The Provision of a Public Good with a direct Provision Technology and a Large Number of Agents

Stefan Behringer
Universität Frankfurt
December 20, 2006

Abstract
This paper provides a limit result for the provision of a public good in a mechanism design framework as the number of agents gets large. What distinguishes the public good investigated in this analysis is its direct provision technology.

1 Motivation

Limit theorems for the provision of goods under asymmetric information about individual valuations with many agents yield strongly opposing results depending on whether the good to be provided is private or public. In the former case the impossibility result of ex-post efficient provision given the presence of both private information and voluntary participation follows from the Myerson and Satterthwaite (1983) Theorem.

In the case of a public good, i.e. a good that is non-rival in consumption, the limit result can be fundamentally different. Intuitively, with more agents each agent will have a lower probability of being ’pivotal’, i.e. bring about

\footnote{This paper is a revised version of Chapter 2 of my PhD dissertation at Universität Mannheim. I am grateful to my supervisor Martin Hellwig, the editor John Conley, an associate editor, and the referees who helped to improve the analysis substantially.}
the provision of the public good. On the other hand a misrepresentation of his type may still leave him with a lower contribution towards the provision of that good. Hence the inefficiency is expected to be larger in the limit. Mailath and Postlewaite (1990) give an example for such an inefficiency result where the probability of provision converges to zero as the number of agents grows out of bounds. Their result hinges on the assumption that the total cost of the public good is linear in the number of agents which prevents this magnitude from having any impact on the necessary level of per-capita contributions.

More recently Hellwig (2003), in a more general analysis that allows for variable quantities, disentangles the two effects and shows that assuming independence of the cost function from the number of agents is sufficient for the negative limit result to be reversed. For a public good that is \textit{excludable}, Norman (2004) shows that even with a linear total cost, a mechanism exits that charges a simple fixed fee from each non-excluded agent and allows for provision in the limit. The magnitude of the implied waste from exclusion relative to first best will be strictly reduced if there is more than one public good that can be bundled as shown in Fang & Norman (2006).

The discussion which assumption is more adequate when looking at public good provision in replica economies goes back to Roberts (1976) analysis. This paper avoids the difficulty of an a choice as the corresponding cost function for the provision of the public good will be determined endogenously. We depart from existing approaches by specifying a benchmark technology that allows for \textit{direct provision} of the public good by the agents. The provision thus does not lie in the hand of some external entity but can be undertaken by each agent individually. Agents have a dichotomous nature as they are beneficiaries as well as direct providers of the public good. This dichotomy allows us to set up a mechanism that consists of separate contracts. Following a standard Bayesian mechanism design approach we seek an incentive compatible, individual rational, and feasible mechanism allowing for side payments to generate the provision of the public good. We derive necessary and sufficient conditions for the existence of such a mechanism.

We show that the conditions for existence of the mechanism will be relaxed as the number of agents gets large. Most importantly, given our technology and a large number of agents we can approximate the first best outcome arbitrarily closely.
2 The setup

There are \( n \) agents with private valuation parameter (type) \( \theta_i \in \Theta, i = 1, 2...n \), \( \Theta \equiv [\underline{\theta}, \bar{\theta}] \) which are realizations of independently and identically distributed (i.i.d.) random variables \( \tilde{\theta}_i \). The random variables \( \tilde{\theta}_i \) are drawn from identical distributions that have a continuous and strictly positive density \( f(\theta_i) \), and a cumulative distribution function \( F(\theta_i) \).

The provision of the public good is an all or nothing decision, i.e. the level of the public good provision is fixed and will take place with some probability \( r(\theta) : \Theta^n \to [0, 1] \) where the total vector of types is denoted by \( \theta \in [\underline{\theta}, \bar{\theta}]^n \). Employing the revelation principle we can restrict ourselves to incentive compatible direct mechanisms.

A mechanism (allocation) is a triple of functions \( \langle r(\theta), p(\theta), z(\theta) \rangle \) where \( p(\theta) \) is the vector of total effort or contributions to the public good with generic elements \( p_i(\theta) : \Theta^n \to [0, 1] \) and \( z(\theta) \) is a vector of net side payments with generic elements \( z_i(\theta) : \Theta^n \to \mathbb{R} \).

Agent’s net utilities will then consist of the real allocations \( r(\theta) \) and \( p(\theta) \) and the allocation of net side payments \( z(\theta) \). Ex-post utilities are denoted as

\[
u_i = \theta_i r(\theta) - p_i(\theta) - z_i(\theta) \tag{1}\]

As the agent is a user as well as a direct provider of the public good the utility function can be decomposed into two parts as

\[
u_i = \underbrace{\theta_i r(\theta)}_{\text{user}} - \underbrace{p_i(\theta)}_{\text{provider}} + \underbrace{z_i(\theta)}_{\text{user}} \tag{2}\]

respectively, so that net side payments are defined as

\[
z_i(\theta) \equiv z_{\text{user}}(\theta) - z_{\text{provider}}(\theta) \tag{3}\]

The direct provision technology implies that each agent \( i \) can provide the good with some probability (effort) \( p_i(\theta) \) directly. The total probability of the good being provided independently with \( n \) agents is thus

\[
r(\theta) = 1 - \prod_{i=1}^{n}(1 - p_i(\theta)) \tag{4}\]

An allocation is feasible if side payments satisfy

\[
\sum_{i=1}^{n} z_i(\theta) \equiv \sum_{i=1}^{n}(z_{\text{user}}(\theta) - z_{\text{provider}}(\theta)) \geq 0 \forall \theta \tag{5}\]
A *first best allocation* is given as a feasible allocation \((r(\theta), p(\theta), z(\theta))\) for which no other feasible allocation achieves a higher aggregate expected surplus\(^1\)

\[
\sum_{i=1}^{n} \int ... \int [(r(\theta)\theta_i - p_i(\theta))] f(\theta_1)f(\theta_2)...f(\theta_n)d\theta_n...d\theta_2d\theta_1 =
\]

\[
\sum_{i=1}^{n} \int [(r(\theta)\theta_i - p_i(\theta))] dF^n(\theta).
\]

The mechanism is divided into *two contracts* corresponding to the agents’ functions. A first contract is written for the agents as providers and a second contract for the agents as users.

