Economic Growth and Development: Problem Set Solutions

1. Justification for the AK model: human capital.

The introduction of human capital has been one of the most important ways of obtaining constant returns to an aggregate measure of capital. Note, however, that having human capital as a factor of production per se does not generate an AK type of function, as we saw in the augmented Solow model.

(a) The equations describing the accumulation of the two types of capital are

$$\dot{K} = s_k Y - \delta_k K \quad \text{and} \quad \dot{H} = s_h Y - \delta_h H \ . \tag{S1.13}$$

Savings can thus be invested in either of these two assets. In equilibrium, the return to both, and hence their net marginal product, must be the same. That is,

$$\alpha \left(\frac{K}{H}\right)^{1-\alpha} - \delta_h = (1-\alpha)\left(\frac{H}{K}\right)^{\alpha} - \delta_k \ ,$$

which yields

$$\left(\frac{K}{H}\right)^{-\alpha}\left(\frac{K}{H} - \frac{1-\alpha}{\alpha}\right) = \frac{\delta_h - \delta_k}{\alpha}. \tag{S1.14}$$

This arbitrage equation implies that the physical to human capital ratio is a constant, denote it by κ . There may be more than one solution. In the case where both types of capital depreciate at the same rate, we get that

$$\frac{K}{H} = \frac{1-\alpha}{\alpha} \ . \tag{S1.15}$$

Note that in this case we need a restriction on the saving rates for a balanced growth path to exist. Since the physical to human capital ratio is constant, both types of capital must grow at the same rate. That is, $\dot{K}/K = \dot{H}/H$. Using (S1.13) and given that both types of capital depreciate at the same rate, $\dot{K}/K = \dot{H}/H$ implies $s_k/s_h = K/H$. Hence it must be that the savings rates satisfy $s_k/s_h = (1-\alpha)/\alpha$.

(b) We can express the production function as $Y = (K/H)^{\alpha} K$, and substitute for the ratio of the two types of capital,

$$Y = \kappa^{\alpha} K. \tag{S1.16}$$

The rate of growth of output is given by

$$\frac{\dot{Y}}{Y} = \alpha \frac{\dot{H}}{H} + (1-\alpha)\frac{\dot{K}}{K},$$

which substituting for the rates of accumulation of H and K as given by equations (S1.13) and for the physical to human capital ratio, gives us the following rate of growth:

$$\frac{\dot{Y}}{Y} = \alpha(s_k \kappa^{-\alpha} - \delta_k) + (1-\alpha)(s_h \kappa^{1-\alpha} - \delta_h).$$

The rate of growth is thus constant and depends on model parameters, such as the saving rates and the depreciation rates. The difference with the augmented Solow model is that in that set-up there were decreasing returns to a combined measure of capital that included human and physical capital. As a result there were diminishing returns to the accumulation of factors and the growth rate tended to zero. On the other hand, the model we have just seen exhibits constant returns to all cumulable factors.

2. Infrastruture

d.(i) The gross interest rate is

$$rgross = (1-\alpha-\beta)A\left(\frac{\gamma}{K}\right)^{\beta}\left(\frac{L}{K}\right)^{\alpha}$$

The budget constraint

$$\gamma = \tau y = \tau A K^{1-\alpha-\beta}\gamma^{\beta}\ \frac{L^{\alpha}}{L}.$$

Hence

$$\frac{\gamma}{k} = \left(\frac{A\tau}{L}\left(\frac{L}{K}\right)^{\alpha}\right)^{1/(1-\beta)}$$

Then

$$rgross = (1-\alpha-\beta)A\left(\frac{A\tau}{L}\right)^{\beta/(1-\beta)}\left(\frac{L}{K}\right)^{\alpha/(1-\beta)}$$

So

$$g = \frac{1}{\sigma}\left[(1-\alpha-\beta)A\left(\frac{A\tau}{L}\right)^{\beta/(1-\beta)}\left(\frac{L}{K}\right)^{\alpha/(1-\beta)} - \rho\right]$$

