Economic Growth and Development: Problems for classes 1

1. Dynamic optimization.

Using the mathematical appendix in Barro and Sala-i-Martin examine how we obtain the first-order conditions for a dynamic optimisation problem.

Use those first-order conditions to solve the following problem:

$$\max \int_0^\infty \log C(t) e^{-\rho t} dt$$

$$\begin{aligned} \text{s.t.} \quad \dot{k} &= k(t)^\alpha - c(t) \\ k(0) &= k_0 \end{aligned}$$

where k_0 is given.

2. The AK model with an exogenous savings rate.

Consider the Romer (1986) production function for firm j

$$Y_{jt} = A_t K_{jt}^\alpha L_{jt}^{1-\alpha}$$

where Y , L and K are output, capital and employment. Suppose that s is the constant saving rate, n is the constant population growth rate, and δ the rate of depreciation of physical capital. There are N is the number of firms.

a. Find the differential equation for k (aggregate capital per worker) when all firms are identical.

b. Suppose that aggregate productivity changes over time because of an externality stemming from aggregate capital such that $A_t = Ak_t^\eta$. What is the equation governing the evolution of capital per worker?

c. Represent graphically the solutions to the model for the cases where the production function exhibits (i) diminishing returns to scale, $\alpha + \eta < 1$, (ii) constant returns to scale, $\alpha + \eta = 1$, (iii) increasing returns to scale, $\alpha + \eta > 1$. What is meant by "the knife-edge property" of the AK model?

d. Examine the effect on the long-run growth rate of a change in the saving rate for each of the three cases.

e. Suppose a shock destroys half of the capital stock of the economy. Examine what happens in each of the three cases to: the growth rate immediately after the shock, the long-run growth rate, and the income level in steady state. When are there permanent effects and when not?

3. The Lucas model.

Consider Lucas model discussed in the lectures, where agents decide which fraction of their time, $u \in [0, 1]$, to devote to production, and how much to devote to the accumulation of human capital, $(1 - u)$. The individual's problem is then

$$\max \int_0^\infty \frac{c^{1-\sigma} - 1}{1 - \sigma} e^{-\rho t} dt \tag{2.1}$$

subject to

$$\dot{k} = Ak^\beta (uh)^{1-\beta} - c - nk. \tag{2.2}$$

$$\dot{h} = z(1 - u)h \tag{2.3}$$

a. Set up the Hamiltonian of the optimization problem faced by a producer-consumer agents, noting that there are two dynamic constraints and two decision variables, c and u. Find the first-order conditions.

b. From the first-order conditions obtain the rate of growth of consumption as a function of h and k. In steady state, u is constant and all other variables grow at a constant rate. Use the dynamic equations for consumption and physical capital to find a relationship between the rate of growth of consumption and of capital. Take logs and differentiate to show that these two variables grow at the same rate. Show that physical and human capital also grow at the same rate.

c. Use the first-order conditions governing the behaviour of the shadow prices of human and physical capital to get

$$g_c^* = \frac{z - \rho}{\sigma}$$

d. Obtain the rate of growth of per capita output and the proportion of time spent in production.

4. The intermediate good sector in the Romer model (Romer 1990). Recall Romer's model where output is produced by

$$Y = H_1^\alpha L^\beta \int_0^A x_i^{1-\alpha-\beta} di \tag{1}$$

P_A is the patent royalty paid by an intermediate goods firm in order to be the only producer of a new variety. The cost of producing a unit of machinery is 1 unit of the final good. Machinery does not depreciate, hence the flow cost is rx_i.

(a) Under the assumption of symmetry, write the production function as a function of the capital stock, $K = xA$.

(b) From the production function obtain the marginal product of skilled labour, unskilled labour and intermediate goods, and write what are the factor prices that final goods firms are willing to pay.

(c) Write the inverse demand for intermediate goods and from it the profit function of a producer in the intermediate goods sector. What is the price elasticity of demand?

(d) Solve the intermediate goods producer's problem and obtain the price charged, the quantity produce and his profits. How much will he be willing to pay for a patent?

5. Welfare analysis in the Romer model (Romer 1990)

Consider the Romer 1990 developed in the lectures. Show that the social optimum differs from the competitive equilibrium.

