Distributional Dynamics in a Neoclassical Growth Model:  

The Role of Elastic Labor Supply*

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Abstract: We examine the evolution of the distributions of wealth and income in a Ramsey model in which agents differ in their initial capital endowment and where the labor supply is endogenous. The assumption that the utility function is homogeneous implies that the macroeconomic equilibrium is independent of the distribution of wealth and allows us to characterize fully income and wealth dynamics. We find that although the dynamics of the distribution of wealth are similar under fixed and flexible labor, those of the income distribution are not. In response to a structural change, income inequality may move in opposite ways depending on whether or not the labor supply is fixed.

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1. Introduction

The Ramsey (1928) model has been widely employed by macroeconomists for almost eighty years, and over time has become the standard framework for addressing a number of important issues in macrodynamics and growth theory. While it typically treats all agents as identical (the representative agent paradigm), if preferences are homothetic the framework can readily incorporate certain forms of agent heterogeneity; see Chatterjee (1994), Caselli and Ventura (2000), and Mailar and Mailar (2001). An important question which then naturally arises concerns the consequences of this heterogeneity for the dynamics of wealth and income distribution. This is particularly pertinent in light of observed long-run changes in wealth and income inequality. While an extensive literature, dating back to Stiglitz’s (1969) seminal contribution, exists on the dynamics of wealth distribution, the evolution of income inequality has received much less attention. In this paper we analyze the dynamics of wealth and income distribution in a Ramsey model with elastic labor supply, examining in particular the significance of the endogeneity of labor for income distribution.

The source of the heterogeneity we introduce is the agent’s initial endowment of capital (wealth). Assuming that preferences are represented by a utility function that is homogeneous in consumption and leisure, allows aggregation as in Gorman (1953) or Eisenberg (1961), and generates the representative-consumer characterization of the macroeconomic equilibrium. We can then represent the macroeconomic equilibrium in terms of a simple recursive structure. First, the dynamics of the aggregate stock of capital and labor supply are jointly determined, independently of distributional concerns. Then, the cross-sections of individual income and wealth and their dynamics are characterized in terms of the aggregate magnitudes, thereby enabling us to address distributional issues in a tractable way.

We characterize both the time paths of the distributions of wealth and income, as well as their steady-state distributions, and compare their responses to a number of structural changes under

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1 For example, Piketty and Saez (2003) document dramatic long-run changes in income distribution for the United States over the period 1913-1998.
2 This type of utility function is common in many areas of macroeconomics, most notably in business cycle theory but also in the endogenous growth literature. See King and Rebelo (1999) for a survey on business cycle theory, and Ladrón-de-Guevara et al. (1997, 1999, 2002), Turnovsky (2000), and García-Peñalosa and Turnovsky (2006a, 2007) for endogenous growth models with leisure.
both fixed and flexible labor supply. We find that although for plausible parameter values the
dynamics of wealth inequality are relatively insensitive to the nature of labor supply, whether labor
supply is fixed or flexible can have a dramatic impact on the dynamics of income distribution.

A crucial mechanism determining the evolution of income inequality is the relationship we
derive between agents’ relative wealth (capital) and their relative allocation of time between work
and leisure. This relationship is very basic and has a simple intuition. Wealthier agents have a lower
marginal utility of wealth. They therefore choose to increase consumption of all goods including
leisure, and reduce their labor supply. Indeed, the role played by labor supply in this model is
analogous to its role in other models of capital accumulation and growth, where it provides the
crucial mechanism by which demand shocks influence the rate of capital accumulation. This
mechanism is also central to empirical models of labor supply based on intertemporal optimization;
see e.g. MaCurdy (1981).

There is substantial empirical evidence documenting this negative relationship between
wealth and labor supply. Holtz-Eakin, Joulfaian, and Rosen (1993) find evidence to support the view
that large inheritances decrease labor participation. Cheng and French (2000) and Coronado and
Perozek (2003) use data from the stock market boom of the 1990s to study the effects of wealth on
labor supply and retirement, finding a substantial negative effect on labor participation. Algan,
Chéron, Hairault, and Langot (2003) use French data to analyze the effect of wealth on labor market
transitions, and find a significant wealth effect on the extensive margin of labor supply. Overall,
these studies and others provide compelling evidence in support of the wealth-leisure mechanism
being emphasized in this paper.

Our analysis shows that endogenizing the labor supply has three substantial consequences for
the dynamics of income distribution. First, whereas with fixed labor supply the distribution of
income evolves continuously, when labor is supplied elastically, the initial jump in labor supply
following a structural change implies that income distribution will also jump, leading to much larger
movements in the distribution. As a result, income inequality may adjust non-monotonically,

3 For example, in the standard Ramsey model, government consumption expenditure will generate capital accumulation
if and only if labor is supplied elastically. With inelastic labor supply it will simply crowd out an equivalent amount of
private consumption. The key factor is the wealth effect and the impact this has on the labor-leisure choice, as
emphasized by both Ortigueira (2000) and Turnovsky (2000).
increasing on impact and declining thereafter, something that is not possible with fixed labor.

Second, it is possible for a structural change that leads to a decline in income inequality with fixed labor supply to lead to greater income inequality when labor is supplied elastically (and vice versa). The reason for this is the change in the distribution of leisure time, and hence of work hours. Since the reduction in the wage due to a higher (average) labor supply induces poorer agents to work (relatively) harder, an aggregate labor supply change will result in a change in the distribution of labor incomes. This effect can be strong enough to offset the impact of declining wealth inequality, which is the only effect to operate when labor supply is fixed. The extent to which this may occur depends upon the form of the production function, as well as the nature of the shock. For example, for a Cobb-Douglas production function, a reduction in the rate of time preference from 0.04 to 0.02 will lead to a long-run decline in income inequality of around 7% with fixed labor supply, but it will yield a long-run increase in income inequality of 10% if labor supply is flexible.

Third, if labor supply is fixed, all three structural changes we consider – a 50% increase in productivity, a 1.5 percentage point decline in population growth, and a 2 percentage point decline in the rate of time preference – have almost identical impacts on income inequality. In contrast, the responses of income inequality with elastic labor supply can vary sharply, depending on the source of the shock. For example, with a high elasticity of substitution in production, a reduction in the rate of population growth or in the rate of time preference would result in a small increase in long-run income inequality with fixed labor; with flexible labor the decrease in the population growth rate would reduce inequality by over 30%, while the preference shock would increase inequality by almost 24%. The conclusion to emerge from these examples is that to treat labor supply as fixed, when in fact it is supplied elastically, may yield seriously misleading conclusions with respect to the dynamic evolution of income inequality.

Our paper contributes to the recent literature characterizing distributional dynamics in growth models.4 This question was first examined in Stiglitz (1969) using a form of the Solow model. Early work examining the evolution of the distribution of wealth in the Ramsey model assumed agents that differ in their rate of time preferences. In this framework, the most patient agent ends up

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4 See Bertola, Foellmi, and Zweimüller (2006) for a survey.
holding all the capital in the long-run, although the presence of progressive taxation or capital market imperfections can prevent a degenerate distribution of wealth; see Becker (1980), Becker and Foias (1987), Sorger (2002).

An extensive literature has examined wealth dynamics when agents have the same rate of time preference but differ in their initial capital endowments. Caselli and Ventura (2000) consider the dynamics of wealth and income distribution in the Ramsey model, though they restrict their analysis to exogenous labor supply. Starting with Chatterjee (1994), several papers have examined the implication of assuming a minimum consumption requirement. Chatterjee finds that under reasonable parameter specifications wealth inequality increases during the process of development. The mechanism driving this result is the existence of a minimum consumption requirement which tends to make the propensity to save increasing in individual wealth, and hence exacerbates inequalities as the economy accumulates capital. This literature has, however, considered only models with fixed labor, and has examined the evolution of wealth but not income inequality.

Closest to our analysis are Sorger (2000) and Maliar and Maliar (2001). Sorger studies the Ramsey model with endogenous labor and heterogeneous wealth endowments. His specification assumes that the utility function is non-homogeneous in its two arguments, and as a result the evolution of aggregate variables depends on the entire distribution of wealth at each point in time. This difference has two important implications. First, Sorger finds a correlation between per capita income levels and the distribution of wealth, which depends on the elasticity of intertemporal substitution. In contrast, for our chosen utility specification, a particular level of per capita output is compatible with any distribution of wealth, depending on the initial distribution. Second, Sorger focuses on the stationary state, as the interdependence of the macroeconomic equilibrium and distribution renders the analysis of the dynamics intractable.

Maliar and Maliar (2001) examine a neoclassical model with endogenous labor and wealth heterogeneity. As in our analysis, homothetic preferences permit a representative-consumer characterization of the macroeconomic equilibrium. However, they ask a different question since they focus on the labor market implications of business cycles when agents are heterogeneous, rather

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5 See also Chatterjee and Ravikumar (1999), Alvarez-Peláez and Díaz (2005), and Obiols-Homs and Urrutia (2005).
than on the dynamics of the wealth distribution. Maliar, Maliar, and Mora (2005) examine the business-cycle behavior of the distributions of income and wealth, and in an earlier version of their paper considered the case of endogenous labor. Their conclusion that endogenous and exogenous labor have the same implications for distributional dynamics is, however, due to the fact that they consider a Cobb-Douglas production function and a technology shock. We also find that under these assumptions both models indeed yield similar distributional dynamics. However, by generalizing the production function and extending the range of shocks we find that the endogeneity of labor has important consequences for the dynamics of income distribution.

The paper is organized as follows. Section 2 describes the economy and derives the macroeconomic equilibrium. Section 3 characterizes the distributions of wealth and income and derives the main results of the paper. Section 4 derives the effects of alternative structural changes on the long-run distributions of wealth and income, which are then illustrated in Section 5 with a number of numerical examples. Section 6 concludes, while insofar as possible the technical details are relegated to an Appendix.