3. The modified Lucas Model

(a) Efficient allocation requires that the marginal value product of the two inputs be equal across sectors. Let q denote the price of a unit of human capital in terms of physical capital. Differentiating the two production functions with respect to the two inputs we have

$$A(\phi k)^{\beta-1}(uh)^{1-\beta} = q((1-\phi)k)^{\beta-1}((1-u)h)^{1-\beta} \tag{4.8}$$

$$A(\phi k)^{\beta}(uh)^{-\beta} = q((1-\phi)k)^{\beta}((1-u)h)^{-\beta} \ . \tag{4.9}$$

These two equations together yield

$$\frac{\phi k}{uh} = \frac{(1-\phi)k}{(1-u)h} \ . \tag{4.10}$$

(b) The net rate of return to capital employed in final-good production is

$$r = \beta A(\phi k)^{\beta-1}(uh)^{1-\beta} \ . \tag{4.11}$$

Alternatively, one could accumulate $1/q$ units of human capital, which yield a return of

$$r^h = (1-\beta)((1-\phi)k)^{\beta}((1-u)h)^{-\beta} + \frac{\dot{q}}{q} \ , \tag{4.12}$$

where $\dot{q}/q$ represents the "capital gain" from investing in human capital. At the optimum, $r = r^h$. In the steady state, equation (4.8) (or equivalently (4.9)) implies that q is constant. Then

$$\beta A \left(\frac{\phi k}{uh}\right)^{\beta-1} = (1-\beta)\left(\frac{(1-\phi)k}{(1-u)h}\right)^{\beta} \ . \tag{4.13}$$

(c) We can solve equations (4.10) and (4.13) together for the capital-labour intensities, and obtain

$$\frac{\phi k}{uh} = \frac{(1-\phi)k}{(1-u)h} = A\frac{\beta}{1-\beta}.$$

Substituting for the capital-labour intensities into either (4.11) or (4.12) we find the steady state interest rate,

$$r^* = (1-\beta)^{1-\beta} (A\beta)^{\beta} \ .$$

The rate of interest is therefore a constant that depends only on model parameters, and not on the stock of human or physical capital.

(d) Since the rate of growth of consumption is given by $g = (r - \rho)/\sigma$, in steady state we have

$$g = \frac{(1-\beta)^{1-\beta} (A\beta)^{\beta} - \rho}{\sigma}. \tag{4.14}$$

The accumulation of human capital increases the marginal product of physical capital, and hence induces its accumulation. But this in turn increases the marginal product of human capital, and more resources are invested in it. Although there are diminishing returns to each factor, there are constant returns to all cumulable factors, which induce continuous levels of net investment and hence sustained growth.

4. The role of the linear R&D function [Based on Jones (1995)]

(a) The equation for the rate of growth of the stock of knowledge is

$$\frac{\dot{A}}{A} = \delta \frac{H_2^{\lambda-1}}{A^{1-\phi}} H_2 \ . \tag{S1.38}$$

In steady state, the rate of growth of designs is constant. Hence, if we take logs and differentiate equation (S1.38) with respect to time we obtain that the steady state rate of growth of the stock of designs is

$$g \equiv \frac{\dot{A}}{A} = \frac{\lambda}{1-\phi} \frac{\dot{H}_2}{H_2} \ . \tag{S1.39}$$

We can now proceed as usual. As in the Lecture we have that the level of x is such that

$$r = \alpha^2 \left(\frac{H_1}{x} \right)^{1-\alpha} \ , \tag{S1.40}$$

and the value of the patent is

$$P_A = \frac{1-\alpha}{\alpha} x \ . \tag{S1.41}$$

What changes is the labor market equilibrium equation, which, from the production function and equation (S1.38) (taking the productivity of researchers as given) yields

$$\delta H_2^{\lambda-1} A^{\phi} P_A = (1-\alpha) H_1^{-\alpha} x^{\alpha} A \ . \tag{S1.42}$$