### 2.1 The Provider contract

In this contract, agents are given payments \(z_{Pi}(\theta_i, \theta_{-i})\) in exchange for the (enforceable) promise to develop and provide the public good with some probability \(p_i(\theta_i, \theta_{-i})\). Interim individual rationality of the contract which guarantees the voluntary acceptance of the contract for each agent given his type requires that

\[
U_{Pi}(\theta_i) = \int (-p(\theta_i, \theta_{-i}) + z_{Pi}(\theta_i, \theta_{-i}))dF^{n-1}(\theta_{-i}) \geq 0 \forall \theta_i
\]

where the outside options have been normalized to zero. This condition can be guaranteed to hold if we make the implementation assumption of ex-post individual rationality by equality between

\[
p_1(\theta) = ... = p_n(\theta) = p(\theta) = z_{Pi}(\theta) \forall \theta
\]

This implementation assumption trivially implies that the weaker condition of ex-ante individual rationality also holds. Clearly the implied duplication of effort implies a very conservative and thus robust starting point when investigating the total welfare consequences of the direct provision technology

\(^1\)The second line follows from the assumed independence of the random variables \(\tilde{\theta}_i\) so that the prior distribution is simply given by the product distribution \(F^n\). Whenever we leave out the limits of the integrals we integrate over the full support of \(\Theta\).
below. We also assume that effort levels $p(\theta)$ are perfectly observable and irreversible. The implementation assumption allows us to write the total provision probability (4) as

$$r(\theta) = 1 - (1 - p(\theta))^n \forall \theta$$  \hspace{1cm} (9)

### 2.2 The User contract

This contract is intended to generate the side payments $z_{Ui}(\theta_i, \theta_{-i})$ necessary to finance the expenses incurred by the provider contract and thus to satisfy the overall feasibility constraint. The form of this contract is more involved as we have to guarantee that no agent has an incentive to misrepresent his type that is unobservable to all others. Again we also guarantee that each agent will participate voluntarily after getting to know his own type. Interim incentive compatibility (IIC) guarantees that no agent has an incentive to misrepresent his type, i.e.

$$\int (r(\theta_i, \theta_{-i})\theta_i - z_{Ui}(\theta_i, \theta_{-i}))dF^{n-1}(\theta_{-i}) \geq 0$$ \hspace{1cm} (10)

Interim individual rationality (IIR) requires

$$\int (r(\hat{\theta}_i, \theta_{-i})\theta_i - z_{Ui}(\hat{\theta}_i, \theta_{-i}))dF^{n-1}(\theta_{-i}) \geq 0 \forall \theta_i \in \Theta, \forall \theta_i$$ \hspace{1cm} (11)

### 2.3 Bayesian-Nash Implementation

For ease of notation we define the interim probability of provision as perceived by agent $i$ as

$$\rho_i(\theta_i) \equiv \int r(\theta_i, \theta_{-i})dF^{n-1}(\theta_{-i})$$  \hspace{1cm} (12)

and the interim expected side payments from the user contract of agent $i$ to the mechanism designer as

$$\xi_{Ui}(\theta_i) \equiv \int z_{Ui}(\theta_i, \theta_{-i})dF^{n-1}(\theta_{-i})$$  \hspace{1cm} (13)
so that the agent’s expected utility from the user contract can be written as

\[ U_{Ui}(\theta_i) = \rho_i(\theta_i)\theta_i - \xi_{Ui}(\theta_i) \quad (14) \]

We now use the following well known result:

**Lemma 1** A mechanism \( \langle r(\theta), z_{U1}(\theta), ..., z_{Un}(\theta) \rangle \) is interim incentive compatible (IIC), iff

a) expected probability of provision \( \rho_i(\theta_i) \) is non-decreasing in \( \theta_i \), and

b) interim utility levels satisfy

\[ U_{Ui}(\theta_i) = U_{Ui}(\theta) + \int_{\theta_i}^{\theta} \rho_i(\eta)d\eta \quad (15) \]

so that high valuation types will receive a higher expected utility level and each agents expected side payments satisfy

\[ \xi_{Ui}(\theta_i) = \xi_{Ui}(\theta) + \int_{\theta_i}^{\theta} \eta d\rho_i(\eta) \forall \theta_i. \quad (16) \]

Proof: Standard.■

If expectations of all side payment functions from the user contract \( \xi_{Ui}(\theta_i) \) satisfy the above and the expectations of the probability of provision \( \rho_i(\theta_i) \) are non-decreasing in type we can then chose a set of side payment functions \( z_{Ui}(\theta) \) such that \( \int z_{Ui}(\theta_i, \theta_{-i})dF\theta_{-i} = \xi_{Ui}(\theta_i) \forall \theta_i \). One obvious choice for such a function is \( z_{Ui}(\theta) = \xi_{Ui}(\theta_i) \). This procedure yields a Bayesian incentive compatible allocation. Note that the IIC constraint determines each agent’s interim expected utility up to some constant of integration.

The IIR condition

\[ \int (r(\theta_i, \theta_{-i})\theta_i - z_{Ui}(\theta_i, \theta_{-i}))dF\theta_{-i}(\theta_{-i}) \geq 0 \forall \theta_i \quad (17) \]

can be rewritten using the IIC condition as

\[ U_{Ui}(\theta_i) = \rho_i(\theta_i)\theta_i - \xi_{Ui}(\theta_i) = \int_{\theta_i}^{\theta} \rho_i(\eta)d\eta - \xi_{Ui}(\theta) + \rho_i(\theta)\theta_i \geq 0 \forall \theta_i \quad (18) \]

As \( \int_{\theta_i}^{\theta} \rho_i(\eta)d\eta \geq 0 \forall \theta_i \) we have a necessary and sufficient condition for IIR to hold given by

\[ U_{Ui}(\theta) = \rho_i(\theta)\theta_i - \xi_{Ui}(\theta) \geq 0 \quad (19) \]
Lemma 2 Given the user contract is IIC, then ex-ante side payments satisfy
\[
\int z_{U_i}(\theta) dF^n(\theta) = -U_{U_i}(\theta) + \int \left( \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) r(\theta) dF^n(\theta)
\] (20)

Proof: See online Appendix.

3 The Mechanisms

We are now in a position to derive a necessary and sufficient condition for the existence of an effort allocation function and an overall development probability that satisfy the key constraints of the analysis. We first weaken the requirement of strict (or ex-post) feasibility to weak (or ex-ante) feasibility in order to simplify the analysis. An allocation is \textit{weakly feasible} if side payments satisfy
\[
\sum_{i=1}^{n} \int z_i(\theta) dF^n(\theta) \equiv \sum_{i=1}^{n} \int (z_{U_i}(\theta) - z_{P_i}(\theta)) dF^n(\theta) \geq 0
\] (21)

We thus have the following Lemma:

Lemma 3 For any probability of provision \( r(\theta) \) such that \( \rho_i(\theta_i) \) is non-decreasing in \( \theta_i \), there exist net side payments \( z(\theta) \) such that \( <r(\theta), z(\theta)> \) is interim incentive compatible, interim individual rational, and weakly feasible iff
\[
\int \left( \sum_{i=1}^{n} \left( \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) r(\theta) - np(\theta) \right) dF^n(\theta) \geq 0
\] (22)

Proof: See online Appendix.
3.1 The Program $\mathcal{P}$

The mechanism designer has to solve the following program $\mathcal{P}$ for the weakly feasible overall mechanism that implements the contracts under interim individual rationality and incentive compatibility and generates non-negative expected social benefit. Using Lemma 3 this program is

$$\begin{align*}
\max_{r(\theta)} \left\{ \int \left( \sum_{i=1}^{n} \theta_i r(\theta) - k_n(r(\theta)) \right) dF^n(\theta), 0 \right\} \quad (23)
\end{align*}$$

s.t. $\rho_i(\theta_i)$ non-decreasing in $\theta_i$ and

$$\begin{align*}
\int \left( \sum_{i=1}^{n} \left( \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) r(\theta) - k_n(r(\theta)) \right) dF^n(\theta) \geq 0 \quad (25)
\end{align*}$$