2. The Analytical Framework

We begin by setting out the components of the model.

2.1 Technology and factor payments

Aggregate output is produced by a single representative firm according to a standard neoclassical production function, so that

\[ Y = F(K, L) \quad F_L > 0, F_K > 0, F_{LL} < 0, F_{KK} < 0, F_{LK} > 0 \]  

where, \( K, L \) and \( Y \) denote the per capita stock of capital, labor supply and output. The wage rate, \( w \), and the return to capital, \( r \), are determined by the marginal physical products of labor and capital,

\[ w(K, L) = F_L(K, L) \quad w_K = F_{LK}, \quad w_L = F_{LL} \]  

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6 That is, both factors of production have positive, but diminishing, marginal physical products and the production function exhibits constant returns to scale, implying \( F_{LL}F_{KK} - F_{LK}^2 = 0 \), from which \( F_{LK} > 0 \) immediately follows.
\[ r(K, L) = F_k(K, L) \quad r_k = F'_{kk} < 0, \quad r_L = F'_{kl} > 0 \quad (2b) \]

2.2 Consumers

At time 0, the economy is populated by \( N_0 \) individuals, represented as a continuum, each indexed by \( I \), and identical in all respects except for their initial endowments of capital, \( K_{i,0} \). Each individual defines a “family.” Population grows uniformly across all families at the exponential rate, \( n \), so that at time \( t \) family \( i \) has grown to \( e^{nt} \) and the total population of the economy is \( N(t) = N_0 e^{nt} \). Each member of a given family has the same capital stock, although the distribution of capital differs across families. From a distributional perspective we are interested in the share of family \( i \)'s capital stock of the total capital stock in the economy. To this end we identify the following quantities:

(i) Individual \( i \) holds \( K_i(t) \) units of capital at time \( t \), so that the amount held by family \( i \) is \( K_i(t) e^{nt} \). This depends upon the capital of each representative member of the family plus the fact that the size of the family is growing exponentially over time.

(ii) The total amount of capital in the economy at time \( t \) is the total capital stock owned by the \( N_0 \) families and can be expressed as

\[ K^T(t) = \int_0^{N_0} K_i(t) e^{nt} \, di \]

(iii) The average per capital amount of capital is

\[ K(t) = \frac{1}{N_0 e^{nt}} \int_0^{N_0} K_i(t) e^{nt} \, di = \frac{1}{N_0} \int_0^{N_0} K_i(t) \, di. \]

Since the economy is growing we must be careful in defining the distribution of the capital stock. We shall define the share of capital owned by family \( i \) as

\[ k_i(t) \equiv \frac{K_i(t) e^{nt}}{K^T(t) / N_0} = \frac{K_i(t) e^{nt}}{\frac{1}{N_0} \int_0^{N_0} K_i(t) e^{nt} \, di} = \frac{K_i(t)}{\frac{1}{N_0} \int_0^{N_0} K_i(t) \, di} = \frac{K_i(t)}{K(t)} \]

With all agents in the different families growing at the same rate, we can express the distribution in terms of relative family shares, \( k_i(t) \). Note that relative capital has mean 1. We denote its initial distribution function by \( H_0(k) \), the initial density function by \( h_0(k) \), and the initial (given) standard
deviation of relative capital by $\sigma_{k,0}$.

We now focus on a particular agent. Each individual is endowed with a unit of time that can be allocated either to leisure, $l_i$, or to work, $1 - l_i \equiv L_i$. The agent maximizes lifetime utility, assumed to be a function of both consumption and the amount of leisure time, in accordance with the isoelastic utility function

$$\max \int_0^\infty \frac{1}{\gamma} (C_i(t)l_i^\gamma)^{\gamma - 1} e^{-\beta t} dt, \quad \text{with} \quad -\infty < \gamma < 1, \eta > 0, \gamma \eta < 1$$  \hspace{1cm} (3)

where $1/(1 - \gamma)$ equals the intertemporal elasticity of substitution.\(^7\) The preponderance of empirical evidence suggests that this is relatively small, certainly well below unity, so that we shall restrict $\gamma < 0$.\(^8\) The parameter $\eta$ represents the elasticity of leisure in utility. This maximization is subject to the agent’s capital accumulation constraint

$$\dot{K}_i(t) = (r(t) - n)K_i(t) + w(t)(1 - l_i(t)) - C_i(t)$$  \hspace{1cm} (4)

The first-order conditions from the consumer problem are standard and are reported in the Appendix; see (A.1) - (A.4).\(^9\) From these equations we obtain

$$\eta \frac{C_i}{l_i} = w(K, l)$$  \hspace{1cm} (5)

$$\left(\gamma - 1\right) \frac{\dot{C}_i}{C_i} + \eta \gamma \frac{\dot{l}_i}{l_i} = \frac{\dot{\lambda}_i}{\lambda_i} = \beta + n - r(K, l)$$  \hspace{1cm} (6)

where $\dot{\lambda}_i$ is agent $i$’s shadow value of capital.\(^10\) Equation (5) equates the marginal rate of substitution between consumption and leisure to the price of leisure, while (6) is the Euler equation modified to take into account the fact that leisure changes over time. The important point about (6) is that each agent, irrespective of capital endowment, chooses the same growth rate for the shadow value of capital. Using (5) we may write the individual’s accumulation equation, (4), in the form

\(^7\) For convenience we focus on a constant elasticity utility function which is homogeneous of degree $\gamma(l + \eta)$ in $C_i$ and $l_i$. In the Appendix, we indicate how the analysis is easily adapted to any arbitrary homogeneous utility function.

\(^8\) See e.g. the discussion of the empirical evidence summarized and reconciled by Guvenen (2006).

\(^9\) Time dependence of variables will be omitted whenever it causes no confusion.

\(^{10}\) We are assuming an internal solution exists, and hence write the Euler equation with equality. On the role of borrowing constraints and corner solutions for the dynamics of wealth see Hernández (1991).
\[
\frac{\dot{K}_i}{K_i} = r(K_i, l) - n + \frac{w(K_i, l)}{K_i} \left( 1 - l_i \frac{1 + \eta}{\eta} \right)
\]  
(7)

Taken together with the corresponding conditions for the aggregate economy we can derive the macroeconomic equilibrium and the dynamics of the aggregate economy. Having determined these, we shall then obtain the dynamics of the distribution of capital and income.

2.3. Derivation of the macroeconomic equilibrium

In general, we shall define economy-wide averages as

\[
Z(t) = \frac{1}{N_0} \int_0^{N_0} Z_i(t) e^{u} \, di = \frac{1}{N_0} \int_0^{N_0} Z_i(t) \, di
\]

Summing over households, equilibrium in the capital and labor markets is described by

\[
K = \frac{1}{N_0} \int_0^{N_0} K_i(t) \, di
\]  
(8a)

\[
L = 1 - l = \frac{1}{N_0} \int_0^{N_0} (1 - l_i(t)) \, di
\]  
(8b)

Equation (8b) gives the relationship between average leisure and the average labor supply. Note that in equations (2) we have defined the wage and the interest rate, \( w, r \), and expressed them as functions of average employment, \( L \). From (8b), we can equally well write them as functions of aggregate leisure time, \( 1 - l \), namely, \( w = w(K, l) \) and \( r = r(K, l) \).

The key element allowing aggregation is that all agents choose the same growth rate for the shadow value of capital, as seen in (6). As a result of this, we can then show (see Appendix) that

\[
\frac{\dot{C}_i}{C_i} = \frac{\dot{C}}{C}; \quad \frac{i_i}{l_i} = \frac{i}{l} \quad \text{for all } i
\]  
(9)

That is, all agents will choose the same growth rate for consumption and leisure, implying further that average consumption, \( C \), and leisure, \( l \), will also grow at the same common growth rates.

Now turn to the aggregates. Summing (5) over all agents, the aggregate economy-wide consumption-capital ratio is
\[ \eta \frac{C}{l} = w(K,l) \quad (5') \]

while summing over (6) and (7) yields the aggregate Euler and capital accumulation equations

\[ (\gamma - 1) \frac{\dot{C}}{C} + \eta \frac{\dot{l}}{l} = \beta + n - r(K,l) \quad (6') \]

\[ \frac{\dot{K}}{K} = r(K,l) - n + \frac{w(K,l)}{K} \left( 1 - l \frac{1 + \eta}{\eta} \right) \quad (7') \]

In the Appendix we show how this can be reduced to a pair of dynamic equations in \( K \) and \( l \), [(A.8) and (A.9)], which are independent of the distributional aspects and identical to those in the representative agent economy.

2.4. Aggregate equilibrium dynamics

The dynamic behavior of the aggregate economy as represented by (A.8) and (A.9) has been studied elsewhere; see, for example Turnovsky (2002). Assuming that the economy is stable, the aggregate quantities converge to a steady state characterized by a constant average per capita capital stock, labor supply, and leisure time, denoted by \( \bar{K} \), \( \bar{L} \) and \( \bar{l} \), respectively. Setting \( \dot{K} = \dot{l} = 0 \), the steady state is summarized by

\[ F_K(\bar{K}, \bar{L}) = \beta + n \quad (10a) \]

\[ F(\bar{K}, \bar{L}) - n\bar{K} = \frac{F_L(\bar{K}, \bar{L})\bar{l}}{\eta} \quad (10b) \]

\[ \bar{L} + \bar{l} = 1 \quad (10c) \]

These equilibrium relations are standard. Equation (10a) is the modified golden rule, equating the marginal product of capital to the discount rate, adjusted for population growth. The second is simply a reformulation of the first-order condition \((5')\) equating the marginal rate of substitution of consumption and leisure to the price of leisure (the real wage), where the left-hand side captures the fact that in steady state consumption is equal to output minus the amount needed to keep per capita
capital constant with a growing population. The third is just the labor market clearing condition.