We rearrange equation (S1.42) and substitute into it (S1.40) and (S1.41), to get

$$\frac{H_1}{H_2} = \frac{r}{\alpha} \frac{1}{\delta H_2^{\lambda} A^{\phi-1}} \ , \tag{S1.43}$$

and, using (S1.38),

$$\frac{H_1}{H_2} = \frac{r}{\alpha} \frac{1}{g} \ . \tag{S1.44}$$

Since in steady state the interest rate and the growth rate are constant, this expression implies that a constant proportion, s, of the labor force is employed the research sector. That is,

$$\frac{H_2}{H} = s \ , \qquad \text{where} \ \ s = \frac{1}{1+\varphi} \ , \quad \text{and} \quad \varphi = \frac{r}{\alpha}\frac{1}{g}.$$

Research employment is proportional to the stock of labor, therefore it must grow at the same rate as the population. Hence, from equation (S1.39) we have that the growth rate of the stock of designs is

$$g = \frac{\lambda n}{1-\phi} \ . \tag{S1.45}$$

This expression pins down all growth rates in the model. It is straightforward to show that the steady state growth rates of per capita consumption, per capita income and per capita capital must be equal to g.

To obtain the value of φ , we use the Ramsey-Cass-Koopmans equation for the rate of growth of consumption, $\dot{c}/c = (r-\rho)/\sigma$ and substitute for the growth rate as defined by equation (S1.45). Then

$$\varphi = \frac{1}{\alpha}\left(\sigma + (1-\phi)\frac{\rho}{\lambda n}\right) . \tag{S1.46}$$

(b) Let's examine (S1.45), which tells us what the steady state growth rate is. First of all, note that when $\phi = 1$, as in the Romer model, no balanced growth path exists since the population is growing. This is solved by assuming that $\phi < 1$

. Second, note that the scale effects have disappeared, and that the growth rate depends on the *rate of growth* of the labor force, instead of on its *level.* Lastly, the greater the degree of duplication, the lower the bgrowth rate is .

(c) The problem faced by the social planner is

$$\max_{C} \int_{0}^{\infty} \frac{C^{1-\sigma}-1}{1-\sigma} e^{-\rho t} dt$$

subject to

$$\begin{aligned} \dot{K} &= K^{\alpha} A^{1-\alpha} H_1^{1-\alpha} - C \\ \dot{A} &= \delta A^{\phi} H_2^{\lambda} \\ H &\geq H_1 + H_2 \end{aligned}$$

Setting up the Hamiltonian and optimizing we obtain that the rates of growth of income, consumption and capital are all equal to the rate of growth of the stock of designs, as given by equation (S.145) above. This means that the social planner chooses the same rate of growth as the competitive economy, despite the presence of externalities. However, the proportion of the labor force engaged in research differs in the two cases due to the presence of imperfect competition and the intertemporal spillover effect. (Check for yourselves that the social planner would choose a constant H_2/H).

The absence of monopoly implies that the term $1/\alpha$ representing the monopoly mark-up disappears from the expression determining the share of labor devoted to research. A second difference is that the planner would take into account the effect of duplication, and hence the labor market equilibrium equation is

$$\lambda \delta H_2^{\lambda-1} A^{\phi} V_A = (1-\alpha) H_1^{-\alpha} x^{\alpha} A \ , \tag{S1.47}$$

where V_A is the value of the innovation. The socially optimum x is determined at the point where marginal benefit is equal to marginal cost. That is, it is given by the expression

$$r = \alpha H_1^{1-\alpha} x^{\alpha-1}.$$

The value of the innovation is then $V_A = H_1^{1-\alpha} x^{\alpha}/r - x = (1-\alpha) H_1^{1-\alpha} x^{\alpha}$. Substituting for the value of the innovation in equation (S1.47) we have that the level of employment in research is

$$\frac{H_2}{H} = \frac{1}{1+\varphi^{**}} \ ,$$

where

$$\varphi^{**} = \frac{1}{\lambda}\left(\sigma + (1-\phi)\frac{\rho}{\lambda n} - \phi\right) \ . \tag{S1.48}$$