The ex-post efficient (or first best) provision rule given by the first order condition for (23) equates

$$\frac{\partial k_n(r(\theta))}{\partial r} = (1 - r)^{\frac{1}{n} - 1} = \sum_{i=1}^{n} \theta_i \quad (26)$$

The provision rule is thus

$$r^*(\theta) = \begin{cases} 
0 & \text{if } \sum_{i=1}^{n} \theta_i \leq 1 \\
1 - \left( \sum_{i=1}^{n} \theta_i \right)^{-\frac{n}{n-1}} & \text{if } \sum_{i=1}^{n} \theta_i > 1
\end{cases} \quad (27)$$

Note that this first best development probability is interior for finite magnitudes. Using an indicator function we have

$$r^*(\theta) = \left( 1 - \left( \sum_{i=1}^{n} \theta_i \right)^{-\frac{n}{n-1}} \right) \mathbf{1}_{\{\sum_{i=1}^{n} \theta_i > 1\}} \quad (28)$$

at first best total costs

$$k_n(r^*(\theta)) = n \left( 1 - \left( \sum_{i=1}^{n} \theta_i \right)^{-\frac{1}{1-n}} \right) \mathbf{1}_{\{\sum_{i=1}^{n} \theta_i > 1\}} \quad (29)$$
A sufficient condition for constraint (24) is that the hazard rate condition holds, i.e. that the "virtual utility"
\[ \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \] is non-decreasing in \( \theta_i \) \hspace{1cm} (30)

Looking at the objective function, note that the function is concave in \( r(\theta) \) and hence first-order conditions are necessary and sufficient for the constraint optimization. Given (30), Program \( \mathcal{P} \) can be solved by pointwise maximization and its solution satisfies the monotonicity constraint.

In the following Lemma we show that the constraints that the user and the provider contract put on the socially optimal outcome of the public good provision is non-trivial. Generically the implementable mechanism cannot achieve the first best outcome.

The second best maximization problem of program \( \mathcal{P} \) can be rearranged using a Lagrange multiplier approach as
\[
\text{Max}_{r(\theta), \lambda} \left\{ (1 + \lambda) \left( \int \left( \sum_{i=1}^{n} \theta_i r(\theta_i) - k_n(r(\theta)) \right) dF^\theta(\theta) \right) - \lambda \int \left( \sum_{i=1}^{n} \left( \frac{1 - F(\theta_i)}{f(\theta_i)} \right) r(\theta) \right) dF^\theta(\theta), 0 \right\}
\] \hspace{1cm} (31)

We can show that the optimal Lagrange multiplier \( \lambda^* \) is bounded away from zero and hence the constraint is non-trivial. The proof parallels Hellwig’s, 2003, proof of Proposition 1 and is relegated to the online appendix.

**Lemma 4** Define the probability of provision as an indicator function\
\[
r_{\lambda}(\theta) \equiv \left( 1 - \left( \sum_{i=1}^{n} \theta_i \right) \right) 1 \left\{ \frac{(1 + \lambda) \times \left( \sum_{i=1}^{n} \theta_i \right) \times \left( 1 - \left( \sum_{i=1}^{n} \theta_i \right)^{-\frac{n}{n-1}} \right) - k_n \left( r_{\lambda}(\theta) = 1 - \left( \sum_{i=1}^{n} \theta_i \right)^{-\frac{n}{n-1}} \right) \times \left( 1 - \left( \sum_{i=1}^{n} \theta_i \right)^{-\frac{n}{n-1}} \right) \lambda \sum_{i=1}^{n} \left( \frac{1 - F(\theta_i)}{f(\theta_i)} \right) \times \left( 1 - \left( \sum_{i=1}^{n} \theta_i \right)^{-\frac{n}{n-1}} \right) \right] > 0 \right\}
\] \hspace{1cm} (32)

Then there exists a unique Lagrange multiplier \( \lambda^* > 0 \) such that
\[
G(\lambda^*) = \int \left( \sum_{i=1}^{n} \left( \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) r_{\lambda^*}(\theta) - k_n(r_{\lambda^*}(\theta)) \right) dF^\theta(\theta) = 0 \hspace{1cm} (33)
\]
given \( n > 1 \). Also \( r_{\lambda}(\theta) \) is non-increasing and \( G(\lambda) \) is non-decreasing in \( \lambda \).
Proof: See online Appendix.

Looking at program $\mathcal{P}$ defined by (31) we can show that the 'shadow cost of the constraint', and therefore the value of $\lambda^n$ that satisfies the complementary slackness condition $\lambda^n G(\lambda^n) = 0$ increases in $n$. Given there are more agents, the probability that any particular agent’s signal will be pivotal for the provision of the public good becomes negligible. It follows that extracting a given side payment under the user contract and still satisfying incentive compatibility and individual rationality becomes increasingly difficult and hence the constraint bears more heavily on the maximization program the more agents are involved. This intuition is formalized by showing that the critical Lagrange multiplier is not uniformly bounded.

**Lemma 5** The Lagrange multiplier of program $\mathcal{P}$ grows out of bounds as $n$ gets large for any $r(\theta) \in [0, 1)$.

Proof: See online Appendix.

4 Limit Results

Given the characteristics of Program $\mathcal{P}$ we have analysed above it is now interesting to find out how the number of agents affects the possibility to obtain a desirable outcome.\(^2\)

Looking at the first best provision rule we find the following Lemma:

**Lemma 6** The first best provision rule $r^*(\theta)$ converges to provision with certainty in economy $n$ as $n$ gets large.

Proof: See online Appendix.

Also

\(^2\)In order to take limits we define an economy $n$ as given by a vector of type distributions $\theta^n = (\theta_1, ..., \theta_n) \in \Theta^n$ and an economy $n+1$ as given by a vector $\theta^{n+1} = (\theta_1, ..., \theta_{n+1}) \in \Theta^{n+1}$ where $\theta_{n+1}$ is a new i.i.d. draw from the distribution $F(\theta_i)$ accordingly.
Lemma 7  The total cost function $k_n(r(\theta))$ in economy $n$ is increasing in $n$, continuous, and strictly concave. It approaches a limit as

$$\lim_{n \to \infty} k_n(r(\theta)) = -\ln(1 - r(\theta)) \quad (34)$$

Proof: See Appendix.

We now proceed with the construction of approximately efficient mechanisms. Assuming for the moment that the type function density $f(\cdot)$ is degenerate and has point mass half at the extremes of the support $\Theta \equiv [\theta, \bar{\theta}]$ and that the random variables are i.i.d.. We may then use an argument by Al-Najjar & Smorodinsky, 2000 and show

Lemma 8  The LHS of each agent’s interim incentive compatibility constraint is bounded above by

$$\frac{1}{\sqrt{2\pi}} \frac{2}{\sqrt{(n-1)}} e^{\frac{1}{2n}}.$$

Proof: See online Appendix.