Using (10a) and (10b), while recalling the homogeneity of the production function, we immediately infer that:

\[ \tilde{l} > \frac{\eta}{1+\eta} \]  

(11)

This inequality yields a lower bound on the steady-state time allocation to leisure that is consistent with a feasible equilibrium. As we will see below, this condition plays a critical role in characterizing the dynamics of distribution.

In order to describe the dynamics of the distribution of capital and income, we first need to obtain the dynamics of the aggregate magnitudes. Linearizing equations (A.8) and (A.9) around steady state yields the local dynamics for \( K \) and \( l \),

\[
\begin{pmatrix}
\dot{K} \\
l
\end{pmatrix} =
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\begin{pmatrix}
K - \tilde{K} \\
l - \tilde{l}
\end{pmatrix}
\]  

(12)

where \( a_{11}, a_{22}, a_{12}, a_{21} \) are defined in the Appendix. There we show that \( a_{11}a_{22} - a_{12}a_{21} < 0 \), implying that the steady state is a saddle point. The stable path for \( K \) and \( l \) can be expressed as

\[ K(t) = \tilde{K} + (K_0 - \tilde{K})e^{\mu t} \]  

(13a)

\[ l(t) = \tilde{l} + \frac{a_{21}}{\mu - a_{22}}(K(t) - \tilde{K}) + \frac{\mu - a_{11}}{a_{12}}(K(t) - \tilde{K}) \]  

(13b)

where \( \mu < 0 \) is the stable eigenvalue. From the sign pattern established in the Appendix, \( (a_{22} - \mu) > 0 \), implying that the slope of stable arm depends inversely upon the sign of \( a_{21} \). The sign of this expression reflects two offsetting influences of capital on the evolution of leisure. On the one hand, an increase in capital lowers the return to capital and hence the return to consumption, thereby reducing the growth rate of consumption and raising the desire to increase leisure. At the same time, the higher capital stock, by reducing the productivity of labor, raises the benefits from increasing labor, thus reducing the growth of leisure. As we show in the Appendix, which effect dominates

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11 We can combine (10a) and (10b) to yield: \( F_L / \eta \tilde{K}(\tilde{l}(1+\eta)-\eta) = \beta \) from which (11) follows.
depends upon the underlying parameters and in particular upon the elasticity of substitution in production. There we demonstrate that for plausible cases [including the conventional case of Cobb-Douglas production and logarithmic utility ($\varepsilon = 1, \gamma = 0$)] $a_{21} < 0$, in which case the stable locus is positively sloped; accumulating capital is therefore associated with increasing leisure.\footnote{See Appendix for further discussion.}

As we will see below, the evolution of average leisure over time is an essential determinant of the time path of wealth and income inequality. For expositional convenience we shall restrict ourselves to what we view as the more plausible case of a positive sloped stable locus, (13b). Since this relationship holds at all times, we have

$$l(0) - \bar{l} = \frac{a_{21}}{\mu - a_{22}} \left( K_0 - \tilde{K} \right)$$  \hspace{1cm} (13b')

Thus consider a situation in which the economy is subject to a structural shock that results in an increase in the steady-state average per capita capital stock relative to its initial level ($K_0 < \tilde{K}$). The shock will lead to an initial jump in average leisure, such that $l(0) < \bar{l}$, so that, thereafter, leisure will increase monotonically during the transition; an analogous relationship applies if $K_0 > \tilde{K}$.

3. **The distribution of income and wealth**

3.1. **The dynamics of the relative capital stock**

To derive the dynamics of individual $i$’s relative capital stock, $k_i(t) \equiv K_i(t)/K(t)$, we combine (7) and (7') to obtain

$$\dot{k}_i(t) = \frac{w(K,l)}{K} \left[ 1 - \theta \left( 1 + \frac{1}{\eta} \right) - \left( 1 - \dot{l} \left( 1 + \frac{1}{\eta} \right) \right) k_i \right]$$  \hspace{1cm} (14)

where $K,l$ evolve in accordance with (13a, 13b) and the initial relative capital $k_{i,0}$ is given from the initial endowment. Since $\dot{l}/l = \dot{l}/l$ we may write

$$l_i = \theta l \quad \text{where} \quad \frac{1}{N_0} \sum_{i=0}^{N} \theta_i \, di = 1$$
and \( \theta_i \) is constant for each \( i \), and yet to be determined.

To solve for the time path of the relative capital stock, we first note that agent \( i \)'s steady-state share of capital satisfies

\[
1 - \theta_i \tilde{I} \left( 1 + \frac{1}{\eta} \right) - \left( 1 - \tilde{I} \left( 1 + \frac{1}{\eta} \right) \right) \tilde{k}_i = 0 \quad \text{for each } i
\]

or, equivalently

\[
\tilde{l}_i - \tilde{l} = \left( \tilde{l} - \frac{\eta}{1 + \eta} \right) (\tilde{k}_i - 1) \quad \text{for each } i \tag{15}
\]

Recalling (11), this equation implies that the higher an agent’s steady-state relative capital stock (wealth), the more leisure he chooses and the less labor he supplies. This relationship is a critical determinant of the distributions of wealth and income and explains why the evolution of the aggregate quantities such as \( K \) and \( l \) are unaffected by distributional aspects. There are two key factors contributing to this: (i) the linearity of the agent’s labor supply as a function of his relative capital, and (ii) the fact that the sensitivity of labor supply to relative capital is common to all agents, and depends upon the aggregate economy-wide leisure. As a consequence, aggregate labor supply depends only on the aggregate amount of capital but not on its distribution amongst agents. It is important to note here that this result holds for any utility function that is homogenous of degree, \( b \) say, in consumption and leisure, and in the Appendix we show that this is indeed the case.

To analyze the evolution of the relative capital stock, we linearize equation (14) around the steady-state \( \tilde{K}, \tilde{I}, \tilde{k}_i, \tilde{l}_i \), in (15). In the Appendix we show that the stable solution to the resulting equation is

\[
k_i(t) - 1 = \delta(t)(\tilde{k}_i - 1) \tag{16}
\]

where

\[
\delta(t) \equiv 1 + \left( \frac{1}{\beta - \mu} \right) \frac{F_L(\tilde{K}, \tilde{L})}{K} \left( 1 - \frac{l(t)}{l} \right), \tag{17}
\]
Setting \( t = 0 \) in (16) and (17), we have

\[
k_{i,0} - 1 = \delta(0)(\tilde{k}_i - 1) = \left( 1 + \frac{1}{\beta - \mu} \right) \frac{F_k(\tilde{K}, \tilde{L})}{\tilde{K}} \left( 1 - \frac{l(0)}{l} \right)(\tilde{k}_i - 1)
\]

(18)

where \( k_{i,0} \) is given from the initial distribution of capital endowments.

The evolution of agent \( i \)'s relative capital stock is determined as follows. First, given the time path of the aggregate economy, and the distribution of initial capital endowments, (18) determines the steady-state distribution of capital, \((\tilde{k}_i - 1)\), which together with (16) then yields the entire time path for the distribution of capital. Using (16) – (18), and equations (13), describing the evolution of the aggregate economy, we can express the time path for \( k_i(t) \) in the form

\[
k_i(t) - \tilde{k}_i = \left( \frac{\delta(t) - 1}{\delta(0) - 1} \right)(k_{i,0} - \tilde{k}_i) = \left( \frac{l(t) - \tilde{l}}{l(0) - \tilde{l}} \right)(k_{i,0} - \tilde{k}_i) = e^{\mu t}(k_{i,0} - \tilde{k}_i)
\]

(19)

from which we see that \( k_i(t) \) also converges to its steady state value at the rate \( \mu \).

We can also determine the time path for the individual’s leisure (labor supply). First, having determined \((\tilde{k}_i - 1)\), (15) yields agent \( i \)'s steady-state leisure allocation, \( \tilde{l}_i \), which, knowing the economy-wide average, \( \tilde{l} \), determines his constant relative leisure time \( \theta_i \), namely

\[
\theta_i - 1 = \left( \frac{1}{\tilde{l}} \left( \frac{\eta}{1 + \eta} \right) \right)(\tilde{k}_i - 1) = \frac{1}{\delta(0)} \left( 1 - \frac{1}{\tilde{l}} \left( \frac{\eta}{1 + \eta} \right) \right)(k_{i,0} - 1)
\]

(20)

Thus, knowing the time path for the aggregate leisure allocation, \( l(t) \), the time path for \( l_i(t) \equiv \theta_i l(t) \) immediately follows. We see from (20), in conjunction with (11), that any agent whose steady-state capital stock exceeds the economy-wide average will enjoy above average leisure time throughout the transition.

Because of the linearity of (16), (18), and (19), we can immediately transform these equations into corresponding results for the standard deviation of the distribution of capital, which serves as a convenient measure of wealth inequality. Specifically, corresponding to these three equations we obtain

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13 Because of the linearity of \( k_i(t) \) in \( k_{i,0} \), the same analysis applies in terms of more conventional Gini coefficients.
\[ \sigma_k(t) = \delta(t) \tilde{\sigma}_k \]  
(16’)

\[ \sigma_{k,0} = \delta(0) \tilde{\sigma}_k \]  
(17’)

\[ \sigma_k(t) - \tilde{\sigma}_k = e^{\mu t}(\sigma_{k,0} - \tilde{\sigma}_k) \]  
(19’)

The allocation of wealth then converges to a long-run distribution. Moreover, from (19) it follows that the ranking of agents according to wealth is the same as in the initial distribution.

