Getting the Right Spin: A Theory of Optimal Viral Marketing

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Summary

I develop a theoretical framework for studying viral communications through a principal agent set-up. The principal derives wealth from a signal in a society while agents in the society talk to each other. A society is likely to create a viral bubble if there is a short loop of positive reinforcement of the message. The principal is more likely to use widespread marketing strategies with a weaker effect if she is impatient. She will however prefer a viral communication strategy with stronger effects if she is patient. Such framework can be used to derive an optimal solution computable in polynomial time. All in all, the model suggests principals have incentives to generate spins and that the way information is spread matters in transitionary dynamics of an economy.

Keywords: Network Theory, Viral Marketing, Principal Agent.

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1 Introduction

This paper tackles a very simple question: what is the best marketing campaign that can be done on a social network? Embedded in this questions lies considerations for two groups of persons. First, the marketing agency who seeks the highest exposure of a given product on the network. Second, the consumers, nodes in a network, who seeks to reproduce the information they like best. These consumers share an opinion that can then be freely reproduced or modulated by listeners, friends or colleagues, of a given consumer. This process of sharing and modulating generates the characteristics of a viral campaign.

While this paper will use a marketing agency, a network of consumers and a generic signal as an application, one may very well apply this principal-agent framework to different setups. For instance, one can think of the principal being a broker, the network his fellow brokers sharing a signal about a stock. This can also be applied in a political framework as well as a tool for central banks announcements. In short, this paper yields a theory of the "optimal spin", an active modification of a statement in a way to swing public opinion in one direction.

The novel aspect of this paper is that it formally embeds economic decisions into a generic network of agents. Optimal solutions on such networks are generally untractable in terms of marginal analysis precisely because of network externalities. This paper exploits the decision-making paradigm of agents to map the problem into a particular algebra where one can perform the typical economic analysis based on marginal utility to find the trade-off between reach of the signal and strength of the signal. This yields also an algorithm to derive the optimal solution on any network.

My principal results are as follows. In a society where everybody knows well their sources of information, there exists a stable equilibrium where the network structure of the society is unimportant. In such equilibrium, the network has no value and broad advertising is the optimal solution. However, when consumers are unable to correctly detect spins from their neighbors, the society creates a viral bubble if there exists a tight loop of consumers that "lead the spin" in the network. The principal will therefore have a trade-off between time to diffuse the signal and an higher value due to the spin. Typical solutions will range between a pure viral marketing strategy and broad advertising. In the first case, the principal uses the spins in the network to build momentum on the signal at the cost of an higher transient time. In the second case, each node is reached in the first place at the cost of a weaker initial signal. There are no assumptions made on the structure of the network employed, except that it is known to the principal (and this is relaxed in extensions). This can thus be applied to some already existing network.

Examples of applications are straight marketing campaigns on web networks, optimal announcement rules for central banks as well as a different framework to understand economic fluctuations.

The remaining sections of this article are organized as follows. In section 2, I present a brief review of what has been said on viral strategies of communication. In section 3, I present the model itself, the concept of solution and some comments about the economic meaning of these solutions. I also present some extensions in section 4. A brief conclusion follows.

2 A Brief Review

The idea of spin has been explored in various ways. Banerjee [5] explored the notion of herding on a linear network. Agents would receive a simple binary signal from the previous person in the line and infer, based on this signal, if it is actually worth buying a product or not. It was show that such process could generate "cascades" of decisions where people would buy the product even though it is not good.

Bala and Goyal [4] also explore the process of learning from neighbors in a bayesian updating sense but where updating is limited to neighbors. Players on a network choose actions based on the information they get about actions from their neighbors. Their main result is that in finite time, people's belief converge to a unique and people derive the same payoff from their actions.

Schiraldi and Liu [14] explore if a company should seek herding or not when lunching a product when they do not know the value of their product. Their main result is that the firm can increase its profit by manipulating the order of the launch sequence in different markets. Products that are highly expected or highly underrated should be launched simultaneously. However, products with an average anticipation should be launched sequentially to use momentum from reports from early markets. Interestingly, they argue that good products can be trapped in a rejection herd, suggesting that the network effects are important for buying a product.

Arthur, Motwani, Shama and Xu [2] present a framework for an optimal pricing strategy over a social network. They seek a non-adaptive optimal solution to the buyer/recommender cascade meaning that the owner can only choose a strategy before marketing actually occurs based on its full knowledge of the network. They thus seek a set of initial persons to which they should seed the product and a discount strategy offered to persons who recommends the product to their neighbors. Their consumer's decision is probabilistic, namely that consumers will buy a node with some probability that depends on the recommendation and the discount. Their main contribution is algorithmic and relates to the set covering problem (see [8]). They find a solution that is guaranteed to be within a factor of the optimal solution the latter being untractable in polynomial time.

This paper and others rely on a probabilistic modelisation of consumers can be thought as cases of diffusion models where the consumers (nodes) do not make decisions *per se* but are somewhat bound to a behavioral rule. These models draw their inspiration from the contagion models used in epidemiology (see Jackson, ch. 7 [11], for their adaptation to a social context for a treatment on random networks).

Yang [17] has also explored the stability of price formation over a network that has a power law distribution. She models agents with a forecast behavior of prices based on a weighted average of forecasts from their adjacent nodes, each forecast being based on a autoregressive function of prices. She finds that highly connected networks and a weighting strategy on sources of information can have a stabilizing effect on prices. She relates this stability to the underlying eigenvalues of the network.

On the empirical side, Leskovec, Adamic and Huberman [13] have studied the dynamics of the recommendation/propagation cascade on a massive database of an online company similar to Amazon. They find that recommendation chains, generated by emails sent on the website, are not long. They also find that people are more likely to listen to their close neighbors. They also find that it is generally a small group of active persons that lead the recommendation chain for many products. They rely on a diffusion model with a statistical behavior of nodes to reproduce their findings.

