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THE CLASS OF ADDITIVELY DECOMPOSABLE INEQUALITY MEASURES

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An additively decomposable inequality measure is one which can be expressed as a weighted sum of the inequality values calculated for population subgroups plus the contribution arising from differences between subgroup means. The paper derives the entire class of measures which are additively decomposable under relatively weak restrictions on the form of the index. The subclass of mean independent measures turns out to be a single parameter family which includes the square of the coefficient of variation and two entropy formulae proposed by Theil.

1. INTRODUCTION

A TYPE OF QUESTION frequently encountered in the analysis of income inequality concerns the extent to which inequality in the total population can be attributed to income differences between major population subgroups. For instance, we may be interested in the quantitative significance of income variations associated with age, sex, race, occupation, the level of education, and so on.[2]

When considering these kinds of questions the advantages of decomposable inequality measures are often mentioned. The entropy formula popularized by Theil is the best known example. If $\mathbf{y}=(y_1, \ldots, y_n)$ is the income distribution vector for a population of n individuals, the Theil index can be written[3]

$$(1)\qquad T(\mathbf{y}; n)=\frac{1}{n}\sum_i \frac{y_i}{\mu}\log\frac{y_i}{\mu}$$

where μ is the mean income $\Sigma_i y_i/n$. Partition the population into G disjoint subgroups where subgroup g consists of $n_g(\geqslant 1)$ individuals with the distribution vector $\mathbf{y}^g=(y_1^g, \ldots, y_{n_g}^g)$ and mean μ_g. Then, using the fact that T is symmetric over $\mathbf{y}$,

$$(2)\qquad \begin{aligned} T(\mathbf{y}; n)&=T(\mathbf{y}^1, \mathbf{y}^2, \ldots, \mathbf{y}^G; n)\\ &=\frac{1}{n}\sum_g\sum_{i=1}^{n_g}\frac{y_i^g}{\mu}\log\frac{y_i^g}{\mu}\\ &=\sum_g \frac{n_g\mu_g}{n\mu}T(\mathbf{y}^g; n_g)+\frac{1}{n}\sum_g n_g\frac{\mu_g}{\mu}\log\frac{\mu_g}{\mu}.^4 \end{aligned}$$

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[2] Similar questions can be raised concerning inequality of wealth holdings and the size distribution of firms.

[3] Theil [**11**, page 91]. The expression given here corresponds to his equation (1.3), and strictly speaking refers to "redundancy" rather than entropy.

[4] This corresponds to Theil [**11**, page 95, equation (1.9)].

The total inequality $T(y; n)$ can be expressed as the sum of a "within-group" inequality term and a "between-group" term, where the within-group contribution is itself a weighted sum of the sub-group inequality values. An inequality measure is said to be *additively decomposable* when it can be decomposed in this way.

Inequality comparisons are invariably sensitive to the choice of inequality index used since alternative measures tend to emphasize inequality at different points in the distribution. Replacing one index by another will therefore almost always change the relative significance of the between- and within-group terms. If we accept the Theil measure because of its attractive decomposition property, then at the same time we have to accept the weights it attaches to transfers between various points in the distribution. These weights may or may not correspond to those we believe to be desirable in an index. If we wish to retain the decomposition property and yet allow for different perceptions of inequality, we are led to enquire about the range of indices which are decomposable. It is this question to which this paper is directed.

The inequality value for a population of n individuals with distribution $\boldsymbol{y}$ is denoted by $I(\boldsymbol{y}; n)$. The following assumptions are made.

ASSUMPTION 1: $I(\boldsymbol{y}; n)$ is continuous and symmetric in $\boldsymbol{y}$.

ASSUMPTION 2: $I(\boldsymbol{y}; n) \geqslant 0$ with equality holding if and only if $y_i = \mu$ for all i.

Assumption 2 is simply the requirement that inequality attains its minimum value of zero when all individuals have the same income.

ASSUMPTION 3: $I(\boldsymbol{y}; n)$ has continuous first-order partial derivatives $I_j(\boldsymbol{y}; n)$.