Having established the existence of a long-run distribution of wealth, we can compare it to the initial distribution. From equations (16’) and (19’) we see that \( \sigma_k(t) > \sigma_{k,0} \) if and only if \( \delta(0) < \delta(t) \), i.e. if and only if \( I(t) < I(0) \), and that \( \tilde{\sigma}_k > \sigma_{k,0} \) if and only if \( \delta(0) < 1 \). Together with (13b’), and since leisure is monotonically increasing or decreasing along the transition path, this implies the following\(^\text{14}\)

**Proposition 1 (Wealth dynamics): The long-run distribution of wealth converges to a steady state distribution. If the economy starts below (above) the steady state, i.e. \( K_0 < \bar{K} \) (\( K_0 > \bar{K} \)), then wealth inequality will decrease (increase) during the transition, and the long-run distribution of wealth will be less (more) unequal than is the initial distribution.**

The intuition for this result can be easily seen by noting, from equation (A.16) in the Appendix, that

\[ \text{sgn}(k_i - \bar{k}_i) = \text{sgn}(1 - \bar{k}_i)(I(0) - \bar{I}) \]

Recall that if the economy converges to the steady state from below, then \( I(0) < \bar{I} \). Then for people who end up above the mean level of wealth, their wealth will have decreased during the transition \( k_{i,0} > \bar{k}_i \), while for people who end up below the mean level of wealth, their wealth will have decreased, \( k_{i,0} > \bar{k}_i \), implying a narrowing of the wealth distribution.

This result contrasts with the evolution of the distribution of wealth in the Ramsey model

\(^{14}\) We restrict our focus to what we have identified as the “normal” case of a positive adjustment between aggregate leisure and capital
with inelastic labor supply. In this case, if the elasticity of substitution is greater than or equal to one, i.e. $\varepsilon \geq 1$, the distribution of wealth will become more equal during the transition from below, but for $\varepsilon < 1$, the distribution could widen. The reason for this is that a low elasticity of substitution implies fast wage growth as the economy accumulates capital. With sufficiently high wage growth, poor consumers may choose to dis-save early in their life-times and finance current consumption with their (high) future wages. As a result, the distribution of capital becomes more unequal. With endogenous leisure, this effect is offset by labor supply responses, as higher future wages tend to increase both current consumption and future leisure. The desire to increase leisure in the future prevents the reduction in the rate of capital accumulation of capital-poor agents, and hence the wealth divergence, that occurs when individuals cannot change work-hours. Nevertheless, capital accumulation will be associated with increasing wealth inequality only for unrealistic values of the elasticity of substitution in production and the intertemporal elasticity of substitution in consumption, and as we will see below, for plausible parameter values, wealth distribution behaves in a similar way with fixed and with flexible labor.

The standard deviation of the distribution of leisure time can be obtained from (20), and using the fact that $\sigma_{k,0} = \delta(0)\tilde{\sigma}_k$, can be expressed as

$$\sigma_\theta = \left(1 - \frac{1}{l} \frac{\eta}{1 + \eta}\right)\tilde{\sigma}_k$$

which in conjunction with (11) implies that there is a positive correlation between the steady state distributions of wealth and leisure time. That is, there is a negative correlation between the distribution of labor supplies and that of wealth, as richer individuals chose to consume more leisure than poorer ones. Note also that the term in brackets in (21) is smaller the lower the steady state level of average leisure, implying that a higher average labor supply will be associated (for given $\tilde{\sigma}_k$) with a more unequal distribution of work-time.

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15 See Caselli and Ventura (2000).
16 See García-Peñalosa and Turnovsky (2006b).
3.2. Income Distribution

We define the income of individual $i$ at time $t$ as $Y_i(t) = r(t)K_i(t) + w(t)(1-l_i(t))$, average economy-wide income as $Y(t) = r(t)K(t) + w(t)(1-l(t))$, and we are interested in the evolution of relative income, defined as $y_i(t) = Y_i(t)/Y(t)$. Letting $s(t) = F_t K/Y$ denote the share of output going to capital, and recalling that $l_i = \theta l$, relative income may be expressed as

\[
y_i(t) - 1 = s(t)(k_i(t) - 1) + (1-s(t)) \frac{l(t)}{1-l(t)} (1-\theta)
\]

The relative income of agent $i$ has two components, relative capital income, captured by the first term in (22), and relative labor income, reflected in the second term. The capital share determines the relative contribution of capital and labor to overall income, for given individual endowments. The endogeneity of labor implies that for a given capital share and distribution of capital, there will be less inequality than under exogenous labor. To see this note that with fixed labor, relative income is given by $y_i(t) - 1 = s(t)(k_i(t) - 1) + (1-s(t))$ as all agents supply the same amount of labor. When labor is flexible, poor agents supply more labor than do the rich (see equation (20)), which partially offsets the effect of the unequal distribution of capital, as we can see if we rewrite (22) as

\[
y_i(t) - 1 = s(t)(k_i(t) - 1) - (1-s(t)) \frac{l(t)}{1-l(t)} \left(1 - \frac{1}{l(t)} \frac{\eta}{1+\eta} \right) (k_i - 1). \tag{22'}
\]

Using equation (15) to substitute for $\theta_i$ and (16), we may write (22) in the form

\[
y_i(t) - 1 = \varphi(t)(k_i(t) - 1), \tag{23}
\]

where

\[
\varphi(t) = 1-(1-s(t)) \left[1 + \frac{l(t)}{1-l(t)} \left(1 - \frac{1}{l(t)} \frac{\eta}{1+\eta} \right) \delta(t) \right]. \tag{24}
\]

Again, because of the linearity of (23) in $(k_i(t) - 1)$ we can express the relationship between relative income and relative capital in terms of corresponding standard deviations of their respective distributions, namely
\[ \sigma_y(t) = \varphi(t)\sigma_k(t) \quad (23') \]

From inequality (11) the term in square brackets in equation (24) is positive and hence \( \varphi(t) < 1 \), implying that income is more equally distributed than is capital.

Letting \( t \to \infty \), we can express the steady-state distribution of income as

\[ \bar{\sigma}_y = \bar{\varphi}\bar{\sigma}_k \quad (23'') \]

where

\[ \bar{\varphi} = \lim_{t \to \infty} \varphi(t) = 1 - \frac{1}{1+\eta} \left( 1 - \tilde{\delta} \right) = 1 - \frac{1}{1+\eta} \frac{F_k(\tilde{K},\tilde{L})}{F(K,L)}. \]

From (23) we can compare the long-run distribution of income to the initial one, namely

\[ \frac{\sigma_y}{\sigma_{y,0}} = \frac{\bar{\sigma}}{\sigma_{0,0}} \frac{\bar{\sigma}_k}{\sigma_{k,0}} = \frac{1+\eta - F_k(\tilde{K},\tilde{L})}{1+\eta - F_k(K_0,L_0)} \frac{\bar{\sigma}_k}{\sigma_{k,0}} \quad (25) \]

where the subscript 0 identifies the initial distribution, from which we infer that in general \( \text{sgn} (\tilde{y}_i - 1) = \text{sgn} (y_{i,0} - 1) \). The distribution of income hence converges to a long-run distribution such that the relative ranking of agents according to income is the same as that of capital, as well as that of the initial income distribution.

Whether the long-run distribution, following a structural change, is more or less unequal than the initial distribution depends on the long-run change in the distribution of capital, as reflected in \( \bar{\sigma}_k/\sigma_{k,0} \), and factor returns, as reflected in \( \bar{\varphi}/\bar{\varphi}_0 \). As we will illustrate in Section 5 below, any shock leads to an initial jump in the distribution of income, after which it evolves continuously, in response to the evolution of the distribution of capital and factor returns. These dynamics can be seen most conveniently by considering the time derivative of equation (22), namely

\[ \frac{dy_i(t)}{dt} = s(t) \frac{dk_i(t)}{dt} + (1-s(t)) \left( 1 - \theta_i \right) \frac{dl(t)}{dt} + \left( k_i(t) - 1 + (\theta_i - 1) \frac{l(t)}{1-l(t)} \right) \frac{ds(t)}{dt} \quad (26) \]

The equation indicates how the evolution of the relative income of agent \( i \) depends upon two factors, the evolution of relative capital income, reflected in the first term in (26), and that of relative labor.
income. The latter can be expressed as a function of the evolution of aggregate leisure (i.e. labor supply), and of the relative rewards to capital and labor, as reflected by the capital share, $s(t)$. Clearly, the term $dl(t)/dt$ is not present when the labor supply is exogenous.

It is useful to start by examining what happens for a Cobb-Douglas production function. In this case the capital share remains constant, and whether income inequality increases or decreases depends on whether the economy converges to the steady state from below or from above. Consider an economy that starts below the steady state, so that $K_0 < \bar{K}$. Then $l(0) < \bar{l}$ and leisure is rising, $dl/dt > 0$, while wealth inequality is decreasing. Consider an agent with above average wealth, $(k_i - 1) > 0$, then $dk_i/dt < 0$ and $(\theta_i - 1) > 0$, implying that the first two terms in (26) are negative and that the relative income of the agent is decreasing during the transition. The opposite would be true for an agent with wealth below average, $(k_i - 1) < 0$, and hence income inequality will decline during the transition to the steady state from below. For an economy that starts above the steady state, i.e. for $K_0 > \bar{K}$, then $l(0) > \bar{l}$, and together with the fact that wealth inequality is increasing (see Proposition 1) income inequality will be rising during the transition.

The evolution of factor shares may reinforce or offset these effects. For an economy that converges from below, a falling capital share, $ds/dt < 0$, would reinforce the impact of the distribution of wealth and income inequality will decline over time. If the capital share rises over time, $ds/dt > 0$, and this effect will be offsetting. If this latter effect dominates, the distribution of income would become less equal over time.