These papers suggest that the way information is carried over a network has some effect on the decision to purchase or not a product. They however lack of generality in some cases or of economic foundation in others. In Banerjee, the model relies on a simple network structure (a linear network). Although it is clear that information should travel linearly, Banerjee provides no answer to how this line is actually generated. Balla and Goyal provides some information about the learning process but remain agnostic about the best adoption strategy. In Schiraldi and Liu, they rely on distinct markets that have no connections between them. They thus provide an explanation over the different markets, but remain silent on the flow of information within each market.

Arthur, Motwani, Shama and Xu have a model close to the question. They provide no general solution that can be computed in polynomial time but more importantly, relies on a probabilistic framework to model their decision. Yang provides an interesting article for its mathematical foundation. She pins down the conditions for stability on a network and in terms of its underlying eigenvalues. However both lack of an economic foundation in their consumer's decisions as consumers/agents do not seek to find the best solution according to their own tastes or preferences.

This article provide such foundation. It is also completely general as it works on any fixed network and provides an economic foundation about the way information flows and signals are built over time. The model remains however simplistic to provide a clear and tractable answer. As such, consumers on the network have only one choice variable (namely which signal they choose) and the rest of their behavior remains unspecified. In section 4, I discuss one extension where consumers can actually change the intensity of the signal they generate. My approach is close in spirit to the DeGroot model (see Jackson, Ch. 8 [11]) as the signal formation relates the transitionary dynamics of the underlying eigenspace of the network. This eigenspace is however depicted in a different algebra where the maximizing behavior of consumer is modeled as an addition operator.

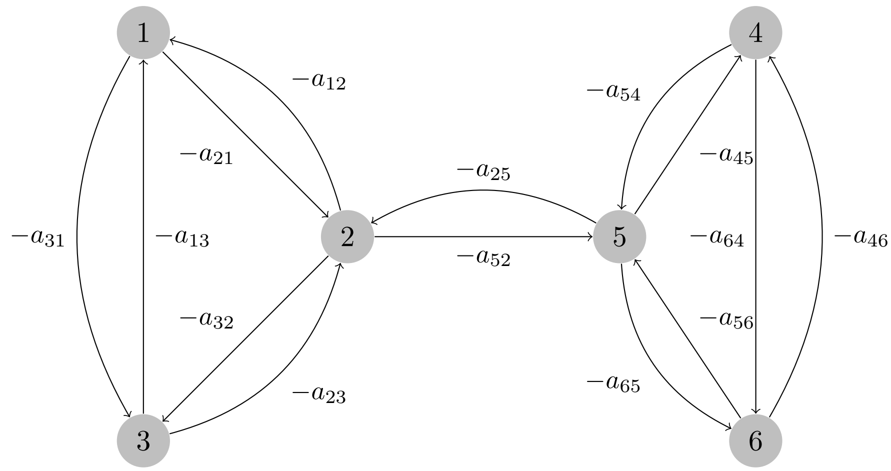

Figure 1: An Example of A Network

3 The Model

In this section, I present the model, the mathematical framework and some interpretations. I start by some generalities about networks and then move on too describing the particulars of this model. I then characterize the optimal solution and comment on its meaning. Throughout the paper, I will use the network described in figure 1 as an example to clarify the meaning of certain concepts.

3.1 The Network

One can think of the network as consumers who shares information about some product. A consumer will thus be seen as a point in the network while arrows between nodes represents the channels of information available to consumers.

Definition 1 (Network, Strongly Connected Network)**.** *A network $G(V, E)$ is a directed graph consisting of a set V of vertexes (nodes) and a set E of edges (links) between vertexes. A particular element of V will be denoted by an index $i \in V$ while an edge going to vertex $i \in V$ from vertex $j \in V$ will be denoted by $e_{ij} \in E$. A strongly connected network is a network such that there exists at least one directed path from any two edges in the network.*

In figure 1, there are six nodes cleverly labelled from one to six. Each directed edge is represented by an arrow starting from node j to node i. The network is strongly connected as one can travel from one node to another by use of the directed arrows. These arrows represent the direction of information channels. Hence, the network can be asymmetric in the sense that one consumer can listen to another consumer, while the opposite does not have to be true.

I will informally refer to nodes as consumers and edges as sources of information for consumers. The next definition is about the inbound neighborhood, which is nothing but the set of sources a given consumer has at is disposal. These sets are exogenous and can be thought as the constraint determining a unique network on which the problem is given.

Definition 2 (Inbound Neighborhood)**.** *The inbound neighborhood of node i (η_i) is the set of incoming links from other nodes.*

$$\eta_i \equiv \{j \in V : e_{ij} \in E\}$$

In figure 1, the inbound neighborhood of consumer 4 is the set of consumers 5 and 6 ($\eta_4 = \{5, 6\}$). This constraint can be thought as if the marketing agency will execute its campaign in the short-run, where the social capital, the structure of the network, is fixed. Hence, consumers can draw information only from sources they have already developed. However, nothing prevents them to stop listening to the sources they already have as its entirely up to them.

3.2 Consumers

I now define the object of value to the marketing agency, a signal, which is what the consumers share over the network. The signal is a theoretical device which captures the value of the product on the real line. It can either be a negative value, meaning that the consumer does not like it, or a positive value meaning the opposite.

Definition 3 (Signal)**.** *A consumer i forms a signal of intensity $s_{it} \in \mathbb{R}$ at time t. I will denote $s_t \in \mathbb{R}^{|V|}$ the vector of all signals at time t.*

Although I work with an unidimensional signal (scalar) produced by consumers, one can imagine multidimensional ones (vectors) embedded in a network where consumers have different neighborhoods for each component of the signal. For the purpose of this article, we keep the model simple and work with scalars, but the vectorial extension is discussed briefly in section 4.

Now I specify how consumers form their signal in this environment and how a spin is generated. The leading idea is quite simple: each consumer has a perception about their neighbors. They might think each neighbor are overoptimistic by nature and amplify the signal. They might also think the opposite, meaning that some neighbors might undersell the signal about the product. Thus, for any source of signal they have, they change the value of the received signal to embed their perception of the spin of each neighbor. They thus subtract some value for each source if the source is optimistic and add some value if they think the opposite.