The condition for additive decomposability may be written

$$(3) \qquad \begin{aligned} I(\boldsymbol{y}; n) &= I(\boldsymbol{y}^1, \ldots, \boldsymbol{y}^G; n) \\ &= \sum_g w_g^G I(\boldsymbol{y}^g; n_g) + B \end{aligned}$$

where w_g^G is the weight attached to subgroup g in a decomposition into G subgroups, and B is the between-group term, assumed to be independent of inequality within the individual subgroups. Making within-group transfers until $y_i^g = \mu_g$ in each subgroup and letting $\boldsymbol{u}_n$ represent the unit vector $(1, 1, \ldots, 1)$ with n components, we obtain

$$I(\mu_1 \boldsymbol{u}_{n_1}, \mu_2 \boldsymbol{u}_{n_2}, \ldots, \mu_G \boldsymbol{u}_{n_G}; n) = B$$

using Assumption 2 and the fact that B is invariant to such transfers. Similarly, the coefficients w_g^G may vary with the vector of subgroup means $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_G)$ and subgroup populations $\boldsymbol{n} = (n_1, \ldots, n_G)$, but are independent of the level of inequality within the subgroups. The additive decomposition assumption can then be stated as follows.

ASSUMPTION 4: Given a population of any size $n \geq 2$ and a partition into G non-empty subgroups, there exists a set of coefficients $w_g^G(\boldsymbol{\mu}, \boldsymbol{n})$ such that

$$I(\boldsymbol{y}^1, \ldots, \boldsymbol{y}^G; n) = \sum_g w_g^G(\boldsymbol{\mu}, \boldsymbol{n}) I(\boldsymbol{y}^g; n_g) + I(\mu_1 \boldsymbol{u}_{n_1}, \ldots, \mu_G \boldsymbol{u}_{n_G}; n) \tag{4}$$

for all $\boldsymbol{y}^1, \ldots, \boldsymbol{y}^G$.

It turns out that additive decomposability places very severe restrictions on the form that inequality measures can take. The entire class of functions satisfying Assumptions 1–4 is given below in equation (15). To derive this result, we first show (Section 2, Theorem 1) that the decomposition coefficients w_g^G have to be of the form given in equation (5). (Incidentally, this does not require the differentiability assumption (Assumption 3).) The proof is then completed in Section 3.

In Section 4 we demonstrate that all these indices satisfy the Pigou-Dalton principle of transfers and the following section investigates the circumstances under which the measures satisfy the "principle of population replication" (or population homogeneity). Section 6 identifies the subclass of measures which are mean independent (i.e., invariant to multiplication of the y_i by a positive scalar). This is a one parameter family, where the parameter can be taken to reflect variations in the perception of inequality (or different degrees of "inequality aversion"). Finally, Section 7 contains some remarks on the use of additively decomposable measures when examining the contribution of differences between subgroups to overall inequality.[5]

2. THE DECOMPOSITION COEFFICIENTS

THEOREM 1: *If $I(\boldsymbol{y}; n)$ satisfies Assumptions* 1, 2, *and* 4, *then there exists a set of functions $\theta(\mu, n)$ such that*

$$w_g^G(\boldsymbol{\mu}, \boldsymbol{n}) = \frac{\theta(\mu_g, n_g)}{\theta(\mu, n)}. \tag{5}$$

PROOF: Take any partition of the population and let subgroup 1 be any subgroup containing two or more individuals. If this subgroup has n_1 individuals with distribution $\boldsymbol{y}^1$ and mean μ_1, define $\boldsymbol{x}^1$ to be another distribution over n_1 individuals with the same mean μ_1 such that

$$I(\boldsymbol{y}^1; n_1) \neq I(\boldsymbol{x}^1; n_1). \tag{6}$$

[5] Some of the results in this paper have been derived independently by Bourguignon [**3**]. The analysis here is more general since Bourguignon assumes at the outset that the inequality measures satisfy population homogeneity and mean independence. He also limits attention to particular types of weighting coefficients w_g^G and concentrates on the two cases when these represent the population proportions in each group and the shares of aggregate income received by the groups. Other recent contributions on related issues include Cowell [**4**], who allows the members of different population subgroups to be treated differently in the evaluation of inequality, thus relaxing the symmetry requirement in Assumption 1; and Blackorby et al. [**2**], who argue that the between group term should reflect inequality within each subgroup as well as the mean incomes of the subgroups.

Assumption 2 ensures that a suitable choice of $\boldsymbol{x}^1$ can always be made.