6. The intermediate good sector in Aghion and Howitt (1992).

Consider the Aghion and Howitt model presented in the lectures. Recall that final output is given by

$$y = A \cdot x^{\alpha}$$

(a) Obtain the price of an intermediate good

(b) Consider the problem of the producer of an intermediate good,

$$\pi_t = \max_x [p_t(x) \cdot x - w_t \cdot x]$$

Obtain the monopoly price, monopoly quantity and profits at an instant in time.

(c) Suppose that vintage t has been invented. What price would the monopolist that owns the patent for vintage t-1 be willing to charge? Compare the price of vintage t and that of vintage t-1. Racalling that the productivity gap between the two vintages is γ, find the condition needed for the owner of vintage t to be able to charge the monopoly price derived in (b). This will be the condition for a drastic innovation. Interprete it.

(d) Suppose that the condition for a drastic innovation is not met. What price will the owner of vintage t charge? What will be his profits?

7. Income distribution, education and growth [Based on Galor and Zeira (1993)].

Consider an OLG economy where agents live for two periods and the population,

L, is constant. There is a single good that can be produced with two constant-returns technologies, one uses unskilled labor and the other skilled labor. The wages of skilled and unskilled workers are respectively, w_u and w_s. Agents can either work as unskilled for both periods, or invest in human capital when young and work as skilled workers when old. One unit of labor is supplied in each period. The investment in human capital is indivisible, that is, either you invest $h > 0$, or not invest. Members of a dynasty are linked through the bequests, b, left to their children. Agents consume, c, only in their second period of life. Assume the utility function takes the following form:

$$U = (1-\beta)\log c + \beta \log b\ , \qquad \text{where } 0 < \beta < 1\ . \tag{Y.22}$$

The only difference between new-born agents is the amount that they have inherited from their parent. Let x_{jt} denote the inheritance received by the agent from dynasty j born at time t, which is equal to the amount bequeathed by her parent, $x_{jt} = b_{jt-1}$. Individuals who borrow in order to invest in human capital face a rate of interest, i, greater than the risk-free rate, r. This is due to the cost of monitoring investment in an intangible asset.

a. There are three types of agents: those who do not invest in human capital, those who invest and lend ($x \geq h$), and those who invest and borrow. Find the utility levels and bequests of these three groups. Assume that

$$w_s - h(1+r) \geq w_u(2+r)\ . \tag{Y.23}$$

How much do individuals need to inherit to invest in h?

b. Let D_t be the distribution of inheritances received by individuals born in period t. Show that it determines the number of skilled workers and income.

c. Obtain the relation between inheritances in period t and $t+1$ for the three types of workers. Find the level to which inheritances converge for both those individuals who do not invest in education and for those that invest and do not need to borrow, denoted $\bar{x}_u$ and $\bar{x}_s$ respectively. Will their descendents ever do anything different from what they did? Show that in the case of individuals that invest in education and need to borrow, all dynasties with initial inheritance greater than

$$z = \frac{\beta\,[h(1+i) - w_s]}{(1+i)\beta - 1} \tag{Y.24}$$

converge to $\bar{x}_s$, and those with inheritance below z will eventually stop investing in education and converge to $\bar{x}_u$. Assume $(1+r)\beta < 1$, in order to ensure the existence of an equilibrium.

Economic Growth and Development: Problem Set for class 2

1. Justification for the AK model: human capital.
Consider a simple model of human capital in which production is given by $Y_t = K_t^{1-\alpha}(A_t L_t)^\alpha$, and A is a measure of the efficiency of labor, such that the productive capacity of the stock of labor, or level of human capital, is $H = AL$. Then $Y_t = K_t^{1-\alpha} H_t^\alpha$. A proportion s_k of income is invested in physical capital, and a proportion s_h in human capital. The depreciation rates are respectively δ_k and δ_h. The population does not grow.

a. Find the equilibrium physical capital to human capital ratio, using the condition that both investments must yield the same return.

b. Show that the production function can be written as an AK function and find the growth rate. Why are the results different from those in the augmented Solow model of Mankiw, Romer and Weil?