Once consumers have modified the signals according to their perceptions, they choose the highest resulting value amongst all those sources. They thus derive utility from what they perceive to be the highest "true signal" about the product. Each consumer then proceeds by sending the resulting choice on top of their own spin, their own modification of the signal. The process is then iterated over time. The definition below summarizes the behavior.

Definition 4 (Consumers's behavior)**.** *For each source in η_j, a consumer $j \in V$ has a set of perceptions $\{a_{ji} \in \mathbb{R}, i \in \eta_j\}$. It thus forms what he believes to be the "true value" of each signal, $s_{ijt} - a_{ji}$ and chooses the highest value amongst them:* $\max_{i\in\eta_j} s_{ijt} - a_{ji}$. *He finally sends his own signal by adding his own spin $a_j^s \in \mathbb{R}$:*

$$s_{j,t+1} = \max_{i\in\eta_j} (s_{ijt} - a_{ji}) + a_j^s.$$

So if consumer 1 has a perception about consumer 2 $a_{12} = 2$, the perception about consumer three $a_{13} = 3$ and signals sent are respectively $s_{12t} = 5$ and $s_{13t} = 3$, he thinks that the actual signals sent are (-3, 0) and thus chooses zero. If consumer one adds no spin to the signal, his signal sent to his neighbors is then $s_{1(t+1)} = 0$.

The spin has thus two dimensions. The first dimension is a_j^s, the change every consumer adds to the signal to sway the signal. It can either be positive (advertising), negative (denial campaigns) or neutral. The second dimension is the perception of consumers about the spin of other consumers in their neighborhood, a_{ji}. They subtract this perception from the signal they receive. Hence, what matters in determining the overall direction of the signal is the "net spin" generated by a particular pair of consumers. The way these spins are elaborated by consumers and how listeners reacts to it remains unspecified for now, so one can think of these as given preferences for both the product and given perception about neighbors. One important case will be treated below, namely when the perception of the spin is actually equal to the spin. Some dynamic extension will also be discussed briefly in section 4.

There are two leading assumptions in the max behavior. First, all consumers value the signal in the same direction as they choose the highest value amongst the group. This is undeniably plausible in a classical advertising perspective (we all want "good" apple juice), but it needs to be interpreted with caution in some other frameworks. For instance, in a political advertising framework, some convinced voters might see a bad signal in whatever the "other candidate" presents and would thus pick the lowest signal in the neighborhood of electors supporting this opponent. Such situation cannot be directly treated with the actual model since the max operator forbids such behavior. If however one sees the voters as people seeking the "best policies", it would seem that everyone would agree on picking the best ones (max operator), but disagreeing on the "goodness" of each policies (spins on the signal).

The second assumption about consumers is that they do not know the structure of the network, but solely their neighborhoods. This means that they derive utility solely on their capacity to choose amongst the set of signals they receive every period. This also means that they do not predict the impact of their own behavior on the whole system. This assumption can be rooted in the classical idea that consumers thinks their own decision have only a marginal effect on the whole society.

3.3 The Principal's Behavior

The marketing agency values a positive total signal amongst consumers and she discounts the value of signals through time at the rate β. This is formalized in the definition below.

Definition 5 (Principal's Utility function)**.** *Let $t \in \mathbb{N}$, $\beta \in (0,1)$ and define de set of all possible signals by :*

$$S \equiv \left\{ \{s_t\}_{t=0}^{\infty} : s_t \in \mathbb{R}^{|V|} \right\}.$$

Let $\imath$ be a $|V| \times 1$ vector of ones. For any $s \in S$, the principal values s by :

$$U(s) \equiv \sum_{t=1}^{\infty} \beta^t \imath' s_t$$

This utility form can be thought as a profit maximizing behavior where the total monetary value of the signal is equal to one.

The agency has one instrument at his disposal, that is the initial vector of signals, s_0, reflecting the first information each consumer gets about the product. A particular signal s_{i0} costs $\frac{s_{i0}^2}{2}$ so the total cost of the initial signal is $\sum_{i \in V} \frac{s_{i0}^2}{2}$. This discussion leads to the following principal problem:

Definition 6 (Principal's behavior)**.** *Let $s_{t+1} = G(s_t)$ denote the law of motion generated by the graph and the consumer's behavior in the network. It then seeks the policy satisfying:*

$$\begin{aligned} &\arg\max_{s_0} U(s) \\ &s.t. \\ &s_{t+1} = G(s_t) \ \forall t \\ &c(s_{i0}) = \sum_{i \in V} \frac{s_{i0}^2}{2} \end{aligned}$$

Denote s^* the solution (if any) to this problem.

3.4 Solving the Model

Before I prove the characteristics of the optimal solution, I will discuss informally two particular cases as they help on building up on intuition. This also allows me to introduce some definitions in a natural way. The first case defines a pure viral marketing approach as an optimal solution when the principal is patient (when β is close to one). The second one defines the broad advertising solution as an optimal solution when the principal is impatient (β close to zero). For any other value of β, the optimal solution will be in between, depending on the impatience of the principal. I will refer to the network in figure 1, which is an numerical version of the first figure.

3.4.1 The Case Where $\beta \to 1$

This case builds-up on intuition as this is when there is no trade-off between time and the strength of the signal. Since the principal is patient enough, she will put time on her side and let the network build the signal for her. She does so with a maximal effect by putting all the strength of the initial signal in only one consumer. This consumer is chosen by its ability to generate the highest long-run spin on all the network.