The distributions $\boldsymbol{y}=(\boldsymbol{y}^1, \boldsymbol{y}^2, \ldots, \boldsymbol{y}^G)$ and $\boldsymbol{x}=(\boldsymbol{x}^1, \boldsymbol{y}^2, \ldots, \boldsymbol{y}^G)$ both have the same vector of subgroup means $\boldsymbol{\mu}$ and subgroup populations $\boldsymbol{n}$. Applying the decomposition equation (4) to both $\boldsymbol{x}$ and $\boldsymbol{y}$, we obtain

$$(7)\qquad \begin{aligned} I(\boldsymbol{y}; n)-I(\boldsymbol{x}; n) &= I(\boldsymbol{y}^1, \boldsymbol{y}^2, \ldots, \boldsymbol{y}^G; n)-I(\boldsymbol{x}^1, \boldsymbol{y}^2, \ldots, \boldsymbol{y}^G; n) \\ &= w_1^G(\boldsymbol{\mu}, \boldsymbol{n})[I(\boldsymbol{y}^1; n_1)-I(\boldsymbol{x}^1; n_1)]. \end{aligned}$$

The left-hand side of (7) does not change if we alter the way in which the population outside subgroup 1 is partitioned. Suppose $(\boldsymbol{y}^2, \ldots, \boldsymbol{y}^G)$ is regarded as a single subgroup. Then

$$(8)\qquad \begin{aligned} I(\boldsymbol{y}; n)-I(\boldsymbol{x}; n) &= I(\boldsymbol{y}^1, (\boldsymbol{y}^2, \ldots, \boldsymbol{y}^G); n)-I(\boldsymbol{x}^1, (\boldsymbol{y}^2, \ldots, \boldsymbol{y}^G); n) \\ &= w_1^2\left(\mu_1, \frac{n\mu - n_1\mu_1}{n-n_1}, n_1, n-n_1\right)[I(\boldsymbol{y}^1; n_1)-I(\boldsymbol{x}^1; n_1)]. \end{aligned}$$

Equating (7) and (8),

$$\begin{aligned} w_1^G(\boldsymbol{\mu}, \boldsymbol{n}) &= w_1^2\left(\mu_1, \frac{n\mu - n_1\mu_1}{n-n_1}, n_1, n-n_1\right) \\ &= \omega(n_1\mu_1, n\mu, n_1, n), \quad \text{say}, \end{aligned}$$

and in general

$$(9)\qquad w_g^G(\boldsymbol{\mu}, \boldsymbol{n}) = \omega(Y_g, Y, n_g, n)$$

where $Y_g = n_g\mu_g$ and $Y = n\mu$.

Now return to the distributions y and x and consider the 2-way partitions into $(\boldsymbol{y}^1, \boldsymbol{y}^2)$, $(\boldsymbol{y}^3, \ldots, \boldsymbol{y}^G)$ and $(\boldsymbol{x}^1, \boldsymbol{y}^2)$, $(\boldsymbol{y}^3, \ldots, \boldsymbol{y}^G)$, where $\boldsymbol{y}^2$ is the distribution in any arbitrary subgroup containing $n_2 \geqslant 1$ individuals. Comparing the two decomposition expressions and substituting (9) gives

$$(10)\qquad \begin{aligned} I(\boldsymbol{y}; n)-I(\boldsymbol{x}; n) = \omega(Y_1+Y_2, Y, n_1+n_2, n)[&I(\boldsymbol{y}^1, \boldsymbol{y}^2; n_1+n_2) \\ &-I(\boldsymbol{x}^1, \boldsymbol{y}^2; n_1+n_2)]. \end{aligned}$$

But, applying (8),

$$(11)\qquad \begin{aligned} &I(\boldsymbol{y}^1, \boldsymbol{y}^2; n_1+n_2)-I(\boldsymbol{x}^1, \boldsymbol{y}^2; n_1+n_2) \\ &\quad = \omega(Y_1, Y_1+Y_2, n_1, n_1+n_2)[I(\boldsymbol{y}^1; n_1)-I(\boldsymbol{x}^1; n_1)] \end{aligned}$$

so

$$(12)\qquad \begin{aligned} I(\boldsymbol{y}; n)-I(\boldsymbol{x}; n) &= \omega(Y_1+Y_2, Y, n_1+n_2, n)\omega(Y_1, Y_1+Y_2, n_1, n_1+n_2) \\ &\quad \times[I(\boldsymbol{y}^1; n_1)-I(\boldsymbol{x}^1; n_1)] \\ &= \omega(Y_1, Y, n_1, n)[I(\boldsymbol{y}^1; n_1)-I(\boldsymbol{x}^1; n_1)] \quad \text{from (8)}. \end{aligned}$$