**Proposition 2 (Income dynamics):** The evolution of income inequality for an economy that converges to its steady state from below, i.e. $K_0 < \bar{K}$, (respectively, from above, i.e. $K_0 > \bar{K}$) is driven by three factors:

(i) decreasing (increasing) wealth inequality, which tends to reduce (raise) income inequality;

(ii) increasing (decreasing) leisure, which tends to increase (decrease) the relative labor income of the capital-poor and hence reduce (raise) income inequality;

(iii) the change in the share of capital in income, which depends both on whether the economy is converging from below or above, and on the elasticity of substitution.
**in production.**

*If the share of capital is constant, income inequality will decrease (increase) during the transition to the steady state from below (above).*

Proposition 2 has two implications. First, an economy may experience episodes of increasing and episodes of decreasing income inequality. To see this, consider an economy which is below its steady state and has an elasticity of substitution greater than one. As we saw above, the first two terms in (26), imply that inequality tends to fall. However, the high elasticity of substitution implies that as capital accumulates the share of capital increases, tending to make the distribution of income more dispersed. At different stages, one or the other effect may dominate, implying episodes of rising or falling inequality.

The second implication concerns the differences between the cases of inelastic and elastic labor supply. With exogenous labor, there is no change in leisure time. Hence, the evolution of the income distribution is driven by two forces, the change in wealth inequality and that of labor share. With a high elasticity of substitution in production, these two forces have opposite signs, and it is possible that the second dominates, making the distribution of income more dispersed. When the labor supply can respond, the changes in average leisure will change the distribution of work-time and tend to reinforce the wealth-distribution effect. The presence of this effect implies that income inequality may move in opposite directions depending on the elasticity of leisure, as we will see in the numerical examples in Section 5.

4. **Long-run adjustments of wealth and income inequality**

To illustrate the dynamic adjustments of wealth and income distribution we analyze three shocks that are of interest: (i) an increase in productivity, (ii) a decrease in the population growth rate, (iii) a decrease in the discount rate. In this section we report the formal expressions for the steady-state responses, and will simulate the dynamic adjustments in the Section 5.

4.1 **Aggregate and distributional effects of an increase in productivity**

We begin by recalling the steady state conditions for the aggregate economy, (10a) – (10c),
where we modify the production function \( Y = AF(K, L) \), and parameterize the productivity increase by an increase in \( A \). Since the distributions of wealth and income depend upon the aggregate economy, we first derive the steady-state responses as follows:

\[
\frac{d\tilde{K}}{\tilde{K}} = \frac{1 - \tilde{L}(1 + \eta)}{1 - \tilde{s}} \left[ 1 + \frac{\tilde{L}(1 + \eta)\varepsilon}{1 - \tilde{L}(1 + \eta)} \right] \frac{dA}{A} \tag{27a}
\]

\[
\frac{d\tilde{L}}{\tilde{L}} = \frac{1 - \tilde{L}(1 + \eta)}{1 - \tilde{s}} (1 - \varepsilon) \frac{dA}{A} \tag{27b}
\]

\[
\frac{d\tilde{Y}}{\tilde{Y}} \left[ 1 + \frac{1 - \tilde{L}(1 + \eta)}{1 - \tilde{s}} + \varepsilon\tilde{K} \left( \tilde{\beta} + (1 - \tilde{s})n \right) \right] \frac{dA}{A} \tag{27c}
\]

where \( \varepsilon \) denotes the elasticity of substitution between capital and labor in production.\(^\text{17}\) An increase in productivity will raise both the steady-state average per capita capital stock, \( \tilde{K} \), as well as output, \( \tilde{Y} \). Its effect on steady-state labor supply depends upon the elasticity of substitution, \( \varepsilon \), raising labor supply if \( \varepsilon < 1 \) and reducing it if \( \varepsilon > 1 \).

To consider the consequences of this for the long-run wealth distribution we recall Proposition 1. Since an increase in productivity raises the long-run capital stock (wealth) it leads to a decrease in the long-run inequality of wealth. To see what this implies for long-run income inequality recall (23), namely

\[
\frac{\tilde{\sigma}_y}{\sigma_{y,0}} = \frac{\tilde{\sigma}_k}{\sigma_{k,0}} = \frac{1 + \eta - F_{\tilde{L}}(\tilde{K}, \tilde{L}) / F(\tilde{K}, \tilde{L})}{1 + \eta - F(L_0, K_0) / F(K_0, L_0)} \frac{\tilde{\sigma}_k}{\sigma_{k,0}}
\]

This breaks down the steady-state change in income inequality into: (i) effect due to the change in wealth inequality (wealth effect), which we have just shown to be negative, and (ii) effect due to the change in labor supply (labor supply effect). The latter depends upon what happens to:

\[
\tilde{\varphi} \equiv 1 - \frac{F_{\tilde{L}}(\tilde{K}, \tilde{L})}{F(\tilde{K}, \tilde{L})} \left( \frac{1}{1 + \eta} \right)
\]

\(^\text{17}\) Note that \( 1 - \tilde{L}(1 + \eta) = I(1 + \eta) - \eta > 0 \) by the transversality condition; \( \tilde{s} \) denotes steady-state share of capital.
Differentiating this with respect to $A$ we obtain

$$\frac{d\tilde{\phi}}{dA} = (\varepsilon - 1) \frac{\tilde{K} \left[ \beta \tilde{L} + (1 - \tilde{s})n \right]}{A \tilde{L}(1 + \eta) \left[ A \tilde{F} - n \tilde{K} \right]} (27d)$$

so that

$$\text{sgn} \left( \frac{d\tilde{\phi}}{dA} \right) = \text{sgn}(\varepsilon - 1)$$

An increase in productivity leads to a reduction in long-run wealth inequality. This will lead to a larger, equal, or smaller decline in long-run income inequality according to whether the elasticity of substitution is smaller than, equal to, or larger than, unity. For a sufficiently large elasticity of substitution long-run income inequality may actually increase.

### 4.2 Aggregate and distributional effects of a decrease in the population growth rate

Second, we consider a decrease in the growth rate of population, yielding the effects

$$\frac{d\tilde{K}}{K} = -\frac{\beta}{D\varepsilon} \left[ \frac{\tilde{L}(1 + \eta)\varepsilon}{1 - \tilde{L}(1 + \eta)} + \tilde{s} \right] dn < 0 \quad (28a)$$

$$\frac{d\tilde{L}}{L} = \frac{\beta}{D\varepsilon} (\varepsilon - \tilde{s}) dn \quad (28b)$$

$$\frac{d\tilde{Y}}{\tilde{Y}} = \frac{\beta}{D\varepsilon} \left[ (\varepsilon - \tilde{s}) - \frac{\tilde{s}\varepsilon}{1 - \tilde{L}(1 + \eta)} \right] dn \quad (28c)$$

where $D\varepsilon \equiv (\beta + n) \left[ A \tilde{F} - n \tilde{K} \right] (1 - \tilde{s}) / \tilde{K} (1 - \tilde{L}) > 0$. The effect of a reduction in the growth rate of population on labor supply depends critically on the relative sizes of the elasticity of substitution and the share of income going to capital. Despite the ambiguity of this response, the reduction in the population growth rate increases the steady state stock of capital, $\tilde{K}$, and hence reduces the inequality of wealth. The net effect on per capita income depends upon the relative sizes of $\varepsilon$ and $s$.

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18 The derivations of a number of these expressions, such as $d\tilde{\phi}/dA$, $d\tilde{\phi}/dn$ involves a lot of detail, making extensive use of equilibrium conditions. They can be expressed in a number of equivalent ways, and we have chosen what we view as the most convenient form. Since these calculations do not have any intrinsic interest, we do not report them, but they are available from the authors on request.
Differentiating $\phi$ with respect to $n$ yields

$$\frac{d\phi}{dn} = \tilde{K}\left\{(1-\varepsilon)\left[\beta\tilde{K}(\beta+n)+(A\tilde{F}/\tilde{L})(1-\tilde{s})(n+\beta\tilde{L})\right]+(1-\tilde{s})(1-\tilde{L})(A\tilde{F}/\tilde{L})\right\}
\frac{A\tilde{F}+(\beta+n)(A\tilde{F}-n\tilde{K})}{(1+\eta)(\beta+n)(A\tilde{F}-n\tilde{K})}$$

(28d)

implying that if $\varepsilon \leq 1$, then $d\phi/dn > 0$. A decrease in the population growth rate leads to a reduction in long-run wealth inequality. This will lead to a larger, equal, or smaller decline in long-run income inequality depending on the elasticity of substitution. If $\varepsilon \leq 1$, the shock results in a larger than proportionate decline in income inequality. For a sufficiently large elasticity of substitution long-run income inequality may actually increase.

### 4.3 Aggregate and distributional effects of a decrease in the discount rate

Consider now the effect of a reduction in the discount rate. In this case,

$$\frac{d\tilde{K}}{\beta} = -\frac{\beta}{D\varepsilon}\left[(1-\tilde{s})\frac{n}{\beta}+1+\frac{\varepsilon\tilde{L}(1+\eta)}{1-\tilde{L}(1+\eta)}\right]d\beta < 0$$

(29a)

$$\frac{d\tilde{L}}{\tilde{L}} = -\frac{\beta}{D\varepsilon}\left[(1-\tilde{s})\frac{n}{\beta}+(1-\varepsilon)\right]d\beta$$

(29b)

$$\frac{d\tilde{Y}}{\tilde{Y}} = -\frac{\beta}{D\varepsilon}\left[(1-\tilde{s})\frac{n}{\beta}+1-\varepsilon + \frac{\tilde{s}\varepsilon}{1-\tilde{L}(1+\eta)}\right]d\beta < 0.$$  

(29c)

The impact of a lower $\beta$ is similar to that of a reduction in $n$. The reduction in the discount rate increases the steady state stock of capital, $\tilde{K}$, and hence reduces the inequality of wealth. The effect on labor supply depends on the elasticity of substitution, while per capita income unambiguously increases. Differentiating $\phi$ with respect to $\beta$ yields

$$\frac{d\phi}{d\beta} = -\frac{\tilde{K}\left\{(\varepsilon-1)\left[\beta\tilde{K}(\beta+n)+(A\tilde{F}/\tilde{L})(1-\tilde{s})(n+\beta\tilde{L})\right]+(1-\tilde{s})(1-\tilde{L})(A\tilde{F}/\tilde{L})n\right\}}{A\tilde{F}(1+\eta)(\beta+n)(A\tilde{F}-n\tilde{K})}$$

(29d)

implying that if $\varepsilon \geq 1$, then $d\phi/d\beta < 0$. As before the decrease in the discount rate leads to a reduction in long-run wealth inequality, and this will lead to a larger, equal, or smaller decline in long-run income inequality depending on the elasticity of substitution. In contrast to the effect of a
reduction in the population growth rate, for an elasticity of substitution $\varepsilon \geq 1$ long-run income inequality will actually increase.