Proposition 1 (Viral Marketing Solution)**.** *Let $\beta \to 1$ in the principal's problem, define λ as the maximal average value of a cycle on the network. Then, there exists an "all for one" solution that approaches s^* and is characterized by:*

1. *the existence of a t^* called the transient time after which the network enters a steady state;*

2. *a cyclical steady state with periodicity p, the longest cycle with the maximal average λ;*

3. *a principal who chooses a node i that maximizes to total amount of spin $\sum_{i=1}^{p} \beta^i \sum_j A^*_{ij}$ over all nodes j of the network given this periodicity p.*

4. *the utility of the principal approaches:*

$$U(s^*) = \frac{\beta^{2t^*}|V|^2}{(1-\beta)^2} + \frac{\sum_{i=1}^{p} \beta^i \sum_j A^*_{ij}}{1-\beta^p} + \lambda \frac{|V|\beta}{(1-\beta)^2} + O\left(\frac{\beta^{t^*}|V|}{1-\beta}\right)$$

where

$$\lim_{\beta \to 1} O\left(\frac{\beta^{t^*}|V|}{1-\beta}\right)\left[\frac{\beta^{2t^*}|V|^2}{(1-\beta)^2}\right]^{-1} = 0$$

The first statement of the proposition asserts that the networks enter a steady state after a finite number of periods. A steady state here describes the moment from which consumers always choose the same source. The second statement says that what defines the cyclical behavior of the network is the cycle (a loop) that has the highest average value of spin increase on the network. This cyclicality induces a constraint on the network as the cycle induce a pattern of behavior that repeats itself after p iterations. Hence, the principal has to choose the node that maximizes the additional spin given this constraint. This is what the third term describe. Each element of the vector of values in A_i^* corresponds to the maximal spin that can be generated given this cyclicality constraint. We will see below that this is a fixed point of the network, namely $s_{t+p} = G(s_t) - \lambda p$.

The last statement characterizes the optimal value in terms of an approximation. It states that the optimal solution is actually the sum of four components. The first term is the value derived from choosing one node that will reach all other nodes in t^*. The second term is the value associated with choosing the node that has the highest marginal spin component in the network. The third component is simply the value given by the increase of signal the network generates at each iteration regardless of which consumer is chosen. Finally, the last term is the value generated by transitionary dynamics and is relatively unimportant compared to others.

The third term can be thought as the intrinsic value of the network. Given *any* initial condition s_0, the network will generate this value of spin. In particular, if the initial condition is the zero vector, the network will still generate some value. So this value is somehow generated for "free" to the principal. This can also be thought as the remuneration the network should get for spinning the message.

The second term can be thought as the intrinsic value of the original node. Choosing this node generates this additional value and this value is the highest one amongst all nodes.

The first and fourth terms come from the signal optimization. For any given signal s_i, the signal will generate

$$\frac{\beta^{t^*}|V|}{1-\beta} s_i + O(\beta^{t^*}|V| s_i).$$

Hence, the principal sets its original signal so that the marginal revenue equals the marginal cost:

$$\frac{\beta^{t^*}|V|}{1-\beta} + O(\beta^{t^*}|V|) = s_i^*.$$

Replacing this value of s_i^* yields the solution. Note that the solution above is not necessarily unique. There might be more than one consumer that maximizes the spin in reaching other nodes.

In figure 2, there are two cycles with the highest average values depicted in red. Both of these cycles have $\lambda = 5$. Each of this cycle has a periodicity of three and the node with the highest value is node number 6. This node has a spin vector of $[-2, -1, -3, 0, 1, 0]'$ for a total value of -5. The total value of other nodes are all lower. The interpretation of these numbers are as follows: it is the net maximal increase the signal can take in reaching node j from node number 6, where the path is constrained to have a length that is a multiple of 3. The number given by the spin vector is given by the overall increase of the signal taken by this path net from the average increase of the network. For instance, reaching consumer 5 from consumer 6 has the following path : 4, 5, 4, 5, 4, 5 with an associated increase of the signal given by 6 + 7 + 2 + 7 + 2 + 7 = 31. However, the signal has increased by an average value of $\lambda = 5$ each period. Hence, the net value is given by 31 - 6·5 = 1. Other numbers are defined likewise.

3.4.2 The Case Where $\beta \to 0$

Here, I describe the second particular case of the problem, namely when $\beta \to 0$. This case is almost trivial and will be discussed briefly.

Proposition 2 (Broad Advertising Solution)**.** *Let $\beta \to 0$ in the principal's problem. Then, there exists a unique solution that approaches s^* and this solution is given by $s^*_{i0} = 1 \ \forall \ i$, $s^*_t = G(s^*_t)$. The principal will derive a utility of $|V|$.*

The solution has an important economic meaning, though, as it shows an impatient principal will choose broad marketing instead of viral marketing and that the strength of the signal (1) is much smaller than when $\beta \to 1$, where the initial signal, net of time effects, is close to $|V|$.

3.4.3 Solving For $\beta \in (0, 1)$

The solution to the model is best understood through a tropical algebra as it allows to write the model as a linear first difference equation, a simple mathematical object.

A tropical algebra redefines the max operator as addition and the addition operator as multiplication. As a result, the scalar product of two vectors and other operators are redefined as well.

$$a_i \oplus b_i \equiv \max(a_i, b_i) \ \forall \ a_i, b_i \in \mathbb{R}_{\max} \equiv \mathbb{R} \cup \{-\infty\}, \qquad \text{(addition in a tropical algebra)}$$

$$a_i \otimes b_i \equiv a_i + b_i \ \forall \ a_i, b_i \in \mathbb{R}_{\max}, \qquad \text{(multiplication in a tropical algebra)}$$

$$\begin{aligned} a \cdot b &\equiv \bigoplus_{i=1}^{n} a_i \otimes b_i \ \forall \ a, b \in \mathbb{R}^n_{\max}, \qquad \text{(scalar product in a tropical algebra)} \\ &= \max_i (a_i + b_i). \end{aligned}$$

If we compare with standard algebra, there is one main difference the reader should know. There is no unique inverse under addition ($a \oplus x = b$ can have more than one solution for x or no solution at all). Otherwise, standard operations are similar to the standard algebra and I will defer readers looking for a formal treatment to the book written by Bacelli, Cohen, J. Olsder and Quadrat while baffled ones might want to look at the appendix of this article for some examples and useful definitions.

I sometimes discuss mathematical statements stated in terms of tropical algebra. To avoid confusion and to keep the language as close as possible to the standard algebra, I will sometimes use the abbreviation "i.t.a.s." to mean "in a tropical algebra sense". For instance, a unit-vector i.t.a.s. is a vector that has a norm equal to the identity element in a tropical algebra (namely, zero).