Intuitively, as $n$ grows large and the discrete Binomial distribution approaches a continuous Normal distribution, the highest probability of having a precise number of occurrences of $\theta_i = \bar{\theta}$ (i.e. of knowing to be the pivotal agent) goes to zero as the Normal distribution has no atoms. For large $n$ the exponential term above converges to one so that we see that the incentive to reveal one’s type truthfully is approximately falling in $(\sqrt{n})^{-1}$.

The perceived probability of being the pivotal agent can be interpreted as the underlying motivation for making contributions towards the provision of the public good in the user contract. We have shown that this motivation is decreasing in $n$ for each individual agent at the rate of $(\sqrt{n})^{-1}$ so that the value of the virtual utility term by a theorem of Lindeberg-Levy satisfies

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) \overset{d}{\to} N(0, \sigma^2) \quad (35)$$

$^3$An extension to non-degenerate type space is in their Section 3.3.
where we normalize $\theta = 0$.\footnote{As one referee has emphasized, such a normalization is typical but not completely innocuous. It implies that there remains the possibility that an agent does not benefit from the provision of the public good at all. This may be empirically plausible for public goods whose benefit can occur privately (without the possibility of some subset of agents being physically excluded) but for which agents are "taxed" (here with the opportunity costs of their effort) collectively. The notion of "taxation" in our context of public good provision may only be disputable for our assumption of participation constraints inasmuch as the designer can retain its coercive powers in a global and competitive environment. As we are concerned with large number results below the normalization is necessary in order to prevent the construction of simpler mechanisms that allow for concurrent welfare results with may agents. Normalizing the lower support to some strictly negative constant on the other hand will lead to asymptotic impossibility.} As the indicator constraint condition in Lemma 4 implies that for the expected aggregate side payments we integrate only over positive sums, the expected value for large $n$ can be derived as $\int_0^\infty xd\Phi(x) \equiv \varphi$ where $\Phi(\cdot)$ is the Normal distribution function. There thus exists a constant $\varphi > 0$ such that the stabilized sum has a limit

$$\lim_{n \to \infty} \frac{1}{\sqrt{n}} \int \left( 1 - \left( \sum_{i=1}^n \theta_i \right)^{-\frac{n-1}{n}} \right) \times$$

$$\text{1}_{\{\sum_{i=1}^n (\theta_i - 1 - F(\theta_i)/f(\theta_i)) > 0\}} \sum_{i=1}^n \left( \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) dF^n(\theta) = \varphi$$

and so the sum of expected virtual utilities that can be collected from the agents if strategic constraints are taken into account will grow out of bounds in $\sqrt{n}$.\footnote{Having exclusion as an additional instrument (as investigated in Hellwig, 2003 and Norman, 2004) allows for a simple, ex-post IR, fixed fee mechanism and a collection from participating agents that grows out of bounds linearly.} The Theorem now shows that even under the constraints imposed, the probability of provision in a second best mechanism will converge in probability to the first best probability as $n$ becomes large.
Theorem 9 Let \( (r_j^n(\theta), p_j^n(\theta), z_j^n(\theta)) \equiv \Psi_j^n(\theta) \) be a \( j^{th} \)-best allocation given the direct provision technology and \( n \) agents. Then for a first best allocation \( \Psi_1^n(\theta) : r_1^n(\theta) \xrightarrow{a.s.} 1 \) i.e. almost surely as \( n \to \infty \). For a second best allocation \( \Psi_2^n(\theta) : r_2^n(\theta) \xrightarrow{p} 1 \) i.e. in probability as \( n \to \infty \). The expected loss from second best relative to first best thus converges to zero.

Proof: See Appendix. ■

Despite convergence to the first best provision rule the direct provision technology implies that there is some waste of effort due to our conservative assumption that the designer requires direct provision from all agents. In the limit this total waste is \(-\ln(1 - r^n(\theta)) - 1\) by Lemma 7 and generically bounded whereas the gross benefit from the mechanism is unbounded above.

5 Conclusion

The analysis has shown that irrespective of an increased incentive to free ride on other agents’ efforts, the conditions for the provision of a public good with a direct provision technology are relaxed as the number of agents gets large. Substituting any ’ad hoc’ choice of the cost function for the provision of the public good with an endogenous motivation we depart from existing approaches by specifying a new provision technology benchmark. Despite our conservative choice of an inherent duplication of efforts in the corresponding Bayesian Mechanisms we are able to approximate the first best outcome arbitrarily closely in the limit. Our result thus sheds light on the conditions for the provision of certain public goods which may be provided directly by its users when there are a large number of them.
6 Appendix I

Proof of Lemma 7:

The first derivative of the total cost function w.r.t $n$ is

$$\frac{\partial k_n(r(\theta))}{\partial n} = 1 - (1 - r(\theta))^{\frac{1}{n}} \left( 1 - \frac{\ln(1 - r(\theta))}{n} \right) \geq 0$$  \hspace{1cm} (37)

i.e. non-negative for all $r(\theta) \in [0, 1)$ which follows from noting that the LHS is increasing in $n$ so that we can focus on the case $n = 1$. Now the first and second bracket are always strictly smaller than one for $r(\theta) \in [0, 1)$ hence the result follows. The second derivative is

$$\frac{\partial^2 k_n(r(\theta))}{\partial n^2} = -(1 - r(\theta))^\frac{1}{n} \left( \frac{\ln(1 - r(\theta))}{n^3} \right)^2 < 0$$

hence the function is strictly concave in $n$.

In the limit, total costs will converge which follows from rewriting

$$(1 - r(\theta))^{\frac{1}{n}} = \exp(-\frac{1}{n} \ln(\frac{1}{1 - r(\theta)}))$$ \hspace{1cm} (39)

where the exponent will go to zero as $n$ becomes large. We then use a Taylor approximation around zero, which implies that from the expansion of the exponential function into $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ we use the first two terms and find

$$(1 - r(\theta))^{\frac{1}{n}} \approx \left( 1 - \frac{1}{n} \ln(\frac{1}{1 - r(\theta)}) \right)$$ \hspace{1cm} (40)

and this will be a strict equality for $n \rightarrow \infty$ as all remaining terms can be neglected. We thus have that the limit of $k_n(r(\theta))$ becomes

$$\lim_{n \rightarrow \infty} \left( n(1 - (1 - r(\theta))^{\frac{1}{n}}) \right) = \ln\left( \frac{1}{1 - r(\theta)} \right) = -\ln(1 - r(\theta)).$$ \hspace{1cm} (41)

Note that total costs will only be uniformly bounded if $r(\theta) < 1$.■
Proof of Theorem 9:
For large $n$ the sum of valuations will be unbounded above for almost all type realizations whereas Lemma 7 reveals an upper bound on total costs for all $n$ so that from a first best perspective the probability of provision approaches certainty almost surely which can be seen from Lemma 6. In order to show convergence in provision probability for a second best allocation we need to show that for any $\varepsilon > 0$, there exists a number of agents $\hat{n}(\varepsilon)$ so that for $n \geq \hat{n}(\varepsilon)$ we have
\[
\Pr \{|r_n(\theta) - 1| > \varepsilon\} \leq \varepsilon
\]
so that as $n$ grows out of bounds $r_n(\theta)$ converges in probability to the first best provision probability.