2. Infrastructure. Past exam question.

Consider the model by Barro (1990) in which government expenditures, which we will call infrastructure, affect the productivity of privately owned factors. The production function takes the form

$$Y_t = AK_t^{1-\alpha}\gamma_t^\alpha \; L_t^\alpha \qquad \text{where } \; 0 < \alpha < 1$$

where K_t is the aggregate stock of capital, L_t the (constant) labour force, and γ_t government expenditure.

The utility function of the representative individual is given by

$$U = \int_0^\infty \frac{c_t^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} dt$$

where c_t is consumption. The representative individual supplies one unit of labour at each point in time and has an initial capital stock K_0.

A proportional tax, τ, is levied on all income and all revenues are spent on the single public good γ. The government cannot borrow, and hence must hold a balanced budget at each point in time

a. Write the production function in per capita terms and obtain the wage rate and interest rate.

b. Normalise population to L=1. Write the government budget constraint and determine γ as a function of τ. Can the production function be written as an AK function? What are the equilibrium wage and interest rate?

c. What is the optimal rate of growth of consumption?

Find the competitive growth rate of this economy, g.

d. Now suppose that Y has the more general form

$$Y_t = AK_t^{1-\alpha-\beta}\gamma_t^{\beta}\, L_t^{\alpha} \qquad \text{where } \; 0 < \alpha < 1, 0 < \beta < 1$$

(i) Write the per capita production function and factor prices. Write the government budget constraint and determine γ as a function of τ. Can the production function be written as an AK function? What are the equilibrium wage and interest rate?

(ii) Obtain the equilibrium growth rate of this economy. Is the rate of growth of constant over time? Explain the economic reasons for your answer. What is the steady state of this economy?

3. Variant of the Lucas model (Based on Rebelo, JPE 1991).

Consider Lucas model discussed in the lectures, where agents decide which fraction of their time, $u \in [0, 1]$, to devote to production, and how much to devote to the accumulation of human capital, $(1 - u)$.

The individual's utility function is

$$\int_0^{\infty} \frac{c^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} dt \tag{2.1}$$

Instead of using Lucas' function for the accumulation of human capital, assume that h is obtained from using human and physical capital. The production technology is identical to that for the final good,

$$\dot{h} = ((1-\phi)k)^{\beta}((1-u)h)^{1-\beta} \;,$$

where $(1 - \phi)$ is the fraction of the stock of physical capital employed in the production of human capital and $(1 - u)$ the fraction of human capital employed in the sector, ϕ,$u \in [0, 1]$. The population is constant and normalized to one.

Output is produced according to

$$y = A(\phi k)^{\beta} \, (uh)^{1-\beta} \, .$$

The final good is used both in consumption and as physical capital, so that the accumulation of physical capital is governed by

$$\dot{k} = y - c \ .$$

The economy has to decide, on the one hand, how much to consume, and, on the other, what proportion of the two types of capital to devote to the production of human capital and final output. Consumers directly operate the economy's technology. They are infinitely lived, and maximize their lifetime utility. The instantaneous utility function is given by $(c^{1-\sigma} - 1)/(1 - \sigma)$, and the discount rate is ρ.

(a) Obtain the static optimality condition that governs the allocation of the existing stocks of physical and human capital across the two sectors.

(b) Obtain the dynamic optimality condition for investment in the two types of capital (that is, equality between the net rates of return). How does the dynamic optimality condition look like in the steady state?

(c) Using the two optimality conditions derived in (a) and (b), show that in the steady state the "capital-labour intensity" in the final-good sector is constant and given by

$$\frac{\phi k}{uh} = A \frac{\beta}{1 - \beta}.$$

Similarly, obtain the capital-labour intensity in the human capital sector. What is the steady state interest rate?

(d) From the dynamic equation for consumption obtain the steady state rate of growth of the economy. Why is it non-zero in the absence of technical change or externalities, and when all cumulable factors exhibit diminishing returns?