The advantages of formulating the problem in this algebra are immediate. The consumer's behavior can be modeled as the scalar product of a linear system. To illustrate this, I introduce the definition of the adjacency matrix of a graph.

Definition 7 (Adjacency Matrix). *Define the weighted adjacency matrix $A \in \mathbb{R}_{\max}^{|V|\times|V|}$ by :*

$$[A_{ij}] = \begin{cases} -a_{ij} & \text{if } j \in \eta_i, \\ -\infty & \text{if } j \notin \eta_i. \end{cases}$$

The Adjacency matrix contains the entries of perceptions that must be subtracted from each signal sent from the neighborhood. Consumers that are not in the neighborhood of the consumer i are weighted with the neutral element $(-\infty)$ under $\oplus$ and are thus never chosen (not accessible). For instance, the matrix of the network presented in figure 1 is given by :

$$A = \begin{bmatrix} -\infty & -a_{12} & -a_{13} & -\infty & -\infty & -\infty \\ -a_{21} & -\infty & -a_{23} & -\infty & -a_{25} & -\infty \\ -a_{31} & -a_{32} & -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty & -\infty & -a_{45} & -a_{46} \\ -\infty & -a_{52} & -\infty & -a_{54} & -\infty & -a_{56} \\ -\infty & -\infty & -\infty & -a_{64} & -a_{65} & -\infty \end{bmatrix}.$$

With such definition, the global behavior of the network can be modeled as an homogenous, first-difference equation:

$$\begin{aligned} s_{t+1} &= A \cdot s_t \otimes a^s \\ &= (A \otimes a^s) \cdot s_t \\ &= A^s \cdot s_t \end{aligned}$$

where a^s is the vector formed of consumer's contributions to the spin and A^s can be thought of the "net adjacency matrix" which embeds both the perceptions and the consumer's spins ($[A^s_{ij}] = -a_{ij} + a^s_j$). It thus depicts the "net spin" for each pair of consumers.

I define below the concept of critical cycle and then state the main mathematical results associated with the tropical algebra representation.

Definition 8 (Cycles, Critical Cycles)**.** *A cycle* $c = \left\{A^s_{ij}, A^s_{jk}, \ldots, A^s_{mn}\tilde{A}_{I'm}\right\} \subseteq E$ *is a repetitionless path of* $|c|$ *edges whose starting point is also its endpoint. Let* $\rho(c) \equiv \sum_{\tilde{A}^s_{ij} \in c} \frac{A^s_{ij}}{|c|}$ *be the average value of a cycle and* $C(G)$ *be the set of all cycles in a network* $G(V, E)$*. A cycle* c_c *is critical if* $c_c \in \arg\max_{c \in C(G)} \rho(c)$*. Denote* $C_c(G)$ *the set of critical cycles.*

For instance, the set of edges $\{e_{12}, e_{31}, e_{23}\}$ is a cycle (a loop) in figure 1. It might be a critical cycle if the average net spin over this loop $\left(\frac{-a_{12}-a_{31}-a_{23}}{3}\right)$ is the highest amongst all possible cycles.

Theorem 1 (Main Mathematical Results)**.** *Let* $G(V, E)$ *be graph and let* A^s *be its adjacency matrix. Then:*

1. A^s *has a unique eigenvalue* $\lambda = \rho(c_c), c_c \in C_c(G)$*;*
2. *there exists eigenvectors* $A^*_1, \ldots A^*_p$*, i.t.a.s. that spans part of the graph space. Such vectors are the different permutations of the cycles belonging to* $C_c(G)$*;*
3. *Define* $A^* \equiv \bigoplus_{i=0}^{t^*} (A^s \otimes \lambda^{-1})^{pi}$ *where* t^* *is the transient time of the graph, then the eigenvectors are represented by the columns in* A^* *for which nodes are in at least one* $C_c(G)$*.*
4. *For any* s_0*, the system will enter a steady state after* t^* *periods. The steady state is characterized by* $s_{t+p} = s_t - \lambda p$*, and* $s_t = A^* \cdot s_0$ *(i.t.a.s) when* $t > t^*$ *.*
5. *The transient time is finite.*

Proof. See Bacelli, Cohen, J. Olsder and Quadrat [3]. □

In short, these results states that we can perform an eigenspace decomposition, as in a standard algebra, to perform some analysis of signal formation. As in standard algebra, an eigenvalue[1] is a solution to $A \cdot v = \lambda \otimes v$. The principal difference is that there exists only one eigenvalue under this algebra. Finally, exponents are defined in a similar fashion as in a classical algebra $A^2 = A \otimes A = A + A$ and $\lambda^{-1} = -\lambda$. All of this is defined formally in the appendix.

[1] Note also that all dynamic programming models can be considerably simplified under this algebra as fixed points are simply (unique) eigenfunctions of the system.

There are two things from this theorem that we can use. First, what drives the increase in the network at each iteration is the value λ and this value is pinned down by a set of cycling consumers. Second, after the transient time, the network enters a steady state where the value of the signal is pinned down by a linear combination of eigevectors i.t.a.s..

The intuition is that the eigenspace mapping separates the "sharing effect" of the network, the dispersion of the intensity signals across nodes, from the "amplitude effect", the total intensity of each signal after the transient time. Since the principal is only interested in the latter, mapping the graph in this space allows for any network to be represented such that only consumers in the critical cycles matters and others listen to these consumers (this will lead to a proposition further below). The value of the spin is increased by λ, the eigenvalue of the system, at each iteration. This makes the mathematical analysis much simpler as the only thing to consider in this space, from the principal's perspective, is the initial cost of each signal.

After the transient time, the system can be represented by the change of coordinates $\tilde{s}_t \equiv A^* \cdot s_t$ and the dynamics of the graph becomes:

$$\tilde{s}_{t+p} = I_\lambda \cdot \tilde{s}_t \qquad \text{(homogenous equation in the eigenspace)}$$

where I_λ is the identity matrix value λ on the diagonal.