The second best probability of provision as given by Program $P$ is
\[
r_n(\theta) \equiv \left(1 - \left(\sum_{i=1}^{n} \theta_i\right)^{-\frac{n}{n-1}}\right) \prod_{i=1}^{n} \left(\frac{1}{\theta_i} \left[1 - F(\theta_i)\right] f(\theta_i)\left[1 - \left(\sum_{i=1}^{n} \theta_i\right)^{-\frac{n}{n-1}}\right]\right)
\]
As the Lagrange multiplier is unbounded above by Lemma 5 for large $n$ we can focus on
\[
r_n(\theta) = \left(1 - \left(\sum_{i=1}^{n} \theta_i\right)^{-\frac{n}{n-1}}\right) \prod_{i=1}^{n} \left(\frac{1}{\theta_i} \left[1 - F(\theta_i)\right] f(\theta_i)\left[1 - \left(\sum_{i=1}^{n} \theta_i\right)^{-\frac{n}{n-1}}\right]\right)
\]
Using Lemma 8 in its modification for a continuous type space we know that individual incentive constraints will be bounded above by some positive constant that is decreasing in $(\sqrt{n})^{-1}$. Whence the term on the LHS of the indicator condition will converge to some strictly positive constant. Using Lemma 7 we know that
\[
k_n \left(\frac{r_n(\theta) = 1 - \left(\sum_{i=1}^{n} \theta_i\right)^{-\frac{n}{n-1}}}{\left(\sum_{i=1}^{n} \theta_i\right)^{-\frac{n}{n-1}} - \left(\sum_{i=1}^{n} \theta_i\right)^{-\frac{n}{n-1}}}\right)
\]
will be bounded above and the RHS of the indicator condition converges to zero. Thus we find that for any \( \varepsilon > 0 \), there exists a number of agents \( \hat{n}(\varepsilon) \) so that for \( n \geq \hat{n}(\varepsilon) \) we have convergence in probability, i.e.

\[
\Pr \{|r_n(\theta) - 1| > \varepsilon\} = \\
\Pr \left\{ \frac{1}{\sqrt{n}} k_n \left( r_n(\theta) = 1 - \left( \sum_{i=1}^{n} \theta_i \right)^{-\frac{n}{n-1}} \right) \times \right. \\
\left. \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} \right) \left( 1 - \left( \sum_{i=1}^{n} \theta_i \right)^{-\frac{n}{n-1}} \right) \right\} < 1 - \varepsilon \leq \varepsilon. \]
7 References


8 Appendix II

Proof of Lemma 2:
Interim side payments have been found above as

$$\xi_{U_i}(\theta_i) = -U_{U_i}(\theta) + \rho_i(\theta_i)\theta_i - \int_\theta^{\theta_i} \rho_i(\eta) d\eta$$  \hspace{1cm} (47)

Now integrating over the full support we find that from the independence assumption

$$\int z_{U_i}(\theta)dF^\theta(\theta) = \int \xi_{U_i}(\theta_i)dF(\theta_i)$$  \hspace{1cm} (48)

so that

$$\int z_{U_i}(\theta)dF^\theta(\theta) =$$  \hspace{1cm} (49)

$$\int \left(-U_{U_i}(\theta) + \rho_i(\theta_i)\theta_i - \int_\theta^{\theta_i} \rho_i(\eta) d\eta\right) dF(\theta_i) =$$

$$-U_{U_i}(\theta) + \rho_i(\theta_i)\theta_i - \int_\theta^{\theta_i} \rho_i(\eta) d\eta dF(\theta_i)$$

Integration by parts of the last term and some rearranging yields

$$\int z_{U_i}(\theta)dF^\theta(\theta) = -U_{U_i}(\theta) + \int \left(\theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)}\right) \rho_i(\theta_i) dF(\theta_i)$$  \hspace{1cm} (50)

Thus

$$\int z_{U_i}(\theta)dF^\theta(\theta) = -U_{U_i}(\theta) + \int \left(\theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)}\right) r(\theta)dF^\theta(\theta).$$  \hspace{1cm} (51)
Proof of Lemma 3:

Only-if-part: Suppose that \( r(\theta), z_{U_1}(\theta), \ldots z_{U_n}(\theta) \) is interim incentive compatible and interim individually rational. Then summing (20) over all \( n \) we have

\[
\sum_{i=1}^{n} \int z_{U_i}(\theta) dF^n(\theta) = - \sum_{i=1}^{n} U_{U_i}(\theta) + \int \sum_{i=1}^{n} \left( \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) r(\theta) dF^n(\theta)
\]

(52)

Using the definition of weak feasibility this becomes

\[
\sum_{i=1}^{n} \int z_{P_i}(\theta) dF^n(\theta) + \sum_{i=1}^{n} U_{U_i}(\theta) - \int \sum_{i=1}^{n} \left( \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) r(\theta) dF^n(\theta) \leq 0
\]

(53)

Using the degree of freedom in the analysis which follows from the constant of integration in Lemma 1 we normalize

\[
U_{U_i}(\theta) = \rho_i(\theta) - \xi_{U_i}(\theta) = 0
\]

(54)

so that the interim individual rationality constraint for the lowest type is strictly binding. Finally using the implementation requirement of the provision contract (8) we find the constraint as

\[
\int \left( \sum_{i=1}^{n} \left( \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) r(\theta) - np(\theta) \right) dF^n(\theta) \geq 0
\]

(55)

If-part: In fact we can show that if (22) is satisfied, then side payments can be chosen so that even strict feasibility holds. The proof is by construction and follows the logic of a Clarke-Groves transfer as set out in Cramton, et. al. (1987)\(^6\). Let the difference in side payments be

\[
z_{U_i}(\theta) - z_{P_i}(\cdot) = p_i(\cdot) - z_{P_i}(\cdot) + \int_{\theta_i}^{\theta_i} \eta d\rho_i(\eta) - \frac{1}{n-1} \sum_{j \neq i} \int_{\theta_j}^{\theta_j} \eta d\rho_j(\eta)
\]

(56)

Taking expectations over all types but \(i\) we find

\[
\int (z_{U_i}(\theta) - z_{P_i}(\cdot))dF^{n-1}(\theta_{-i}) = p_i(\cdot) - z_{P_i}(\cdot) + \int \int_\theta^\theta_i \eta d\rho_i(\eta)dF^{n-1}(\theta_{-i}) - \frac{1}{n-1} \sum_{j \neq i} \int \int_\theta^\theta_j \eta d\rho_j(\eta)dF^{n-1}(\theta_{-i})
\]

(57)

the first double integral term does not contain any \(\theta_{-i}\) and hence is constant when we integrate over \(dF^{n-1}(\theta_{-i})\). The second term is more involved. We integrate over the \(-i\) types so that only \(\theta_j\) matters and the term becomes

\[
\frac{1}{n-1} \sum_{j \neq i} \int \int_\theta^\theta_j \eta d\rho_j(\eta)dF(\theta_j)
\]

(58)