Thus

$$\omega(Y_1, Y, n_1, n) = \omega(Y_1, Y_1+Y_2, n_1, n_1+n_2)\omega(Y_1+Y_2, Y, n_1+n_2, n).$$

Keeping $Y=\bar{Y}$ and $n=\bar{n}$ constant and defining

(13) $\theta(\mu_g, n_g)=\omega(n_g\mu_g, \bar{Y}, n_g, \bar{n})$

we obtain

$$\omega(Y_1, Y_1+Y_2, n_1, n_1+n_2)=\frac{\omega(Y_1, \bar{Y}, n_1, \bar{n})}{\omega(Y_1+Y_2, \bar{Y}, n_1+n_2, \bar{n})}$$
$$=\frac{\theta(\mu_1, n_1)}{\theta((n_1\mu_1+n_2\mu_2)/(n_1+n_2), n_1+n_2)}.$$

Thus, in general

$$w_g^G(\boldsymbol{\mu}, \boldsymbol{n})=\omega(Y_g, Y, n_g, n)=\frac{\theta(\mu_g, n_g)}{\theta(\mu, n)}.$$

This completes the proof whenever $n>n_g\geqslant 2$. If $n_g=1$ then $I(\boldsymbol{y}^g; 1)=0$. The coefficient w_g^G is therefore irrelevant and $\theta(\mu, 1)$ can be chosen arbitrarily.

REMARK: Had we imposed the extra restriction that the coefficients w_g^G sum to unity, Theorem 1 could be further strengthened. It can be easily demonstrated that if $w_g^G(\boldsymbol{\mu}, \boldsymbol{n})$ satisfies (5) and $\Sigma_g w_g^G=1$, then

(14) $$w_g^G(\boldsymbol{\mu}, \boldsymbol{n})=\frac{n_g(1-\lambda+\lambda\mu_g)}{n(1-\lambda+\lambda\mu)}$$

for some value of λ. When $\lambda=1$ we have "income share" weights, as in the Theil index decomposition equation (2). When $\lambda=0$ the weights correspond to the population shares in the corresponding subgroups.

3. THE CLASS OF ADDITIVELY DECOMPOSABLE INEQUALITY MEASURES

THEOREM 2: *$I(\boldsymbol{y}; n)$ satisfies Assumptions 1–4 if and only if*

(15) $$I(\boldsymbol{y}; n)=\frac{1}{\theta(\mu, n)}\sum_i[\phi(y_i)-\phi(\mu)]$$

where $\theta(\mu, n)$ is positive; $\theta_1(\mu, n)$ and $\phi'(\mu)$ are continuous; and $\phi(\cdot)$ is strictly convex.

PROOF: Let i, j be any two individuals and choose any partition in which $\boldsymbol{y}^1=(y_i, y_j)$. Let $\boldsymbol{x}^1=(\mu_1, \mu_1)$ where $\mu_1=(y_i+y_j)/2$. Then substituting (5) into (7) gives

$$I(\boldsymbol{y}; n)=I(y_i, y_j, \boldsymbol{y}^2, \ldots, \boldsymbol{y}^G; n)$$
$$=\frac{\theta(\mu_1, 2)}{\theta(\mu, n)}I(y_i, y_j; 2)+I(\mu_1, \mu_1, \boldsymbol{y}^2, \ldots, \boldsymbol{y}^G; n).$$

Differentiating with respect to y_i, y_j and subtracting, we obtain

$$(16)\qquad \theta(\mu, n)[I_i(\mathbf{y}; n) - I_j(\mathbf{y}; n)]$$
$$= \theta\left(\frac{y_i + y_j}{2}, 2\right)[I_1(y_i, y_j; 2) - I_2(y_i, y_j; 2)] = f(y_i, y_j), \quad \text{say,}$$

and

$$(17)\qquad f(y_i, y_j) + f(y_j, y_k) = \theta(\mu, n)[I_i(\mathbf{y}; n) - I_k(\mathbf{y}; n)]$$
$$= f(y_i, y_k) \quad \text{for all} \quad y_i, y_j, y_k.$$

