distance between its element. The recursive formula (20) is, actually, quite reminiscent of Weitzman's method discussed in the preceding section.

Another structure that can be imposed on the family of attributes is to be made of sets that are *interval* with respect to some underlying ordering of the universal set of objects. For instance one could think of objects as political opinions which are completely ranked on a left-right scale, or as animals ranked in terms of their average mass, or life expectancy. In this

[9]It is on the other hand easy to verify that δ satisfies the triangle inequality.

setting, an attribute is an interval with respect to the ordering if, any time it contains two objects, it also contains all objects that are "in between" those two objects for the given ordering. Examples of such an interval attributes in the context of political opinions expressed on a left-right scale would be, "being leftist" (as defined with respect to some "center") or "being more right-wing than some reference point". On the other hand the attribute "being extremist", in the sense of being either far to the right, or far to the left, would not be part of an interval attribute structure in such a setting.

Another result of Nehring and Puppe (2002) (theorem 3.2) is that requiring the structure of attributes to be made of intervals with respect to an ordering of objects is equivalent, again in the approach to diversity appraisal provided by (18), to requiring the function I_d to write:

$$I_d(\{x_1, x_2, ..., x_m\}) = I_d(x_1) + \sum_{i=2}^{m} \delta(x_i, x_{i-1})$$

for every objects $x_1, ..., x_m$ increasingly ordered by the underlying ordering of objects. Here again, it happens to be possible to express, within the realm of the multi-attributes approach to diversity associated to formula (18), diversity judgements as aggregation of pairwise dissimilarities between the objects as measured by the pseudo-distance δ.

Obviously, this possibility of expressing diversity comparisons as aggregations of pairwise dissimilarities measured by the (pseudo) distance function δ is closely related to the assumptions made on the underlying attributes structure (for instance the fact that they are taxonomic hierarchies or interval). Other attribute structures may not lead to an appraisal of diversity as aggregation of dissimilarities. For instance, Nehring and Puppe (2002) have shown that if the attributes can be described by a vector of zeros and ones, each component of which being interpreted as a dichotomous indicator of whether or not the object possesses a property, then the diversity of a set can *not* be expressed as an aggregation of the pairwise dissimilarities, as appraised by δ, between its elements. Attributes that have this "zero-one" structure, which can be geometrically depicted as an hypercube of length 1, are not rare in economic theory of differentiated products. In the hedonic housing price literature for instance, initiated by the work of Rosen (1974), housing are very often described by 0-1 variables: It has or not a main exposition to the south, it has or not an equipped kitchen, etc.

Nehring and Puppe (2002) have identified a property of the attributes structure that is necessary and sufficient for the value of an diversity indicator, as defined by (19), to depend *only* upon the information on the pairwise dissimilarities between the elements (in the sense that two sets which generates the same list of (pseudo) distances between their elements as measured by δ should be considered equivalent). The property of the attribute structure identified by Nehring and Puppe (2002) is the absence of triples of objects that have the properties that each subset of the triple possesses an attribute of its own. Consider for instance the interval attribute structure discussed above in the context of the political left-right scale. Take any set of three political opinions in this framework. The three elements of this set can by definition be ordered on the left-right scale. No matter what are the interval attributes that have been defined, it is clear that the two "extreme" political opinions of the triple can not have an attribute that the "central" opinion does not have if the attributes are intervals. Hence, in the interval attribute structure, no triple of objects has the property that a specific attribute can be assigned to every of its subsets. As shown by Nehring and Puppe (2002) (theorem 4.1), this property happens to be necessary and sufficient for the ability to express diversity judgements, as per the indicator I_d defined by (19), as aggregation of pairwise dissimilarity as measured by δ.

While it is beyond the scope of this survey to provide a detailed description of all the results contained in Nehring and Puppe (2002), there are two comments that are worth making about the multi attributes approach to diversity appraisal proposed by these authors. First, in its current state of development, it requires, from the part of the appraiser, some information on both the relevant attributes to be considered and the (cardinally meaningful) valuation function of these attributes. As discussed in section 2 and above, this information is required because it can not been obtained uniquely from the mere property of the ranking of sets, even if one accepts the axioms of contraction consistency and weak monotonicity. For many practical problems of diversity appraisal, this information is likely to be difficult to obtain.

Second, from a more philosophical point of view, and except for some degenerate case, the additive multi-attribute approach described in this section is bound to lead to the conclusion that some singletons will have more diversity than sets containing many objects. While there is no formal problem with this, it is a feature of the approach that may hurt several intuitions about the very meaning of diversity. How can *one* object be more diverse than *several* objects ? The answer to this question is of course straightforward within the multi-attribute approach. A single object will be more diverse than a collection of many objects if the attributes possessed by the single object have a larger value than the attributes that are realized in the collection of many objects. This answer suggest however that, in the multi-attribute approach, it is not so much the diversity of a set that is evaluated but, rather, the attributes possessed by the objects. Are the two notions - set diversity, and attributes possessed by the objects - synonymous ? I remain unconvinced that the answer to this question should always be positive.