4. Romer 1990 and the role of the linear R&D function. [Based on Jones, 1995].

Consider Jones version of the Romer model. There is only one type of labour, which can be employed in final good production, H_1, or in research, H_2. Then $H = H_1 + H_2$. Output is given by

$$Y = H_1^{1-\alpha} \int_0^A x_i^\alpha di$$

All other functions are identical except for the R&D function. There are constant returns to the level of employment in the sector, H_2. Suppose, however, that the externality due to past innovations exhibits diminishing returns, that is, only A^ϕ of past designs can be used in the generation of new ideas, where

$0 \leq \phi \leq 1$. Assume also that since there are several firms doing research, some of them duplicate the research that others do. Hence, labor productivity is given by $A^{\phi} H_2^{\lambda-1}$, where $0 < \lambda \leq 1$ is the inverse of the degree of duplication. The aggregate R&D equation is thus

$$\dot{A} = \delta(A^{\phi} H_2^{\lambda-1}) H_2.$$

The term in brackets represents the total productivity of the H_2 workers employed in the sector, which depends on A (spillover) and the level of employment (duplication). The population grows at a constant rate n.

a. Solve the model with the above R&D equation.

b. Discuss the main features of the growth rate in this model.

c. Find the level of R&D employment chosen by a social planner, and the corresponding growth rate. How do they differ from the competitive outcome?

5. The Big Push [Based on Murphy, Shleifer, and Vishny 1989]. Past exam question.

Consider a two-period model of industrialization with no capital as in the Big Push model.

Consumers

The utility function is of the form

$$U = \left[\int_0^1 x_1^{\gamma}(q) dq \right]^{\theta/\gamma} + \beta \left[\int_0^1 x_2^{\gamma}(q) dq \right]^{\theta/\gamma}$$

where $x_1(q)$ is the amount of good q consumed in the first period, $x_2(q)$ is the amount of good q consumed in the second period, $1/(1-\gamma)$ measures the substitutability across different goods, $1/(1-\theta)$ measures the intertemporal elasticity of substitution, and β is the rate of time preference. Note that we have normalized the number of goods to 1.

There are L consumers and hence L units of labour each period. The wage is equal to 1.

Firms

Each good can be produced by one of two alternative production technologies.

On the one hand, there is a traditional technology, or cottage production, which exhibits constant returns to scale. One unit of labour produces one unit of output. Since the wage is equal to 1, the price of a unit of good produced in this sector is 1.

On the other hand, there is mass production, which consists of an industrial technology with increasing returns to scale. This technology has two characteristics.

(i) The firm must invest F units of labour in the first period.

(ii) The firm will then produce output *in the second period.* One unit labour produces α units of output, where $\alpha > 1$.

Since the industrial technology exhibits increasing returns, it will be used by only one firm in each sector, that is, there will be a monopolist in each sector.

The interest rate is r, and will be endogenously determined. The interest factor is then $R = 1/(1 + r)$.

1. Since all goods are symmetric, consumers will purchase the same amount of all goods, that is $x_t(q) = x_t$ for all q. Rewrite the utility function under symmetry.

2. Denote aggregate income in periods one and two by y_1 and y_2. Write per capita consumption of each good in the two periods, x_1 and x_2, as a function of aggregate income.

3. Given the levels of consumption obtained above, write down the consumer's optimization problem. From the first order condition, express the interest factor R in terms of the consumption levels x_1 and x_2, and β. Interpret this relationship.

4. Consider a monopolist using the industrial technology. What is the price chosen by the monopolist in a particular sector? Note that the same good can always be produced by the cottage technology. What are the profits of the monopolist? For notational simplicity, you can define $a = (1 - 1/\alpha)$.

5. The low-income equilibrium

   (i) Obtain the levels of consumption in the low-income equilibrium in which the industrial technology is not used, and find the interest factor R that ensures that this situation is an equilibrium.

   (ii) Find the parameter restriction on β, a, L, b, and F under which a low income equilibrium may exist.

6. The high-income equilibrium

   (i) Obtain the levels of consumption in the high-income equilibrium in which the industrial technology is used by all firms

(ii) Find the interest factor R that ensures that this situation is an equilibrium. Is the equilibrium interest rate higher or lower than in the low-income equilibrium? Why?

7. Multiple equilibria

   (i) Find the parameter restriction under which a low-income and a high-income equilibrium may coexist. Can multiple equilibria occur? When? Explain why.

   (ii) Suppose that there is a single large bank with sufficient funds to lend F to n the firms in this economy. Discuss whether the presence of such a bank can ensure that the economy will be in a high-income equilibrium.