5. Numerical Simulations

To obtain further insights into the dynamics of wealth and income distribution we simulate the economy in response to these shocks. The simulations are based on the following functional forms and standard parameter values, characterizing the benchmark economy.

<table>
<thead>
<tr>
<th>Production function:</th>
<th>$Y = A(\alpha K^{-\rho} + (1 - \alpha)L^{-\rho})^{-1/\rho}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Utility function:</td>
<td>$U = (1/\gamma)(CL^\rho)^\gamma$</td>
</tr>
<tr>
<td>Basic parameters:</td>
<td>$A = 1, \alpha = 0.4$</td>
</tr>
<tr>
<td></td>
<td>$\rho = 1/3, 0, -0.2$ (elast of sub $\varepsilon = 0.75, 1, 1.25$)</td>
</tr>
<tr>
<td></td>
<td>$\beta = 0.04, \gamma = -1.5, n = 0.015$</td>
</tr>
<tr>
<td>Endogenous labor</td>
<td>$\eta = 1.75$</td>
</tr>
<tr>
<td>Exogenous labor</td>
<td>$\bar{L} = 0.316, 0.278, 0.193$ corresponding to $\varepsilon = 0.75, 1, 1.25$</td>
</tr>
</tbody>
</table>

Preferences are specified by a constant elasticity utility function, with intertemporal elasticity of substitution $1/(1 - \gamma) = 0.4$, while the elasticity of leisure in utility is 1.75. The production function is of the CES form, where we allow the elasticity of substitution to assume the values 0.75, 1, and 1.25, while the distributional parameter is $\alpha = 0.4$. Population grows at the rate of 1.5% per annum, while $A = 1$ scales the initial level of productivity.

We assume that the economy is initially in a steady state in which aggregate fraction of time devoted to leisure is $l_0$ and the average stock of capital is $K_0$. For the benchmark Cobb-Douglas economy with endogenous labor supply, $l_0 = 0.722$, which is plausible and implies a labor supply of $L = 0.278$. To preserve comparability, in the case of inelastic labor supply we normalize the fixed labor supply to the same level, i.e. $\bar{L} = 0.278$; implying that all aggregate magnitudes will be the same in the two cases. Similar adjustments are made for other values of $\varepsilon$, as indicated.

We define the initial steady state distributions of wealth (capital) and income (prior to any shock) by the quantities $\bar{\sigma}_{k,0}$ and $\bar{\sigma}_{y,0} = \left[1 - F_{\bar{L}}(\bar{K}_0, \bar{L}_0) \frac{1}{1 + \eta} \right] \sigma_{k,0}$, respectively. Before examining the effects of the shocks, consider the steady-state relationship between income and wealth,
Table 1 reports this ratio for various values of the elasticities of labor supply and of substitution in production and reveals that it is highly sensitive to both. The first column reports the case of an inelastic labor supply, i.e. $\eta = 0$. The table indicates that as $\eta$ increases, income inequality declines relative to the fixed labor supply economy, and this is the case for all values of $\varepsilon$. For the benchmark value $\eta = 1.75$, the ratio of income to wealth inequality is about 50% of what it would be if labor supply is fixed. This is because of the negative relation between individual wealth and individual labor supply as captured by equation (15). Since wealthier individuals supply less labor than do the poorer ones, their relative labor income is lower and hence partially offsets the distributional impact of wealth inequality. The higher $\eta$, the stronger is the (negative) correlation between wealth and labor supply, and the more equal is the income distribution for any given level of asset inequality. This suggests that treating labor supply as fixed is likely to seriously overstate the amount of income inequality.

Consider now the effects of the three shocks discussed in section 4. Starting from the initial steady state, $\tilde{\sigma}_{k,0}, \tilde{\sigma}_{y,0}$, we shall investigate the time paths of the economy in response to three structural changes:

(i) An increase in the level of technology $A$ from 1 to 1.5 (Fig. 1);
(ii) A decrease in the rate of population growth rate, $n$, from 1.5% to 0 (Fig. 2)
(iii) A decrease in the discount rate, $\beta$, from 4% to 2% (Fig. 3).

We plot the time paths for the distribution of wealth and income, relative to their respective initial values, namely, $\sigma_{k}(t)/\tilde{\sigma}_{k,0}$ and $\sigma_{y}(t)/\tilde{\sigma}_{y,0}$, where we further normalize $\tilde{\sigma}_{k,0} = 1$. In the left-hand panels of these figures we plot the time paths of the wealth and income inequality when the labor supply is exogenous, i.e. for $\eta$, while the right-hand panels present the case with endogenous labor, under the assumption that $\eta = 1.75$.

5.1 Increase in $A$ from 1 to 1.5

From Fig. 1 we see that a productivity increase always reduces long-run wealth inequality.

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19 In effect we are graphing $\delta(t)/\delta(0)$ in the case of wealth and $\varphi(t)\delta(t)/\delta(0)$ in the case of the distribution of income.
Long-run income inequality also declines if $\varepsilon < 1$, though it increases if $\varepsilon = 1.25$. These long-run responses are virtually identical for both elastic and inelastic labor supply, as is the entire dynamic time path of the distribution of wealth. In contrast, while $\eta$ has little effect on the change in the long-run distribution of income, it does have a significant impact on its time path.

To see this, it is convenient to focus first on the Cobb-Douglas production function, illustrated by the middle pair of figures in Fig 1. As we showed in Section 4, for $\varepsilon = 1$, an increase in productivity $A$ will leave the long-run aggregate (average) employment unchanged, but will raise aggregate capital. The effect of the increase in the long-run capital stock is to cause the distribution of capital to become gradually more equal over time. For this parameter set, wealth inequality as measured by the standard deviation, decreases uniformly, declining by around 8%, asymptotically for both elastic and inelastic labor supply. Although long-run income inequality also declines by the same proportion in the two cases, with elastic labor supply its transition is very different. The short run decline in average leisure time (increase in labor supply) leads to an increase in short-run income inequality. This can be seen most directly from equation (20). The correctly anticipated decline in long-run wealth inequality reduces (increases) the amount of leisure time, chosen by people with above (below) average wealth. That is, wealthier people initially increase their work time, while poorer people work less and income inequality increases. Over time, as average leisure increases the relative income of agents having above-average wealth declines, for reasons noted in Section 3.2, and income inequality declines over time, eventually catching up to the decline in wealth inequality.

For a lower elasticity of substitution ($\varepsilon = 0.75$), the less flexible production function implies that the long-run accumulation in the aggregate capital stock results in a larger reduction in the return to capital and increase in the wage rate. The time path for the distribution of wealth is relatively unaffected, but over time income inequality declines more than does wealth inequality. In the case of the higher elasticity of substitution ($\varepsilon = 1.25$) the reduction in the long-run capital stock again implies a gradual decline in wealth inequality; in contrast, the long-run distribution of income becomes more unequal, moving in the opposite direction to the wealth distribution. The reason for this is the decrease in the labor share induced by a higher capital-labor ratio when $\varepsilon > 1$, which always dominates the effect of the more equal distribution of capital.
As in the Cobb-Douglas case, there is a sharp difference in the dynamics of income inequality between fixed and flexible labor supply, due to the initial jump in the labor supply which generates a short-run response of income inequality that is of opposite sign to its long-run response. In the case where $\varepsilon = 0.75$, the short-run increase in income inequality with $\eta = 1.75$ means that over time it declines more rapidly than does wealth inequality, overtaking the decline in the latter after around 20 years. In contrast, with $\varepsilon = 1.25$, income inequality under fixed labor increases monotonically to a level 3.8% higher that its initial value; with endogenous labor, it first falls by 8% and then rises sharply, increasing by 3.2% asymptotically. In both cases, the possibility of a labor supply response implies much larger movements in income inequality than with fixed labor, as well as non-monotonic behavior.

One further point we see is that if $\varepsilon = 1.25$, income distribution exhibits some mild non-monotonicity during its transition. This can be understood by recalling (26) and the fact that income distribution is responding to three factors: declining wealth inequality and relative employment effect, $\theta$, both of which tend to reduce inequality, and increasing capital share, which tends to raise it. Initially, the increasing capital share dominates and inequality rises, but as the capital-labor ratio approaches steady state value the effect of factor shares becomes weaker and income inequality is driven by the evolution of the distribution of capital, leading to a period of declining inequality.

5.2 Decrease in the rate of population growth from 1.5% to 0

When labor supply is fixed, the effects of reducing the population growth rate from 1.5% to 0 are qualitatively similar to those of an increase in productivity, with wealth inequality decreasing for all values of the elasticity of substitution and income inequality falling in all cases but that of a high elasticity in production ($\varepsilon = 1.25$). The response with flexible labor shows two sharp differences. First, as long as $\varepsilon > \tilde{\varepsilon}$, a restriction that holds in all three cases, the labor supply declines both in the short-run and long-run. As a result, income inequality exhibits an initial short-run drop due to the decline in labor supply, and keeps falling as the wealth distribution becomes more equal. The consequence of the labor supply response is that although the reduction in wealth inequality is virtually identical for fixed and for flexible labor, income inequality declines much more in the latter
case. For example, for the Cobb-Douglas production, long-run income inequality falls by 7% with \( \eta = 0 \) and by about 37% for \( \eta = 1.75 \).