Hence, the value of the first eigenmode in the eigenspace will approach (in standard algebra):

$$\begin{aligned}\sum_{i=0,p,2p,\dots} \sum_{i=1}^{|V|} \tilde{s}_{i0} &= \sum_{i=1}^{|V|} \tilde{s}_{i0} + \beta^p(\tilde{s}_{i0} + p\lambda) + \beta^{2p}(\tilde{s}_{i0} + 2p\lambda) + \dots + \beta^{ip}(\tilde{s}_{i0} + ip\lambda) + \dots \\ &= \sum_{i=1}^{|V|} \left[\frac{\tilde{s}_{i0}}{1-\beta^p} + \lambda \frac{\beta^p}{(1-\beta^p)^2} \right]\end{aligned}$$

where $\tilde{s}_0 = A^* \cdot s_0$. Similarly, the second eigenmode in the eigenspace will be given by :

$$\sum_{i=1,p+1,2p+1,\dots} \sum_{i=1}^{|V|} \tilde{s}_{i1} = \sum_{i=1}^{|V|} \left[\frac{\beta \tilde{s}_{i1}}{1-\beta^p} + \lambda \frac{\beta^{p+1}}{(1-\beta^p)^2} \right]$$

where $\tilde{s}_1 = AA^* \cdot s_0$. Finally the $p-1$th eigenmode will yield:

$$\sum_{i=p-1,2p-1,4p-1,\dots} \sum_{i=1}^{|V|} \tilde{s}_{i1} = \sum_{i=1}^{|V|} \left[\frac{\beta^{p-1} \tilde{s}_{ip-1}}{1-\beta^p} + \lambda \frac{\beta^{2p-1}}{(1-\beta^p)^2} \right]$$

where $\tilde{s}_{p-1} = A^{p-1}A^* \cdot s_0$. Hence, summing over of all modes yields :

$$\sum_{i=0}^{\infty} \sum_{i=1}^{|V|} \tilde{s}_{i0} = \sum_{i=0}^{|V|} \sum_{j=0}^{p} \frac{\beta^j \tilde{s}_{ij}}{1-\beta^p} + \lambda \frac{|V|\beta}{(1-\beta)^2}.$$

One can see the main effects from which the principal gets value in this space. The first one is the value derived from the strength of the original signal while the second one is the intrinsic value of the network. These results will be interpreted further in the section below.

Now, this signal $\tilde{s}_{i0}$ is a linear transformation through the matrix of eigenvectors:

$$\tilde{s}_{i0} = A^* \cdot s_{i0} = \max\left[s_{10} + A^*_{i1}, s_{20} + A^*_{i2}, \ldots, s_{|V|0} + A^*_{i|V|}\right] \ \forall i,$$
$$\Rightarrow \tilde{s}_0 = A^*_1 \otimes s_{10} \oplus A^*_2 \otimes s_{20} \oplus \cdots \oplus A^*_{|V|} \otimes s_{|V|0},$$

so the goal is to find the linear combination of eigenvectors i.t.a.s. which produces the highest value vector. One can see that when the principal is patient, she can neglect what happens before t^* and simply optimize based on what happens after t^* which is the essence of the first proposition. This happens when β is close to one and she then selects the eigenvector with the highest column sum first. For β close enough to one, all values in the adjacency matrix will be relatively small with respect to $\frac{\beta^{t^*}}{1-\beta}$. Hence, when the principal chooses the value of the signal a to set it equal to its marginal value, it will reach all nodes as any A^*_{ij} will not influence the choice under the max operator. Thus, a signal a will generate :

$$\frac{|V|\beta^{t*}a}{1-\beta} + \frac{\sum_{i=0}^{1} A^*_{0j}}{1-\beta^p} + \cdots + \frac{\sum_{i=0}^{1} A^*_{p-1j}}{1-\beta^p}.$$

Hence, setting the marginal cost equal to the marginal revenue commands:

$$a^* = \frac{|V|\beta^{t*}}{1-\beta}.$$

This is, in essence, how you derive the results of proposition one.

When β is not close to one, there is no guarantee that $\frac{a}{1-\beta}$ is high enough to avoid all other choice variables. And the characterization of the solution must remain in terms of the initial condition. The previous discussion leads to the following theorem:

Theorem 2 (General Solution)**.** *Let $\beta \in (0,1)$ in the principal's problem. Then s^* solves :*

$$U(s) = \frac{\beta^t \sum_i \sum_{j=0}^{p-1} \beta^j (A^s - \lambda)^j \cdot (A^* \cdot s_0)}{1-\beta^p} + \lambda \frac{\beta|V|}{(1-\beta)^2} + \sum_{t=0}^{t^*-1} \beta^i \left(A^t \cdot s_0\right)$$

Computing the solution efficiently can be achieved through standard optimization technique. One can then use various methods for optimizing non-differentiable functions (see Shor [15] or Bonnans, Gilbert and Lemaréchal [6]) that converges in polynomial time.

3.4.4 An Example

In this section, I provide an example based on figure 2, which is itself a numeric version of figure 1. The adjacency matrix $(A \otimes a^s)$ of such network is given by :

$$(A \otimes a^s) = \begin{bmatrix} -\infty & 5 & 3 & -\infty & -\infty & -\infty \\ 3 & -\infty & 5 & -\infty & 1 & -\infty \\ 5 & 3 & -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty & -\infty & 2 & 6 \\ -\infty & 1 & -\infty & 7 & -\infty & 2 \\ -\infty & -\infty & -\infty & 2 & 2 & -\infty \end{bmatrix}$$

There are two critical cycles in the system (in red in the figure) with the associated eigenvalue $\lambda = 5$ and the transient time is equal to 8. These two critical cycles have both length three, so if there is any periodicity it is of length three. The matrix of eigenvalues is given by :

$$A^* = \begin{bmatrix} 0 & -2 & -4 & -2 & -6 & -2 \\ -4 & 0 & -2 & -4 & -5 & -1 \\ -2 & -4 & 0 & -3 & -4 & -3 \\ -8 & -6 & -7 & 0 & -4 & 0 \\ -4 & -5 & -6 & 1 & 0 & 1 \\ -8 & -9 & -7 & -3 & -4 & 0 \end{bmatrix}$$

