1 Lucas model of human capital

Uzawa (1965) and Lucas (1988)

Accumulation of human capital: "time".
Output: $F(K, L^e)$, where $L^e = uhL$.
$u \in [0, 1]$
Per capita output

$$y = Ak^{\beta} (uh)^{1-\beta}$$

Human capital accumulation

$$\dot{h} = zh(1-u)$$

Then

$$Max \int_0^{\infty} \frac{c^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} dt \tag{1}$$

$$\text{s.t. } \dot{k} = Ak^{\beta}(uh)^{1-\beta} - c - nk \tag{2}$$

$$\dot{h} = z(1-u)h \tag{3}$$

Two control variables, c and u, and two state variables, k and h.

First order conditions

$$\frac{\partial H}{\partial c} = 0 \ , \quad \frac{\partial H}{\partial u} = 0 \ ,$$
$$\frac{\partial H}{\partial k} = -\dot{\lambda} \ , \frac{\partial H}{\partial h} = -\dot{\mu}$$

The Hamiltonian

$$H = \frac{c^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} + \lambda(Ak^{\beta}(uh)^{1-\beta} - c - nk) + \mu(z(1-u)h) \ ,$$

f.o.c.'s

$$\begin{aligned}
\lambda &= c^{-\sigma}e^{-\rho t} && (4)\\
\mu z h &= \lambda(1-\beta)Ak^{\beta}u^{-\beta}h^{1-\beta} && (5)\\
-\dot{\lambda} &= \lambda(A\beta k^{\beta-1}u^{1-\beta}h^{1-\beta}-n) && (6)\\
-\dot{\mu} &= \lambda(1-\beta)Ak^{\beta}u^{1-\beta}h^{-\beta} && (7)\\
&\quad +\mu z(1-u)
\end{aligned}$$

From (4) and (6),

$$\frac{\dot{c}}{c} = \frac{A\beta k^{\beta-1}(uh)^{1-\beta}-\rho-n}{\sigma} \tag{8}$$

Since (8) always holds

$$\beta A k^{\beta-1}(uh)^{1-\beta} = \sigma g_c^* + \rho + n \tag{9}$$

From equation (2)

$$g_k^* = \frac{Ak^{\beta}(uh)^{1-\beta}}{k} - \frac{c}{k} - \frac{nk}{k}$$

These two equations together imply

$$\frac{c}{k} = \frac{\sigma g_c^* + \rho + n}{\beta} - g_k^* - n$$

Take logs

$$\log c - \log k = \log\left(\frac{\sigma g_c^* + \rho + n}{\beta} - g_k^* - n\right)$$

and differentiate with respect to time

$$\frac{d\log c}{dt} - \frac{d\log k}{dt} = 0,$$

hence

$$g_c^* = g_k^*. \tag{10}$$

Rewrite (9) as

$$\left(\frac{h}{k}\right)^{1-\beta} u^{1-\beta} = \frac{\sigma g_c^* + \rho + n}{\beta A}$$

Take logs and differentiate with respect to time to get

$$g_k^* = g_h^*\ . \tag{11}$$

Rewrite (5) as

$$\frac{\lambda}{\mu} = \frac{z}{(1-\beta)A}\left(\frac{uh}{k}\right)^{\beta}. \tag{12}$$

Take logs, differentiate with respect to time, and use the fact that $g_k^* = g_h^*$ to get

$$\frac{\dot{\lambda}}{\lambda} = \frac{\dot{\mu}}{\mu} \tag{13}$$

Use (4) to get

$$\frac{\dot{\lambda}}{\lambda} = -\sigma g_c^* - \rho \ , \tag{14}$$

and (12) and (7) to get

$$\frac{\dot{\mu}}{\mu} = -z \tag{15}$$

Then, from (13), (14) and (15) we have

$$g_c^* = \frac{z-\rho}{\sigma} \tag{16}$$

Recall that $g_h^* = g_c^*$, then

$$z(1-u^*) = g_h^* = \frac{z-\rho}{\sigma}$$

which implies that the steady state level of $(1-u)$ is

$$1 - u^* = \frac{z-\rho}{z\sigma} \tag{17}$$

and

$$g^* = z(1-u^*)$$