6 Conclusion

What can be concluded from the attempts, summarized in this chapter, of defining diversity ? Beside the general principles mentioned in section 2, we have compared the following four categories of approach:

1) The biologically inspired approach, that defines diversity as generalized entropy.

2) The approaches that define diversity as an aggregation of a cardinally meaningful information about the pairwise dissimilarities objects conveyed by a distance function.

3) The approaches that define diversity as an aggregation of an ordinal information about pairwise dissimilarities between objects conveyed by a quaternary relation.

4) The approaches that define the diversity of a set as a sum of the values of the attributes that are realized by the objects in the set.

Given the current state of justification and elaboration of all these approaches, I would feel inclined to give my preference to approach 3). The generalized entropy approach to diversity appraisal suffers clearly from the fact that it treats all species symmetrically. Yet it seems that an essential ingredient of any plausible notion of diversity is that it should not treat grouping objects as being symmetric and should account for the varying dissimilarities that exist between objects.

The approaches 2, 3 and 4 can be viewed as alternative ways of introducing this dissimilarity. Approach 2, which requires the diversity appraiser to have available a cardinally meaningful distance function, strikes me as being very demanding from an informational point of view, even for discipline as sophisticated and developed as biology. Since judgements about the proximity between the objects are likely to be difficult to make, it appears safer to require them to take the milder form of an ordinal ranking of the various pairs of objects in terms of their proximity, and to base the diversity evaluation on the information conveyed by this ordinal ranking.

The multi-attributes approach is also a promising way for apprehending diversity. Yet, as it has been developed so far, it requires a rather demanding selection of the relevant attributes of the objects as well as the evaluation of each of these attributes by a cardinally meaningful valuation function. Of course, theorem 2 of this chapter tells us that such an attribute structure, as well as the valuation function, can be revealed from the ranking on the sole basis that it satisfies the properties of weak monotonicity and contraction consistency. But there are many collections of attributes, and many rankings of these attributes by a valuation function, that can be revealed by a given ordering. Without a criterion for selecting the "right" collection of attributes, and the "right" valuation function, the ordinal representation theorem 2 can not, by itself, provide a justification in favour of the multi-attribute approach. Given our current state of knowledge, the only available way to apply the additive multi-attributes approach to diversity is to select both the family of relevant attributes and the valuation function of the attributes. The informational burden put on the diversity appraiser for making this selection is, in my view, quite significant.

The relative independence between the additive multi-attributes approach of Nehring and Puppe (2002) on the one hand and the aggregation of ordinal dissimilarity approach on the other is worth noticing. For some structure of the attributes, and as shown by Nehring and Puppe (2002), it is impossible to view diversity appraisal as resulting from the aggregation of pairwise dissimilarities. Conversely, there are diversity rankings of sets resulting from the aggregation of an ordinal notion of pairwise dissimilarities which violate the property of contraction consistency and which, for this reason, can not be thought of as resulting from comparing the sum of the values of attributes. The somewhat radical difference of these two approaches make the choice between them uneasy. In my view, the small information required by an approach based on the aggregation of an ordinal notion of dissimilarity represents a clear advantage.

Of course, this ordinal dissimilarity approach needs to be further developed. The rankings to which it has led so far, that is the maxi-max and the lexi-max criteria, are far too extreme. It is therefore important to develop other criteria for ranking sets based on an ordinal notion of pairwise dissimilarities.

A priority for future research in the area would be to explore other approaches for defining diversity than those, considered in approach 2 and 3, of defining diversity as "aggregate pairwise dissimilarity". Suppose that we are interested in comparing the diversity of car models offered by various retailers and that we adopt the hedonic perspective of viewing a car as a combination of values taken by, say, k characteristics (such as size, degree of comfort, speed, fuel consumption, etc.) numerically measurable. This amounts to thinking of a model of car as to a point in $\mathbb{R}^k$ and to a car retailer as a to a (finite) set of points in $\mathbb{R}^k$. A reasonably natural notion of dissimilarity between cars in this perspective could be given by the ranking of pairs of points (cars) induced by the comparisons of the Euclidian distances between them. Furthermore, an equally plausible notion of diversity of car retailers in this setting could be given by the ranking of set of points (retailers) induced by the comparison of the dimension of the subspace spanned by these sets of points. Yet it is clear that this dimension can not be deduced from the information of the distance between these points alone.[10] It would therefore be nice to find the axiomatic properties of a ranking of sets which would characterize the fact that this ranking could be

[10] I am indebted to Jean-François Laslier for the mathematical backbone of this example. Notice that this example is valid no matter what is the distance notion used. One can not recover the dimension of the subspace spanned by using only information on the pairwise distance between these points no matter what this distance is.

thought of as resulting from the aggregation of the pairwise dissimilarity of its elements for some underlying dissimilarity quaternary relation.

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