If β is close to one, what matters is choosing the node that has the highest potential in reaching others. In this case, it is the last node as it has the highest column sum. This will thus yield an optimal solution close to :

$$\frac{\beta^8 36^2}{(1-\beta)^2} + \frac{-5 - \beta 11 - \beta^2 23}{1-\beta^3} + 5\frac{6\beta}{(1-\beta)^2} + O\left(\frac{6}{1-\beta}\right).$$

3.5 Economic Analysis

In this section, I focus on the economic meaning of the previous solutions. Altough the previous section yields the optimal solution for any given network, I will rely on the two particular extreme cases to derive intuition about what is going on. Recall that the first one selects mainly one consumer and lets the network share a stronger signal as in a viral marketing strategy. The second one chooses all consumers equally and represents a broad advertising approach. The previous solutions have been showed to be optimal solutions when the principal has extreme preferences for time.

It turns out that the broad marketing strategy can also be an optimum regardless of the principal's preference for time. This is the case when the network has no intrinsic spin in it. I call this a perfect neighbor foresight.

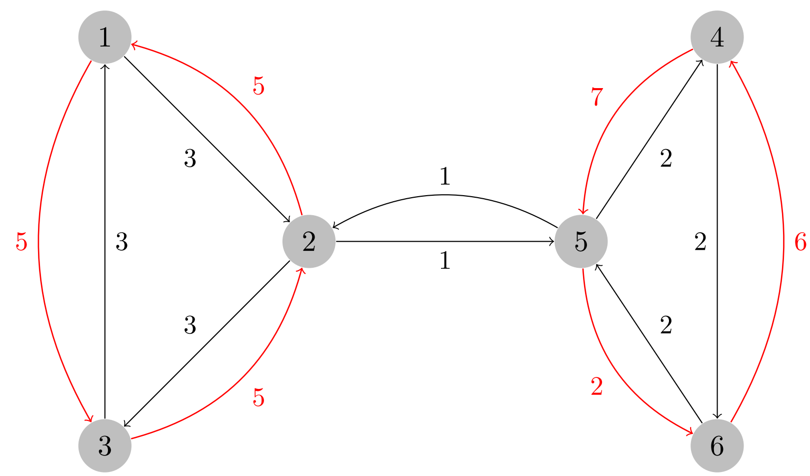

Figure 2: **Numeric Version of the Network in Figure 1.**

Definition 9 (Perfect Neighbor Foresight)**.** *A perfect neighbor foresight behavior is when* $a_j^s = a_{ij}\ \forall i, j$.

In words, a perfect neighbor foresight is when each consumer counterbalances perfectly the spin sent by each of the sources in his neighborhood. In particular, this means that the matrix $A \otimes a^s$ is a permutation matrix (only zeros and $-\infty$) and each row sums i.t.a.s. to zero.

Proposition 3 (Perfect Neighbor Foresight Equilibrium)**.** *If consumers have a perfect neighbor foresight, then:*

1. *The network value is zero* ($\lambda = 0$)*;*
2. *There is only one eigenvector (the zero vector);*
3. *The optimal initial value* s_0^* *is all for one with value* $s_{i0}^* = \frac{1}{1-\beta}\ \forall i$.
4. *Consumers are indifferent about their sources of information;*
5. *This optimal solution is the long run equilibrium.*
6. *The value the marketing agency derives is* $\frac{|V|}{1-\beta}$.

Proof.

1. Since the matrix $A \otimes a^s$ contains only zeros, the highest average cycle in the matrix is 0. Thus $\lambda = 0$ and the network effect is null.

2. By construction, the matrix A^* is the zero matrix.

3. Since there is a unique eigenvector, each node must be chosen equally and each bring the value $\frac{s_0}{1-\beta}$. Hence, the FOC commands that $s_0^* = \frac{1}{1-\beta}$.

4. Consumers face the same signal for any neighborhood. They thus pick any source in their inbound neighborhood.

5. The optimal solution is a linear combination of the eigenvector. Hence, it is a fixed point of the system.

6. This is a direct consequence of the last two points.

□

If we phrase this otherwise, the structure of the network has no importance in a PNF equilibrium and thus, the optimal marketing structure is broad advertising. One interpretation of the PNF equilibrium is that the network has been iterating for a long period of time so that consumers have been able to accurately discern the spin value each member of their neighborhood. This as a "steady state" interpretation, where the structure has no impact in the steady state. This result can also be interpreted as perfect information, in a game theoretical sense, meaning that the way information is shared has no impact whatsoever.

When there there is no perfect foresight, we fall back on the general case found in the previous section. E.g., there is a viral spin in the network ($\lambda \neq 0$). It can either be an positive or negative spin and obviously, the principal prefers positive ones. This is solely due to the usage of the max operator. A min operator type of algebra would have generated similar results in principle.

Definition 10 (Imperfect Neighbor Foresight With Positive Spin.)**.** *A Imperfect Neighbor Foresight With Positive Spin is when there is at least one loop of consumers such that the average value of the loop is greater than zero. Or in other terms, $\lambda > 0$.*

In such case, there is at least one cycle of consumers that "lead the spin" in increasing the signal by a average value of λ each period. The number of eigenvectors will depend on the number of cycles that actually have the highest average increase (λ) on the network. If there are more than one, they will generate a new cycling dimension in the network. This availability of different cycles will allow the principal for a trade-off between spread and spin. By choosing different cycles in the optimal solution, the principal can choose different regions of the network and thus spread the message faster. This trade-off depends on the value of the discounting rate and the diameter of the network. A large diameter and a low discounting factor might lead to a solution closer to broad advertising (selecting more than one eigenvector). If the value of β is high and there is a "small world" property, this trade-off will certainly go in favor of an all for one strategy (as in the example).