This however can be written as

\[
\frac{1}{n-1} \sum_{j \neq i} \int \int_\theta^\theta_j \eta dF(\theta_j)d\rho_j(\eta) = \frac{1}{n-1} \sum_{j \neq i} \int \int_\theta^\theta (1 - F(\eta))\eta d\rho_j(\eta)
\]

(59)

so that we have

\[
\int (z_{U_i}(\theta) - z_{P_i}(\cdot))dF^{n-1}(\theta_{-i}) =
\]

\[
p_i(\cdot) - z_{P_i}(\cdot) + \int \int_\theta^\theta_i \eta d\rho_i(\eta) - \frac{1}{n-1} \sum_{j \neq i} \int \int_\theta^\theta (1 - F(\eta))\eta d\rho_j(\eta)
\]

(60)

so that by incentive compatibility of the user contract (16)

\[
\xi_{U_i}(\theta_i) - \xi_{U_i}(\theta) = \int \int_\theta^\theta_i \eta d\rho_i(\eta)
\]

(61)

the allocation \(\langle r(\theta), z_{U_1}(\theta), ..., z_{U_n}(\theta)\rangle\) is incentive compatible.

Furthermore, summation over \(n\) yields

\[
\sum_{i=1}^n (z_{U_i}(\theta_{-i}) - z_{P_i}(\cdot)) = n(p_i(\cdot) - z_{P_i}(\cdot))
\]

(62)
so that given strict feasibility holds the contract is also implementable as

\[
\sum_{i=1}^{n} \int_{\theta}^{\theta_i} \eta d\rho_i(\eta) = \sum_{i=1}^{n} \frac{1}{n-1} \sum_{j \neq i}^{\theta_j} \int_{\theta}^{\theta_j} \eta d\rho_j(\eta)
\]  (63)

We can now solve explicitly for the side payments requiring that IIR for the lowest type holds, i.e. that

\[
\sum_{i=1}^{n} \int (z_{U_i}(\theta, \theta_{-i}) - z_{P_i}(\cdot))dF^{n-1}(\theta_{-i}) \geq 0
\]  (64)

which can be shown to be implied by (22). We then chose the constant that determines agent’s utility levels of the first contract of the form

\[
p_i(\cdot) - z_{P_i}(\cdot) = \frac{1}{n} \sum_{i=1}^{n} \int (z_{U_i}(\theta, \theta_{-i}) - z_{P_i}(\cdot))dF^{n-1}(\theta_{-i}) + \int_{\theta}^{\theta_i} \eta d\rho_i(\eta) - \frac{1}{n-1} \sum_{j \neq i}^{\theta_j} [1 - F(\eta)]\eta d\rho_j(\eta)
\]  (65)

and as with this choice of a constant the lowest type has

\[
\int (z_{U_i}(\theta, \theta_{-i}) - z_{P_i}(\cdot))dF^{n-1}(\theta_{-i}) = \frac{1}{n} \sum_{i=1}^{n} \int (z_{U_i}(\theta, \theta_{-i}) - z_{P_i}(\cdot))dF^{n-1}(\theta_{-i}) \geq 0
\]  (66)

the IIR of the overall mechanism holds for all types. Via the implementation assumption of the first contract, the above condition specifies the generic elements of the effort contribution scheme \( p \) as

\[
p_i(\cdot) = z_{P_i}(\cdot) + \frac{1}{n} \sum_{i=1}^{n} \int (z_{U_1}(\theta, \theta_{-i}) - z_{P_i}(\cdot))dF^{n-1}(\theta_{-i}) + \int_{\theta}^{\theta_i} \eta d\rho_i(\eta) - \frac{1}{n-1} \sum_{j \neq i}^{\theta_j} [1 - F(\eta)]\eta d\rho_j(\eta)
\]  (67)

so that the allocation \( \langle r(\theta), p \rangle \) is interim incentive compatible, interim individual rational, and strictly feasible.
Lemma 4: Define the probability of provision as an indicator function

\[ r_\lambda(\theta) \equiv 1 - \left( \sum_{i=1}^{n} \theta_i \right)^{-\frac{n}{n-1}} \left\{ \begin{array}{ll}
(1 + \lambda) \times \\
\left( \sum_{i=1}^{n} \theta_i \right) \times \left( 1 - \left( \sum_{i=1}^{n} \theta_i \right)^{-\frac{n}{n-1}} \right) - \\
k_n \left( r_\lambda(\theta) = 1 - \left( \sum_{i=1}^{n} \theta_i \right)^{-\frac{n}{n-1}} \right) \times \\
\lambda \sum_{i=1}^{n} \left[ \frac{1 - F(\theta_i)}{f(\theta_i)} \right] \times \left( 1 - \left( \sum_{i=1}^{n} \theta_i \right)^{-\frac{n}{n-1}} \right)
\end{array} \right\} > 0 \] 

Then there exists a unique Lagrange multiplier \( \lambda^* > 0 \) such that

\[ G(\lambda^*) \equiv \int \left( \sum_{i=1}^{n} \left( \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) r_\lambda^*(\theta) - k_n(r_\lambda^*(\theta)) \right) dF^n(\theta) = 0 \]  

given \( n > 1 \). Furthermore \( r_\lambda(\theta) \) is non-increasing and \( G(\lambda) \) is non-decreasing in \( \lambda \).

Proof:

Following arguments similar to Myerson and Satterthwaite (1983) and Hellwig, (2003), we show that \( \lambda \to G(\lambda) \) is increasing and negative for \( \lambda = 0 \) as show below that \( G(0) < 0 \) for \( n > 1 \) and \( G(\lambda) \) is continuous.

As \( \lambda \) gets large the indicator function above approaches one of the form

\[ r_\lambda(\theta) \equiv 1 - \left( \sum_{i=1}^{n} \theta_i \right)^{-\frac{n}{n-1}} \sum_{i=1}^{n} \left( \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) \right\} > 0 \]

so that there exists some \( \hat{\lambda} > 0 \) for which we have \( G(\hat{\lambda}) > 0 \) given \( k_n(r_\lambda(\theta)) = 1 - \left( \sum_{i=1}^{n} \theta_i \right)^{-\frac{n}{n-1}} \) \( = n \left( 1 - \left( \sum_{i=1}^{n} \theta_i \right)^{-\frac{n}{n-1}} \right) < n \) which holds for all \( n < \infty \). Note that this indicator function \( r_\lambda(\theta) \) is chosen such that for \( \lambda = 0 \) we get the first best result as in that case \( \left( \sum_{i=1}^{n} \theta_i \right) \times \left( 1 - \left( \sum_{i=1}^{n} \theta_i \right)^{-\frac{n}{n-1}} \right) - \\
k_n \left( r_\lambda(\theta) = 1 - \left( \sum_{i=1}^{n} \theta_i \right)^{-\frac{n}{n-1}} \right) > 0 \) is indeed the unconstrained maximum welfare.