The steady state share of labor devoted to R&D differs from the competitive outcome for three reasons: the monopoly mark-up term disappears, the intertemporal spillover effect is internalized, and the effect of duplication that led the competitive economy to overinvest in research is taken into account (as represented by the term $1/\lambda$). The first two effects imply that there is too little research employment in a competitive economy, the third that too much is undertaken. Either effect can in principle dominate.

5. The Big Push.

1. Since all gods are symmetric,

$$U = x_1^\theta + \beta x_2^\theta$$

2. Since there is no capital, aggregate consumption must be equal to aggregate income $\Rightarrow x_1 = y_1/L$, $x_2 = y_2/L$.

3. We have

$$\max U = [x_1]^\theta + \beta [x_2]^\theta$$

$$\text{s.t.} \quad \frac{y_1}{L} + \frac{1}{1+r}\frac{y_2}{L} = x_1 + \frac{1}{1+r}x_2$$

Usual condition

$$\frac{U_1(x_1,x_2)}{U_2(x_1,x_2)} = \frac{\beta}{R} \Rightarrow \left(\frac{x_1}{x_2}\right)^{\theta-1} = \frac{\beta}{R}$$

where $R = 1/(1+r)$

4. Limit pricing, so the price is 1. The monopolist's profit

$$\pi = \frac{1}{1+r}\left(y_{2-}\frac{y_2}{\alpha}\right) - F = Ray_2 - F$$

5. Low-income equilibrium

(i) Low income equilibrium $y_1 = y_2 = L$. This requires perfect smoothing, that is $R = \beta$.

(ii) For this to be an equilibrium, we need $\pi = \beta aL - F < 0$.

6. High-income equilibrium

(i) All sectors industrialize

$$\begin{aligned} y_1 &= L - F \\ y_2 &= \alpha L > L \end{aligned}$$

and then $x_1 = 1 - F/L$ and $x_2 = \alpha$.

(ii) Will the consumer accept the resulting utility level? We find the interest factor under which he does:

$$\left(\frac{x_1}{x_2}\right)^{\theta-1} = \frac{\beta}{R} \Rightarrow R = \beta\left(\frac{\alpha L}{L-F}\right)^{\theta-1} \leq \beta$$

(iii) For this to be an equilibrium,

$$\pi = Ra\alpha L - F = \beta\left(\frac{\alpha L}{L-F}\right)^{\theta-1} a\alpha L - F > 0$$

$$\beta\left(\frac{\alpha L}{L-F}\right)^{\theta-1} a\alpha L > F \qquad ((*))$$

7. Multiple equilibria

(i) Multiple equilibria exist if the two conditions can be simultaneously satisfied

$$\beta a L < F < \frac{F}{(1-F/L)^{1-\theta}} < \alpha^{\theta}\beta a L$$

For intermediate values of the fixed cost, both conditions can simultaneously be satisfied. Note that the discount rate cannot be too large nor too small.

When condition (*) holds, the interest rate does not rise too much when consumption grows. As a results, there exists an equilibrium in which firms expect other firms to invest and income to rise, and all firms then invest. The possibility of the big push comes from the coexistence of both equilibria for the same parameter values.

The key to the coexistence of both equilibria is the fact that a firm's profits are not an adequate measure of its contribution to the demand for manufactures. A firm that invests, even if it loses money, it reduces period 1 income and increases period 2 income. Period 1 income is irrelevant for investment decisions, but period 2 income matters, so the firm is raising demand for other firms manufactures and hence their profitability. Investment by one firm makes investment by other firms more attractive. This is ok as long as the interest rate does not rise too much.