If we define

$$\phi'(y_i) = f(y_i, 0)$$

then

$$(18)\qquad f(y_i, y_j) = f(y_i, 0) - f(y_j, 0) \quad \text{from (17)}$$
$$= \phi'(y_i) - \phi'(y_j).$$

Substituting (18) into (16) gives

$$\theta(\mu, n)I_i(\mathbf{y}; n) - \phi'(y_i) = \theta(\mu, n)I_j(\mathbf{y}; n) - \phi'(y_j)$$

or

$$(19)\qquad g_i(\mathbf{y}; n) = g_j(\mathbf{y}; n) \quad \text{for all } i, j$$

where

$$(20)\qquad g(\mathbf{y}; n) = \theta(\mu, n)I(\mathbf{y}; n) - \sum_{i=1}^{n} \phi(y_i).$$

From (19) we deduce that

$$g(\mathbf{y}; n) = \beta(\mu, n).$$

But

$$\beta(\mu, n) = g(\mu \mathbf{u}_n; n) = -\sum_{i=1}^{n} \phi(\mu).$$

Hence (20) becomes

$$I(y; n) = \frac{1}{\theta(\mu, n)}\left[\sum_i \phi(y_i) + \beta(\mu, n)\right] = \frac{1}{\theta(\mu, n)} \sum_i [\phi(y_i) - \phi(\mu)].$$

That $\theta_1(\mu, n)$ and $\phi'(\mu)$ are continuous follows from Assumption 4. Without loss of generality we may take $\theta(\mu, n) > 0$ for some $n \geq 2$. Assumption 2 implies that $\phi(\cdot)$ is strictly convex. It then follows from Assumption 2 that $\theta(\mu, n)$ is positive for all $n \geq 2$; and $\theta(\mu, 1)$ can be taken to be positive, since the coefficient w_1^G is always irrelevant.

This completes the necessity part of the proof. The demonstration of sufficiency is straightforward and is left to the reader. Note that in Assumption 4 the decomposition coefficients $w_g^G = \theta(\mu_g, n_g)/\theta(\mu, n)$ are all positive.

4. THE PRINCIPLE OF TRANSFERS

We now consider further properties that we should like the index to satisfy, beginning with the Dalton-Pigou *principle of transfers.* This can be stated as follows:

ASSUMPTION 5: If a transfer of $\Delta > 0$ is made from an individual with income y_j to another with income y_i, where $y_j - \Delta > y_i + \Delta$, then inequality decreases.

The condition says that a mean preserving transfer from richer to poorer must reduce inequality, an essential characteristic if the index is to be a measure of inequality. It could be argued that this condition should have been introduced at the outset. The assumption would, however, have been redundant since it *automatically holds* once the index satisfies Assumptions 1–4.

THEOREM 3: *All indices of the form given by* (15) *satisfy the principle of transfers* (*Assumption* 5).

PROOF: From (15), the change in inequality corresponding to a transfer of $\Delta > 0$ from individual j to individual i is given by

$$\Delta I = \frac{\phi(y_i + \Delta) - \phi(y_i) + \phi(y_j - \Delta) - \phi(y_j)}{\theta(\mu, n)}. \tag{21}$$

If $y_j - \Delta > y_i + \Delta$, strict convexity of $\phi(\cdot)$ ensures that the numerator of (21) is negative. Since $\theta(\mu, n)$ is positive, inequality decreases.

5. POPULATION HOMOGENEITY

Since $I(\mathbf{y}; n)$ is defined for different population sizes, we have been seeking a family of indices $I(\mathbf{y}; 2)$, $I(\mathbf{y}; 3)$ and so on. This would usually introduce the problem of specifying when an index defined over a population of k individuals was "the same type of index" as one defined for n individuals. Fortunately it has not been necessary to make such a decision, and the family of indices given in (15) are recognizable as being essentially the same for different population sizes, except for the variation of $\theta(\mu, n)$ with n.

The family resemblance of $I(\mathbf{y}; n)$ for different population sizes can be established more clearly if we introduce the principle of *population replication* (see, for example, Dasgupta *et al.* [**5**]):

ASSUMPTION 6: $I(\mathbf{y}, \mathbf{y}, \ldots, \mathbf{y}; rn) = I(\mathbf{y}; n)$ for any positive integer r.