Strict monotonicity of \( G(\lambda) \) is shown by looking at \( r_{\lambda_1}(\theta) \neq r_{\lambda_2}(\theta) \) for any \( \lambda_1 \neq \lambda_2 \). W.l.o.g. let \( r_{\lambda_1}(\theta) = 1 - \left( \sum_{i=1}^{n} \theta_i \right)^{-\frac{n}{n-1}} \), then from the indicator
function this implies that for small $\lambda$

$$\left(\sum_{i=1}^{n} \theta_i\right) \times \left(1 - \left(\sum_{i=1}^{n} \theta_i\right)^{-\frac{n}{n-1}}\right) - k_n \left(r_\lambda(\theta) = 1 - \left(\sum_{i=1}^{n} \theta_i\right)^{-\frac{n}{n-1}}\right) > 0$$

(71)

and for $r_{\lambda_2}(\theta) = 0 < r_{\lambda_1}(\theta)$ and large $\lambda$ the condition

$$\sum_{i=1}^{n} \left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right) \times \left(1 - \left(\sum_{i=1}^{n} \theta_i\right)^{-\frac{n}{n-1}}\right) - k_n \left(r_\lambda(\theta) = 1 - \left(\sum_{i=1}^{n} \theta_i\right)^{-\frac{n}{n-1}}\right) \leq 0$$

(72)

holds and thus $r_{\lambda_1}(\theta) \leq r_{\lambda_2}(\theta)$ as $\lambda_1 \geq \lambda_2$.

Furthermore by definition we have

$$G(\lambda_1) - G(\lambda_2) = \int \left(\left(r_{\lambda_1}(\theta) - r_{\lambda_2}(\theta)\right) \sum_{i=1}^{n} \left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right) - k_n (r_{\lambda_1}(\theta) - k_n (r_{\lambda_2}(\theta)))\right) dF^n(\theta)$$

(73)

Using the monotonicity of $k_n(r(\theta))$ in $r(\theta)$ for $n > 1$ and (72) we find the final result that $G(\lambda_1) \leq G(\lambda_2)$ implies that $\lambda_2 \geq \lambda_1$ has to hold.$\blacksquare$

**Intermediate result for Lemma 4:**

See that $G(0) < 0$ for $n > 1$.

Proof: Note that for $n = 1$ the problem of providing a public good or a private good are isomorphic. Following Güth and Hellwig, (1986), Proposition 5.4.$^7$ we assume $n > 1$ agents rewriting condition (33) using the ex-post efficient provision probability $r(\theta)^*$ as given by the indicator function (28) as

$$G(0) = \int \left(\sum_{i=1}^{n} \left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right) r(\theta)^* - k_n (r(\theta)^*)\right) dF^n(\theta) \geq 0$$

(74)

Our aim is to show that this condition will not hold. We rewrite the condition by integrating the ex-post efficiency condition into the limits of the integral as

\begin{align*}
\int_{\Delta} \left( \sum_{i=1}^{n} \left( \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} - k_n(r(\theta)) \right) \right) dF^n(\theta) \geq 0 
\tag{75}
\end{align*}

where
\begin{equation}
\Delta \equiv \left\{ \theta \in \Theta^n \middle| \sum_{i=1}^{n} \theta_i > k_n(r(\theta)) \right\} 
\tag{76}
\end{equation}

We aim to show that this leads to a contradiction. In order to separate out integrals, let \( k = 1 \ldots n \) and define two critical type vectors as

\begin{equation}
\Delta^k \equiv \left\{ \theta^k \in \Theta^k \middle| \sum_{i=1}^{k} \theta_i > \frac{k_n(r(\theta))}{r(\theta)} \right\} 
\tag{77}
\end{equation}

and

\begin{equation}
\Delta^{k-1} \equiv \left\{ \theta^{k-1} \in \Theta^{k-1} \middle| \sum_{i=1}^{k-1} \theta_i + 1 > \frac{k_n(r(\theta))}{r(\theta)} \right\} 
\tag{78}
\end{equation}

so that \( \Delta \equiv \bigcup_{k=1}^{n} \left[ \Delta^k \times [0, 1]^{n-k} \right] \) as we only need to integrate over the types for which the constraint holds. We can rewrite the constraint \((25)\) using the independence assumption as

\begin{equation}
\sum_{k=1}^{n} \int_{\Delta^k \times [0, 1]^{n-k}} \left( \sum_{i=1}^{n} \left( \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} - k_n(r(\theta)) \right) \right) dF^n(\theta) \geq 0 
\tag{79}
\end{equation}

For \( k > 1 \) we decompose the integral and write

\begin{align*}
\int_{\Delta^{k-1}} \int_{k_n(r(\theta)}^1 \left( \sum_{i=1}^{k} \left( \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} - k_n(r(\theta)) \right) \right) \times (80) 
\end{align*}

\begin{align*}
dF_k(\theta_k) dF^{k-1}(\theta^{k-1}) + F^k(\Delta^k) \sum_{j=k+1}^{n} \int \left( \theta_j - \frac{1 - F(\theta_j)}{f(\theta_j)} \right) dF_j(\theta_j)
\end{align*}

For \( j = k + 1 \ldots n \) the expected value of the virtual utility is \( \bar{\theta}(=0) \) and hence the second term falls out.
Integrating by parts we can show that the first part becomes

\[
\int_{k_n(r(\theta))}^{1} \left( \theta_u - \frac{1 - F(\theta_u)}{f(\theta_u)} \right) dF_u(\theta_u) = \tag{81}
\]

\[
\left( k_n(r(\theta)) - \sum_{i=1}^{k-1} \theta_i \right) \left( 1 - F_k(k_n(r(\theta)) - \sum_{i=1}^{k-1} \theta_i) \right)
\]

As we have the first-best choice in the integral condition we find

\[
G(0) = \int_{\Delta} \left( \sum_{i=1}^{n} \left( \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) - k_n(r(\theta)) \right) dF^n(\theta) = \int_{\Delta} \left( \sum_{i=1}^{k-1} \left( \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) - k_n(r(\theta)) \right) \times \left( 1 - F_k \left( k_n(r(\theta)) - \sum_{i=1}^{k-1} \theta_i \right) \right) dF^{k-1}(\theta^{k-1}) = \]

\[
- \int_{\Delta} \sum_{i=1}^{k-1} \left( \frac{1 - F(\theta_i)}{f(\theta_i)} \right) \left( 1 - F_k \left( k_n(r(\theta)) - \sum_{i=1}^{k-1} \theta_i \right) \right) dF^{k-1}(\theta^{k-1}) \leq 0 \tag{82}
\]

thus the first best outcome violates the constraint. If \( n > 1 \) and (at least) \( k > 1 \), \( \Delta^{k-1} \) has positive measure and thus the inequality is strict so that we have \( G(0) < 0 \). One way of interpreting this result is that for a non-trivial number of agents a mechanism that imposes the first best outcome would imply a loss to the designer and hence the public good can only be provided with a subsidy. ■
Proof of Lemma 5:

By contradiction: The program $\mathcal{P}$ can be written as

$$\text{Max}_{r(\theta), \lambda} \left\{ \int \left( \sum_{i=1}^{n} \left( \theta_i - \frac{\lambda}{1+\lambda} \left( \frac{1-F(\theta_i)}{f(\theta_i)} \right) \right) r(\theta) - k_n(r(\theta)) \right) dF^m(\theta), 0 \right\}$$