The second important difference is that with a high elasticity of substitution \( \varepsilon = 1.25 \) long-run income inequality may increase or decrease depending on the value of \( \eta \). To understand this, again recall equation (26). There are three ways in which a shock affects relative income: the change in the distribution of wealth, the change in leisure, and the change in the labor share. With exogenous labor, there is no change in leisure time. Hence, when \( \varepsilon > 1 \), there are two opposite forces driving the evolution of the income distribution: the reduction in the labor share tends to increase income inequality while the more equal distribution of capital tends to reduce it. For our parameter set, the first effect dominates, resulting in a more unequal distribution of income. When the labor supply can respond, the lower rate of population growth will lead to more leisure time (as long as \( \varepsilon > \bar{s} \)). This effect is stronger for rich individuals and hence they will supply (relatively) less labor, while poor individuals will supply relatively more labor, tending to reduce income inequality. For the chosen parameter set, the effects through the distributions of capital and labor time are stronger than that operating through the reduction in the labor share, leading to lesser income inequality. That is, the decline in population growth will increase income inequality if labor supply is fixed but increase it when labor is supplied elastically.

### 5.3 Decrease in the rate of discount from 4% to 2%

With fixed labor supply, the effects of reducing the rate of time preference from 4% to 2% are again qualitatively similar to those in the previous cases. The differences in the response of income inequality with and without flexible labor are highly sensitive to the elasticity of substitution in production. For the Cobb-Douglas production \( \varepsilon = 1 \) long-run income is reduced if labor is fixed but increases if labor is flexible. This is because with endogenous labor, the reduction in \( \beta \) leads to a reduction in average leisure time, and hence to a more equal distribution of work time. This effect is sufficiently strong to offset the decline in wealth inequality, leading to a more dispersed distribution of income.

For a low elasticity \( \varepsilon = 0.75 \) long-run income inequality declines in both cases; however,
with elastic labor it first jumps to a higher value and then declines as it converges to its steady state value. Moreover, the long-run level of income inequality is higher with elastic labor, again due to the labor supply response. In the case of a high elasticity in production ($\varepsilon=1.25$) income inequality exhibits a similar behavior in both cases, although the increase in inequality will be small for fixed labor, 5%, and large for flexible labor, over 23%.

6. Concluding comments

This paper has studied the dynamics of wealth and income distributions in a Ramsey model with endogenous labor supply. We have utilized the fact that the homogeneity of the utility function facilitates aggregation and leads to a macroeconomic equilibrium having a simple recursive structure. First, the aggregate dynamics are determined, independently of distribution. Then, having determined aggregate behavior, the dynamics of wealth and income distribution are obtained in terms of the equilibrium aggregate quantities. The aggregate behavior of the model hence collapses to that of a representative-consumer setup, implying that existing results of conventional representative-agent growth models with homogeneous preferences are robust to the introduction of wealth heterogeneity.

Two general results emerge from our analysis of distributional dynamics. First, in the absence of impediments such as fixed costs or capital market imperfections, the accumulation of capital tends to reduce the degree of wealth inequality. Second, the dynamics of income inequality are driven by three factors: the dynamics of wealth inequality, of factor shares, and of leisure. The initial (discrete) adjustment of leisure following a structural change is the crucial factor accounting for the impact of elastic labor supply on the time path of income distribution.

The specific structural changes we introduce provide a range of illustrations of the sensitivity of the long-run response of income inequality to the production characteristics. These have been supplemented with numerical solutions which enable us to characterize the dynamic time path, and in particular, the crucial role played by the elasticity of labor supply. Overall, the simulations indicate that although the behavior of wealth inequality is very similar for elastic and inelastic labor, income distribution dynamics can differ substantially. Three general features stand out. First, with
fixed labor the distribution of income evolves continuously; with flexible labor, the initial jump in labor supply in response to a shock implies that income distribution will also jump, leading to much larger movements in the distribution. Second, the figures illustrate the possibility that, in response to a shock, the distribution of income becomes more equal or less equal depending on whether labor is elastic or not. Third, with fixed labor, all three shocks have an almost identical impact on income inequality. In contrast, under flexible labor, the shocks can generate diverse responses in long-run income inequality.

Finally, taken in conjunction with other contributions, our analysis highlights how sensitive existing results on distributional dynamics are to the particular assumptions made regarding the form of utility function and the technology. For example, Sorger (2000) and Ghiglino and Sorger (2002) find that with endogenous labor supply, but non-homogenous utility, the distribution of wealth affects aggregate income dynamics so that the dynamics of the aggregate economy and distribution become simultaneously determined. By contrast, in García-Peñalosa and Turnovsky (2006a, 2007) we consider homogeneous preferences, but introduce a technological externality resulting in an AK technology. In this case, the present recursive equilibrium structure continues to prevail. But now wealth distribution remains fixed at its exogenously given initial level, and income distribution, although endogenously determined, remains constant over time. In the absence of wealth dynamics, the response of income inequality is determined solely by the (constant) adjustment in labor supply. The present paper can be viewed as an intermediate case which offers the advantage of providing a tractable framework for studying the dynamics of wealth and income inequality.
Table 1
Steady-state income inequality

<table>
<thead>
<tr>
<th></th>
<th>η = 0</th>
<th>η = 1</th>
<th>η = 1.75</th>
<th>η = 2.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>ε = 0.75</td>
<td>0.244</td>
<td>0.155</td>
<td>0.131</td>
<td>0.117</td>
</tr>
<tr>
<td>ε = 1</td>
<td>0.400</td>
<td>0.254</td>
<td>0.215</td>
<td>0.192</td>
</tr>
<tr>
<td>ε = 1.25</td>
<td>0.652</td>
<td>0.418</td>
<td>0.353</td>
<td>0.316</td>
</tr>
</tbody>
</table>

Income inequality is expressed relative to steady-state wealth inequality.
Fig 1: Increase in A from 1 to 1.5

Inelastic labor supply

Elast of Subs = 0.75

Flexible labor supply

Elast of Subs = 1

Elast of Subs = 1.25

capital income
Fig 2: Decrease in n from 0.015 to 0

Inelastic labor supply

Flexible labor supply

Elast of Subs = 0.75

Elast of Subs = 1

Elast of Subs = 1.25

--- capital

--- income
Fig 3: Decrease in $\beta$ from 0.04 to 0.02

Inelastic labor supply
Elast of Subs =0.75

Flexible labor supply

Elast of Subs =1

Elast of Subs =1.25

---
capital
income
Appendix

This Appendix is devoted to the derivation of several technical details

A.1 Derivation of the macroeconomic equilibrium

The consumer’s problem defined by (3) and (4) yields the corresponding first-order conditions

\[ C_i^{t+1} l_i^{t+1} = \lambda_i \]  \hspace{1cm} (A.1)  

\[ \eta C_i^{t+1} l_i^{t+1} = w \lambda_i \]  \hspace{1cm} (A.2)  

\[ r - n = \beta - \frac{\lambda_i}{\lambda_i} \]  \hspace{1cm} (A.3)  

where \( \lambda_i \) is agent \( i \)'s shadow value of capital, together with the transversality condition

\[ \lim_{t \to \infty} \lambda_i K_i e^{-\beta t} = 0 \]  \hspace{1cm} (A.4)  

The first-order conditions (A.1)-(A.3) imply equations (5) and (6) in the text. Taking the time derivative of (5) yields

\[ \frac{\dot{C}_i}{C_i} - \frac{\dot{I}_i}{I_i} = \frac{w_i(K,l)K}{w(K,l)} + \frac{w_i(l)l}{w(l)} \frac{\dot{I}_i}{I_i} \]  \hspace{1cm} (A.5)  

Now consider equations (6) and (A.5) for individuals \( i \) and \( k \). We obtain

\[ (\gamma - 1) \left( \frac{\dot{C}_i}{C_i} - \frac{\dot{C}_k}{C_k} \right) + \eta \gamma \left( \frac{\dot{I}_i}{I_i} - \frac{\dot{I}_k}{I_k} \right) = 0 \]

\[ \left( \frac{\dot{C}_i}{C_i} - \frac{\dot{C}_k}{C_k} \right) - \left( \frac{\dot{I}_i}{I_i} - \frac{\dot{I}_k}{I_k} \right) = 0 \]

from which we infer

\[ \frac{\dot{C}_i}{C_i} = \frac{\dot{C}_k}{C_k} \quad \frac{\dot{I}_i}{I_i} = \frac{\dot{I}_k}{I_k} \quad \text{for all } i, k \]  \hspace{1cm} (A.6)
Now turn to the aggregates. Combining
\[
\frac{\dot{C}(t)}{C(t)} = \frac{\int_0^{N_i} \dot{C}_i(t) \, di}{\int_0^{N_i} C_i(t) \, di}
\]
with (A.6) implies
\[
\frac{\dot{C}_i}{C_i} = \frac{\dot{C}}{C}; \quad \frac{\dot{I}_i}{I_i} = \frac{\dot{l}}{l}
\quad \text{for all } i \quad (A.7)
\]

To obtain the goods market clearing condition, substitute for the equilibrium rate of return on capital, \(r(K,l)\), and the wage rate, \(w(K,l)\), into (7'). Recalling the linear homogeneity of the production function yields
\[
\dot{K} = F(K,L) - \frac{F_L(K,L)l}{\eta} - nK \quad (A.8)
\]

Second, substituting (A.6) into (6) and taking the time derivative of (5') to substitute for \(\dot{C}/C\), yields the aggregate modified Euler equation
\[
\left[ (\gamma - 1) \left( 1 + \frac{w_l(K,l)}{w(K,l)} \right) + \eta \gamma \right] \frac{\dot{l}}{l} + (\gamma - 1) \left( \frac{w_K(K,l)K}{w(K,l)} \right) \dot{K} = \beta + n - r(K,l)
\]