This principle states that if r groups, each containing n individuals and having an identical distribution y, are aggregated into a single population of rn individuals, then aggregate inequality is the same as in each of the constituent groups.

THEOREM 4: *Indices of the form given in* (15) *satisfy Assumption* 6 *if and only if*

$$\theta(\mu, n) = n\,\alpha(\mu) \tag{22}$$

where $\alpha(\cdot)$ *is positive and differentiable.*

PROOF: From (15) we obtain

$$I(y; n) = \frac{1}{\theta(\mu, n)} \sum_{i=1}^{n} [\phi(y_i) - \phi(\mu)],$$

$$I(y, \ldots, y; rn) = \frac{r}{\theta(\mu, rn)} \sum_{i=1}^{n} [\phi(y_i) - \phi(\mu)].$$

Thus Assumption 6 holds if and only if

$$\theta(\mu, rn) = r\theta(\mu, n) \quad \text{for all } n \geqslant 2$$

or

$$2\theta(\mu, n) = \theta(\mu, 2n) = n\theta(\mu, 2)$$

which gives

$$\theta(\mu, n) = n\frac{\theta(\mu, 2)}{2} = n\alpha(\mu), \quad n \geqslant 2,$$

where $\alpha(\mu)$ is positive and continuously differentiable.

As already mentioned, $\theta(\mu, 1)$ can be defined arbitrarily, so to be consistent with (22) we can set $\theta(\mu, 1) = \alpha(\mu)$.

Under Assumption 6 the decomposition coefficients (5) become

$$w_g^G(\boldsymbol{\mu}, \boldsymbol{n}) = \frac{n_g \alpha(\mu_g)}{n\alpha(\mu)}.$$

From (14) we see that these sum to unity only when

$$\alpha(\mu) = a + b\mu \tag{23}$$

for some constants a, b.

6. MEAN INDEPENDENCE

A further condition often imposed on inequality measures is the property of *mean independence* or *income homogeneity*. This is the requirement that the value of the index remains unchanged when the y_i are all multiplied by the same positive scalar.

ASSUMPTION 7: $I(ky; n) = I(y; n)$ for all $k > 0$.

To derive the mean independent indices we shall need to slightly strengthen the differentiability condition on $I(\cdot)$.

ASSUMPTION 3′: $I(y; n)$ has continuous second derivatives $I_{ij}(y; n)$.

THEOREM 5: *$I(y; n)$ satisfies Assumptions 1, 2, 3′, 4, and 7 only if it has the form*

$$I(y; n) = \frac{A_n}{c(c-1)} \sum_i \left[\left(\frac{y_i}{\mu} \right)^c - 1 \right], \quad c \neq 0, 1,$$

or

$$(24) \qquad I(y; n) = A_n \sum_i \log \frac{\mu}{y_i}$$

or

$$I(y; n) = A_n \sum_i \frac{y_i}{\mu} \log \frac{y_i}{\mu}$$

where $A_n > 0$.

PROOF: Under Assumption 7 the index is homogeneous of degree zero in the y_i. Applying Euler's theorem to (15) gives

$$(25) \qquad \theta(\mu, n)[\sum_i y_i\phi'(y_i) - \mu\phi'(\mu)] - \mu\theta_1(\mu, n) \sum_i [\phi(y_i) - \phi(\mu)] = 0.$$

Subtracting the derivative of (25) with respect to y_t from the derivative with respect to y_s, we obtain

$$(26) \qquad y_s\phi''(y_s) - y_t\phi''(y_t) + \left[1 - \frac{\mu\theta_1(\mu, n)}{\theta(\mu, n)} \right] (\phi'(y_s) - \phi'(y_t)) = 0.$$

Since n and μ can be varied while y_s and y_t are kept constant, it follows that the term in square brackets in (26) is independent of μ and n. Hence

$$(27) \qquad \frac{\mu\theta_1(\mu, n)}{\theta(\mu, n)} = c, \quad \text{a constant,}$$

and

$$(28) \qquad \theta(\mu, n) = K_n\mu^c$$

where $K_n > 0$. Substituting (28) into (26) and rearranging, we obtain

$$y_s\phi''(y_s) + (1-c)\phi'(y_s) = y_t\phi''(y_t) + (1-c)\phi'(y_t)$$

which holds for all y_s, y_t. Hence

$$(29) \qquad y\phi''(y) + (1-c)\phi'(y) = b$$

where b is some constant.