Denote $r_{\lambda^{**}}(\theta)$ as the provision rule that maximizes the probability of provision and we assume that $\lambda = \bar{\lambda}$ is constant, so that $r_{\lambda^{**}}(\theta) = r_{\lambda}(\theta)$. For a constant multiplier $\bar{\lambda}$, $r_{\lambda}(\theta) \overset{a.s.}{\rightarrow} 1$ if

$$\lim_{n \to \infty} \left( \frac{k_n(r(\theta))}{\sum_{i=1}^{n} \left( \theta_i - \frac{\lambda}{1+\lambda} \left( \frac{1-F(\theta_i)}{f(\theta_i)} \right) \right)} \right) \overset{a.s.}{\rightarrow} 0 \quad (83)$$

for all $r(\theta) \in [0, 1)$. Looking at

$$(1 - (1 - r_{\lambda}(\theta))^{\frac{1}{\lambda}}) = \frac{1}{n} \sum_{i=1}^{n} \left( \theta_i - \frac{\bar{\lambda}}{1+\lambda} \left( \frac{1-F(\theta_i)}{f(\theta_i)} \right) \right) \quad (84)$$

we solve into

$$1 - r_{\lambda}(\theta) = \left( 1 - \frac{1}{n} \sum_{i=1}^{n} \left( \theta_i - \frac{\bar{\lambda}}{1+\lambda} \left( \frac{1-F(\theta_i)}{f(\theta_i)} \right) \right) \right)^n \quad (85)$$

Using the LLN in the numerator we know that the realization of the sum of random variables converges to its expectation almost surely, i.e.

$$\lim_{n \to \infty} \left( \frac{1}{n} \sum_{i=1}^{n} \left( \theta_i - \frac{\bar{\lambda}}{1+\lambda} \left( \frac{1-F(\theta_i)}{f(\theta_i)} \right) \right) \right) = \frac{E \{ \theta \}}{1+\lambda} < 1 \quad (86)$$

i.e. a constant, which follows from an integration by parts. Hence for large $n$ the equality can only be satisfied if we have that $r_{\lambda}(\theta) \overset{a.s.}{\rightarrow} 1$, given a constant multiplier. This contradicts the assumption, as almost sure convergence implies that for large $n$, the provision probability $r(\bar{\theta}_i, \theta^{n-1}_i)$ cannot vary very much with the signal of the $i^{th}$ agent and hence the side payment extracted from him under the user contract will be close to zero. As this holds for any agent the feasibility constraint will be violated. Thus the Lagrange multiplier cannot be constant and therefore $\lambda^{**}$ has to grow out of bounds as the number of agents gets large. We conclude that $\lim_{n \to \infty} \lambda^{**} = \infty$ has to hold so that both numerator and denominator go to zero which prevents $r_{\lambda}(\theta) \overset{a.s.}{\rightarrow} 1$.
**Proof of Lemma 6:**
The first best provision rule satisfies

$$\lim_{n \to \infty} r^*(\theta) = \lim_{n \to \infty} \left( 1 - \left( \sum_{i=1}^{n} \theta_i \right)^{-\frac{n}{n-1}} \right) 1_{\{\sum_{i=1}^{n} \theta_i > 1\}} = 1. \quad (87)$$

**Proof of Lemma 8:**
The interim incentive compatibility of a mechanism (satisfying the two contracts) guarantees that no agent has an incentive to misrepresent his type, i.e.

$$\int (r(\theta_i, \theta_{-i}) \theta_i - p(\theta_i, \theta_{-i}))dF^{n-1}(\theta_{-i}) \geq \int (r(\hat{\theta}_i, \theta_{-i}) \theta_i - p(\hat{\theta}_i, \theta_{-i}))dF^{n-1}(\theta_{-i}) \forall \theta_i, \hat{\theta}_i \in \Theta, \forall \theta; \quad (88)$$

Consider only mechanisms which treat all agents symmetrically and let $r(\theta) \in \left\{0, \left(1 - \left(\sum_{i=1}^{n} \theta_i \right)^{-\frac{n}{n-1}}\right) 1_{\{\sum_{i=1}^{n} \theta_i > 1\}} \right\}$ be non-decreasing in $k \equiv \#\{i | \theta_i = \bar{\theta}\}$. The strictest interim incentive compatibility condition is given as

$$\int (r(\theta_i = \bar{\theta}, \theta_{-i}) - r(\theta_i = \bar{\theta}, \theta_{-i}))dF^{n-1}(\theta_{-i}) \geq \frac{1}{n\bar{\theta}} \int (k_n(r(\theta_i = \bar{\theta}, \theta_{-i})) - k_n(r(\theta_i = \bar{\theta}, \theta_{-i})))dF^{n-1}(\theta_{-i}) \quad (89)$$

Due to symmetry we can write $r$ as a function of $k$ and there is at most one value of $k$ (defining the ‘pivotal’ agent $k^*$) for which

$$(r(\theta_{k^*} = \bar{\theta}, \theta_{-k^*}) - r(\theta_{k^*} = \bar{\theta}, \theta_{-k^*})) = \left(1 - \left(\sum_{i=1}^{n} \theta_i \right)^{-\frac{n}{n-1}}\right) 1_{\{\sum_{i=1}^{n} \theta_i > 1\}} \quad (90)$$

Let $h$ be the number of agents besides $i$ to have $\theta_i = \bar{\theta}$ we can rewrite the incentive compatibility using the Binomial distribution for equal probability.
as
\[
\max_h \left[ \frac{1}{2} \right]^{n-1} \frac{(n-1)!}{(n-1-h)!h!} \geq \frac{1}{n\theta} \int (k_n(r(\theta_i = 1, \theta_{-i})) - k_n(r(\theta_i = 0, \theta_{-i})))dF_{n-1}(\theta_{-i})
\]
(91)

where the LHS is now the probability of being the pivotal agent \( k^* \). An upper bound on the LHS is found by choosing \( h = (n-1)/2 \) (as in the simple majority rule case) so that using Stirling’s formula given as
\[
n! = \sqrt{2\pi n} \left( \frac{n}{e} \right)^n e^{\frac{1}{12n}} \text{ for some } \varphi(n) \in (0, 1)
\]
(92)
we have
\[
\frac{1}{2} \left( \frac{1}{2} \right)^{n-1} \frac{\sqrt{2\pi(n-1)} \left( \frac{n-1}{e} \right)^{n-1} e^{\frac{1}{12n}}}{\sqrt{2\pi(n-1)} \left( \frac{n-1}{e} \right)^{n-1} e^{\frac{1}{12n}}} \leq \frac{1}{2} \left( \frac{1}{2} \right)^{n-1} \frac{\sqrt{2\pi(n-1)} \left( \frac{n-1}{e} \right)^{n-1} e^{\frac{1}{12n}}}{\sqrt{2\pi(n-1)} \left( \frac{n-1}{e} \right)^{n-1} e^{\frac{1}{12n}}} = \frac{1}{\sqrt{2\pi(n-1)} e^{\frac{1}{12n}}}.
\]
(93)