Finally, substituting (A.8) into this equation, and recalling the expressions \(w_L\), etc. implies
\[
\dot{l} = \frac{F_K(K,L) - \beta - n - (1 - \gamma) \frac{F_{KL}(K,L)}{F_L(K,L)} \left[ F(K,L) - \frac{F_L(K,L)}{\eta} - nK \right]}{G(l)} \quad (A.9)
\]

where
\[
G(l) \equiv \frac{1 - \gamma(1 + \eta)}{l} - \frac{(1 - \gamma)F_{LL}}{F_L} \quad (A.10)
\]

Equations (A.8) and (A.9) are autonomous equilibrium dynamic equations in the economy-wide average quantities of capital and leisure (or employment). Setting \(\dot{K} = \dot{l} = 0\) in (A.8) and (A.9), we can then express the steady state as in (10).
A.2 Labor supply with a homogeneous utility function

Suppose the utility function $U(C_i, l_i)$ is homogenous of degree $b$, so that we can write $U(C_i, l_i) = l_i^b U(C_i/l_i, l_i/l_i) \equiv l_i^b v(C_i/l_i)$. The first-order conditions for utility maximization with respect to consumption and leisure are now $U_c(C_i, l_i) = \lambda_i$ and $U_l(C_i, l_i) = w\lambda_i$, which can be expressed as

$$w = \frac{b v(C_i/l_i) - v'(C_i/l_i)(C_i/l_i)}{v'(C_i/l_i)}$$

Since the right-hand side of this expression depends only on the consumption-leisure ratio, we can invert it and express the consumption-leisure ratio as a function of the wage, $C_i/l_i = \phi(w)$. Note that $C_i/K_i = \phi(w)l_i/K_i$ and substitute for the consumption-capital ratio in the individual budget constraint to get

$$\frac{\dot{K}_i}{K_i} = r - n + \frac{w}{K_i} \left( 1 - \frac{\phi(w)}{w} l_i \right),$$

which yields the aggregate accumulation equation

$$\frac{\dot{K}}{K} = r - n + \frac{w}{K} \left( 1 - \frac{\phi(w)}{w} l \right).$$

To derive the dynamics of individual $i$’s relative capital stock, $k_i(t) \equiv K_i(t)/K(t)$, we combine these two equations to obtain

$$\dot{k}_i = \frac{w}{K} \left[ 1 - \left( 1 + \frac{\phi(w)}{w} \right) \theta l - \left( 1 - \left( 1 + \frac{\phi(w)}{w} \right) l \right) k_i \right]$$

Imposing steady state to solve for $\theta$ and substituting back into the above expression we can express the relative labor supply as

$$\tilde{l}_i - \bar{l} = \left( \tilde{l} - \frac{1}{1 + \phi(w)/w} \right) \left( k_i - 1 \right)$$

which implies that the aggregate labor supply is independent of the distribution of capital. This
equation becomes (15) in the text with the particular form we have assumed for the utility function.

### A.3 Linearization of the aggregate dynamic system

Linearizing equations (A.8) and (A.9) around steady state yields

$$\begin{pmatrix} \dot{K} \\ \dot{l} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} K - \dot{K} \\ l - \dot{l} \end{pmatrix} \tag{A.11}$$

where

$$a_{11} = F_K - F_{KL} \frac{l}{\eta} - n; \quad a_{12} = -F_L \left(1 + \frac{1}{\eta}\right) + F_{LL} \frac{l}{\eta} < 0$$

$$a_{21} = \frac{1}{G(l)} \left[F_{KK} - (1 - \gamma) \frac{F_{KL}}{F_L} a_{11}\right]; \quad a_{22} = \frac{1}{G(l)} \left[-F_{KL} - (1 - \gamma) \frac{F_{KL}}{F_L} a_{12}\right]$$

and

$$G(l) \equiv \frac{1 - \gamma(1 + \eta)}{l} - \frac{(1 - \gamma)F_{LL}}{F_L} > 0$$

By direct calculation we show that

$$a_{11}a_{22} - a_{12}a_{21} = F_{KK} F_L \left(1 + \frac{1}{\eta}\right) - F_{KL} \beta < 0$$

implying that the steady state is a saddle point.

We immediately see that $a_{12} < 0$. In order to determine the likely signs of the other elements, it is useful to express them in terms of dimensionless quantities such as the elasticity of substitution in production, $\varepsilon \equiv F_K F_L / FF_{KL}$ and $s \equiv F_K K / F$, the share of output going to capital.

Thus, using the steady-state equilibrium conditions, we may write

$$a_{11} = \frac{1}{\varepsilon} [n(s - 1) + \beta(\varepsilon - 1)]$$

$$G(l)a_{2i} = \frac{F_{KL}}{F_L} \left\{ \frac{1-s}{s} \left[ \beta + n \left[ 1 - (1 - \gamma) \frac{s}{\varepsilon} \right] + \beta(1 - \gamma) \frac{\varepsilon - 1}{\varepsilon} \right] \right\}$$

the signs of which involve tradeoffs between $\varepsilon$ and the other parameters. From these two equations
we find that $a_{11} < 0, a_{21} < 0$ if and only if $\varepsilon$ lies in the range

$$s(1-\gamma)\left[n(1-s) + \beta\right] < \varepsilon < \frac{n(1-s) + \beta}{\beta}$$  \hspace{1cm} (A.12)

which is certainly met in the case of Cobb-Douglas production and logarithmic utility ($\varepsilon = 1, \gamma = 0$).

More generally, assuming $n = 0.015, \beta = 0.04, s = 0.33, \gamma = -2$, as highly plausible parameters, we see that (A.12) restricts $\varepsilon$ to lie in the range $0.65 < \varepsilon < 1.25$, which is consistent with virtually all empirical evidence. Thus while we view $a_{11} < 0, a_{21} < 0$ as most plausible, and $a_{11} > 1$ as improbable, we cannot dismiss $a_{21} > 0$, which will occur if $\varepsilon$ is sufficiently small to violate the left hand inequality in (A.12). Finally,

$$G(l)a_{22} = -\frac{F_{KL}}{\eta} \left(-1 + \gamma(1+\eta) + (1-\gamma)\frac{F_{LL}}{F_{L}}\right) > 0$$

hence $a_{22} > 0$.

**A.4 The dynamics of the relative capital stock**

To obtain the dynamics of individual capital we linearize equation (14) around the steady-state $\tilde{K}, \tilde{I}, \tilde{k}, \tilde{i}$. This is given by

$$k_i(t) = \frac{w(\tilde{K}, \tilde{I})}{\tilde{K}} \left[1 + \frac{1}{\eta} \left(\tilde{k} - \theta_t \right)(l(t) - \tilde{I}) + \left(\tilde{I}(1+\frac{1}{\eta}) - 1\right)\frac{k_i(t) - \tilde{k}_i}{\tilde{I}}\right]$$  \hspace{1cm} (A.13)

Using the equilibrium condition, $w(\tilde{K}, \tilde{I}) = F_{L}(\tilde{K}, \tilde{L})$ and combining the steady-state conditions (10a) and (10b) to show

$$\frac{F_{L}(\tilde{K}, \tilde{L})}{\tilde{K}} \left[\tilde{I} \left(1 + \frac{1}{\eta}\right) - 1\right] = \beta$$

the stable solution to (A.13) can be written as

$$k_i(t) = \tilde{k}_i + \left(\frac{1}{\mu - \beta}\right) \frac{F_{L}(\tilde{K}, \tilde{I})}{\tilde{K}} \left(1 + \frac{\eta}{\eta}\right) \left(\tilde{k} - \theta_t\right) l(0) - \tilde{I} e^{\mu t}$$  \hspace{1cm} (A.14)
Setting \( t = 0 \) in (A.14) and noting that \( k_{i,0} \) is given, we obtain

\[
k_{i,0} = \tilde{k}_i + \left( \frac{1}{\mu - \beta} \right) F_i(\tilde{K}, \tilde{L}) \left( \frac{1 + \eta}{\eta} \right) (\tilde{k}_i - \theta_i) \left[ l(0) - \tilde{l} \right]
\]

(A.15)

Thus, having determined \( \tilde{K}, \tilde{L} \), equations (15) and (A.15) jointly determine \( \theta_i, \tilde{k}_i \). We can express equation (15) as

\[
\theta_i = \frac{(1 - \tilde{k}_i)}{\tilde{l} \left( \frac{1 + \eta}{\eta} \right)} + \tilde{k}_i
\]

thus enabling us to rewrite (A.15) as

\[
k_{i,0} = \tilde{k}_i + \left( \frac{1}{\beta - \mu} \right) F_i(\tilde{K}, \tilde{L}) \left( 1 - \tilde{k}_i \right) \left[ l(0) - \tilde{l} \right]
\]

(A.16)

Given \( k_{i,0} \) and the evolution of the aggregate economy, this equation determines the steady-state distribution of capital. Substituting (A.16) into (A.14) the time path for relative capital is given by

\[
k_i(t) = \tilde{k}_i + \left( \frac{1}{\beta - \mu} \right) F_i(\tilde{K}, \tilde{L}) \left( 1 - \tilde{k}_i \right) \left[ \frac{l(0) - \tilde{l}}{\tilde{l}} \right] e^{\mu t}
\]

(A.17)

i.e.

\[
k_i(t) = \tilde{k}_i + \left( \frac{1}{\beta - \mu} \right) F_i(\tilde{K}, \tilde{L}) \left[ 1 - \frac{l(t)}{\tilde{l}} \right] (\tilde{k}_i - 1)
\]

(A.18)

which we can re-express in the form of equation (16) in the text.
References