The solution to the differential equation (29) can be written

$$\text{(30)} \qquad \phi(y)=\frac{A}{c(c-1)}y^c+By+C, \quad c\neq 0, 1,$$

$$\phi(y)=-A\log y+By+C, \quad c=0,$$

$$\phi(y)=Ay\log y+By+C, \quad c=1,$$

where A, B, C are arbitrary constants (but $A>0$ in order that $\phi(\cdot)$ be convex). The linear terms vanish when these functions are substituted into (15). Making use of (28) and combining the two arbitrary positive constants A and K_n so that $A_n=A/K_n$, we obtain the three alternative forms given in (24).

If the subclass of mean independent measures also satisfies population replication (Assumption 6) then

$$\theta(\mu, n)=K_n\mu^c=n\alpha(\mu)$$

from (22) and (28). Thus K_n is proportional to n and in (24) we may write $A_n=A/n$. For most purposes the precise positive value of A is of no significance and we can set $A=1$. The additively decomposable indices satisfying both mean independence and population replication therefore comprise a one parameter family whose members are identified by the value of c:

$$\text{(31)} \qquad I_c(y)=\frac{1}{n}\frac{1}{c(c-1)}\sum_i\left[\left(\frac{y_i}{\mu}\right)^c-1\right], \quad c\neq 0, 1,$$

$$I_0(y)=\frac{1}{n}\sum_i\log\frac{\mu}{y_i}, \quad c=0,$$

$$I_1(y)=\frac{1}{n}\sum_i\frac{y_i}{\mu}\log\frac{y_i}{\mu}, \quad c=1.$$

The square of the coefficient of variation corresponds to $c=2$. I_1 is the Theil index given earlier in equation (1). I_0 is another entropy expression suggested by Theil.[6]

All the indices given in (31) are defined for distributions of positive incomes and, as long as incomes are positive, satisfy the principle of transfers (Theorem 3).

[6] See Theil [**11**, pp. 126–7]. The set of measures given by (31) also includes monotonic transformations of the entire Atkinson [**1**] family of indices. For example,

$$1-\exp(-I_0)=1-\prod_i\left(\frac{y_i}{\mu}\right)^{1/n},$$

the form of Atkinson's index corresponding to a utilitarian social welfare function with logarithmic utility of income.

Indices corresponding to positive even integers are also defined (and satisfy the principle of transfers) when some incomes are negative.[7]

The additively separable form of (31) make the transfer properties of the indices particularly easy to investigate. If $T(y_s, y_t, \Delta)$ denotes the change in the value of the index corresponding to a transfer of Δ from an individual with income y_t to another with income y_s, then

$$T(y_s, y_t, \Delta) \simeq \begin{cases} \dfrac{\Delta}{n(c-1)\mu^c}[y_s^{c-1} - y_t^{c-1}], & c \neq 1, \\ \dfrac{\Delta}{n} \log \dfrac{y_s}{y_t}, & c = 1, \end{cases}$$

for small Δ, and the incomes of everyone apart from the donor and the recipient are irrelevant (except for their influence on the mean). As c decreases, the index becomes more sensitive to transfers lower down the distribution. For example, the square of the coefficient of variation gives roughly the same weight to a transfer of £10 from a person with £10,000 to another with £2,000 as a £1 transfer from someone with £100,000 to another with £20,000[8] since, if $c = 2$,

$$T(£2{,}000, £10{,}000, £10) \simeq T(£20{,}000, £100{,}000, £1).$$

When $c = 1$ (the Theil index),

$$T(£2{,}000, £10{,}000, £10) \simeq T(£20{,}000, £100{,}000, £10),$$

and when $c = 0$,

$$T(£2{,}000, £10{,}000, £10) \simeq T(£20{,}000, £100{,}000, £100).$$

As c declines further, the index requires larger and larger transfers at the top end to compensate for a given transfer lower down in the distribution. In the limit as $c \to -\infty$, the index focusses exclusively on the extreme of the lower tail and the associated inequality rankings of distributions correspond to those generated by Rawls' maximin criterion.

[7] While the lower bound on the range of I is always zero, the upper bound varies widely with the value of c. If all incomes are positive and $c > 0$, I_c has an upper bound given by

$$\bar{I}_c = \frac{1}{c(c-1)}(n^{c-1} - 1), \quad c > 0, c \neq 1,$$

$$\bar{I}_1 = \log n.$$

For this range of c the index can be appropriately normalized so that it takes values in the interval $[0, 1]$. These will have the linearly decomposable form (24), but not that of (31). Thus normalization is achieved at the cost of giving up the population replication condition (Assumption 6). If $c \leq 0$ the indices are unbounded above, even when all incomes are positive. These cannot be normalized without abandoning the property of additive decomposition.

[8] This rather perverse result illustrates why the coefficient of variation is extremely sensitive to changes in the upper tail of the distribution. In fact the transfer properties of indices corresponding to $c > 2$ become even stranger. Although they still satisfy the principle of transfers, they show little concern for equalization except among the very rich. This had led Kolm [7] and Love and Wolfson [**8**] to question whether they should not be eliminated from consideration as inequality measures, as would be the case if Kolm's "principle of diminishing transfers" were adopted.

7. CONCLUDING REMARKS

It has been demonstrated that the additive decomposability assumption imposes a severe restriction on the form of the inequality measure. Combining it with the mean independence and population replication conditions leads to the single parameter family indicated by (31). Let us now restrict attention to this set of measures and examine the implications for the decomposition of total inequality into the contribution of inequality within subgroups and that attributable to differences between groups.

The decomposition coefficients for these indices are given by

$$w_g^G(\boldsymbol{\mu}, \boldsymbol{n}) = \frac{n_g}{n}\left(\frac{\mu_g}{\mu}\right)^c \tag{32}$$

and sum to unity only when $c = 0$ or $c = 1$. Thus, in general, the total within group contribution to inequality

$$\sum_g w_g^G(\boldsymbol{\mu}, \boldsymbol{n})\, I(\boldsymbol{y}^g)$$

will not be a weighted average of the subgroup values $I(\boldsymbol{y}^g)$. This may not be regarded as a major handicap, but Theil [**11**, p. 125] has pointed out a more serious objection. It can be shown that $1 - \Sigma_g w_g^G$ is proportional to the between group term in the corresponding decomposition equation. Thus, apart from the two measures proposed by Theil ($c = 0$ and $c = 1$), the decomposition coefficients are not independent of the between group contribution.

When inequality measures are used to assess the contribution of one particular factor to total inequality, another problem arises in the different interpretations that can be placed on statements like "X per cent of inequality is due to *Y*."[9] Consider, for example, the question "How much inequality can be attributed to age variations in income." This may be interpreted as meaning: (i) How much less inequality would we observe if age variations were the only source of income differences; (ii) by how much would inequality fall if age-income differences were eliminated.

A considerable amount of confusion in the literature can be traced to such ambiguities of interpretation and to not recognizing that alternative interpretations will generally produce different results.[10] Interpretation (i) suggests a comparison of total inequality with the amount which would arise if inequality was zero within each age group, but the difference in mean income between age groups remained the same. For the additively decomposable indices this would eliminate the total within group term and leave only the between group contribution B. Interpretation (ii) suggests a comparison of total inequality with the inequality value which would result if the mean incomes of the age groups were made identical, but inequality within each age group remained unchanged. This

[9] The potential ambiguities are discussed to some extent in Davies and Shorrocks [**6**].

[10] For example, compare Nelson's [**9**] comments on the appropriate decomposition of the Gini coefficient with the analysis of Paglin [**10**].

eliminates the between group term in the decomposition equation; but the reduction in inequality is not simply B because, in general, changing the age group means will also affect the decomposition coefficients and hence the total within group contribution. Only when these coefficients do not depend on the subgroup means will (i) and (ii) produce the same answer. Of the family of measures (31), one alone satisfies this requirement—the index I_0, for which the corresponding decomposition coefficients are the population shares n_g/n. For this reason, I_0 is the most satisfactory of the decomposable measures, allowing total inequality to be unambiguously split into the contribution due to differences between subgroups

$$B = \frac{1}{n} \sum_g n_g \log \frac{\mu}{\mu_g}$$

and the contribution due to inequality within each subgroup $g = 1, \ldots, G$,

$$C_g = w_g I(\mathbf{y}^g) = \frac{1}{n} \sum_{i=1}^{n_g} \log \frac{\mu_g}{y_i^g},$$

in such a way that total inequality is the sum of these $G + 1$ contributions.

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