The value that the marketing agency derives is in part given by the "spin effect" of the network $\lambda\frac{\beta}{(1-\beta)^2}$. In other words, even a minimalist campaign where consumers have a chance to see the product without advertising will yield some value which is independent of the marketing signal. This can be qualified somehow as the intrinsic value of the network. A remarkable result is that it does not depend on the size of the network or on its structure. It solely depends on the average effect of the loop of consumers leading the spin. In this stylistic framework, this means that as little as two consumers can completely derail a perfect foresight equilibrium. It seems however more realistic that these cycles should be hard to detect for consumers as if they are all in thee same neighborhood, it seems they could figure it out.

But perhaps the most important interpretation is the following one. If one thinks of this framework as being a linear approximation around a perfect neighborhood foresight equilibrium, the model suggests that the principal who generates value only through signals has some incentive to push the system out of equilibrium. Indeed, this is when he can capture the additional value $\lambda\frac{\beta}{(1-\beta)^2}$. If the signal is thought as a general market index, some broker has incentives in generating some spin to extract the value out of the network. In the actual version of the model, the principal has no tool to increase the spins on the network, but this could well be added to the framework with ease as all it changes is the $\lambda\frac{\beta}{(1-\beta)^2}$. This extension is discussed in the next section.

3.6 Network Efficiency

A standard star configuration network is when there is one node acting as an hub and all other nodes are attached to it (see figure 3). This network is the efficient network configuration for a given number of nodes under a vast number of circumstances (see Jackson ch. 6.3 for a description). Intuitively, this is the most efficient went people can organize themselves to minimize

I define below a constrained star network and show that in the steady state, any graph $G(V, E)$ where nodes have a max-plus behavior will have a constrained star network configuration.

Definition 11 (Constrained Star Network)**.** *Let $G(V, E)$ be any graph with at least one cycle. Then, a constrained star network is defined as a collection of subgraphs where each of them has a critical cycle as the hub and all other nodes that are not in an hub are connected to one hub through a path that maximizes the average increase of the signal.*

In other words, a constrained star network tries to mimic a star network given the constraints generated by the original graph. The branches of the star are such that for all

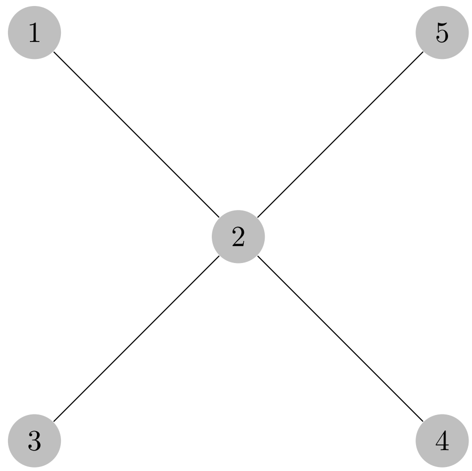

Figure 3: **A simple undirected star network.**

possible paths from the hub to a given node, the one selected is the one the maximizes the average increase of the signal over the number of edges.

This definition then leads to the following proposition:

Proposition 4. *After the transient time t^*, the choice of sources made by consumers on the network generates a constrained star network structure.*

Proof. Obviously, any consumer in a critical cycle will choose one of its neighbor in the critical cycle as it is the highest signal increasing strategy at each period. Now, assume that there is a node i that is not on a cycle and such that its source choice j does not receive any signal from any path from the hub. This means that on average, any signal sent by j is increased by a lower value than the hub, by the definition of a critical cycle. Thus, after t^* this node cannot be selected and there thus exists a directed path from the hub to node i. By the same token, the path between the hub and i has to be maximizing the increase of the signal on average. □

This means that even if consumers are not aware of the whole structure of the network, a structure of spin efficiency emerges. There is a cycle of consumers that lead the spin and other consumers attach themselves to this cycle in the way that maximizes the increase of the spin.

This efficiency result has to be interpreted with caution here. It means that the networks organizes itself in such a way that it maximizes the increase of the signal. Nothing says, from a normative point of view, if maximizing the increase of the signal is a good thing as well. For instance, if the signal is the price of an asset, this might not be an "efficient" result in a classical sense. This could however serve as a basis of explanation for market bubble behaviors.

4 Possible Extensions

In this section, I briefly present five directions in which the model can be extended. The first three can be directly tackled with minor changes to the model. The third case represents a numerical example of dynamic adjustments of the vector a^s while the fourth case is a generalization to a network where there is both consumers with max and min behaviors.

4.1 A Spin Increasing Device

Imagine that the principal has an additional tool to her set of instruments: she can also increase the spin of some consumers on the network at a quadratic cost (say an increase $\Delta\lambda$ costs $\frac{(\Delta\lambda)^2}{2}$). Given the previous analysis, she would obviously choose one of the consumer on a critical cycle and reinforce the spin (or diminish it when the spin is negative) up to the point where $\Delta\lambda = \frac{1}{|c_c|}\frac{\beta}{1-\beta}$. One can see that this formula implies that if there are many critical cycles, the principal will choose the smallest one since it yields the highest marginal return.This addition formalizes the idea that the principals has incentives to generate bigger spins and that small groups of persons should be

4.2 Multiple Signals

One can imagine that there is not only one, but many signals on the network. Moreover, the consumer's neighborhood of sources might not be the same for each signal. Some signals might have some additive properties (enhancing the spin) as some other might have anihilative ones (destroying the spin).

So this means there are still $|V|$ consumers, but $E_1, E_2, \dots E_n$ edges set, each of them being perhaps different for each signal. Each of these signals can enter in the signal formation of each consumer in an positive or negative fashion. In terms of analysis, nothing changes except the dimensionality of the problem. This can either be represented through an "adjacency tensor" or through an adjacency matrix generated by the kronecker product (with some minor adaptations). Hence, this case has been implicitly treated although there could be some particular analysis done on signal enhancement/destruction.

4.3 Random Networks

It may seem unrealistic that the principal knows everything about the network. Although this may be true on a social network like Facebook, it is not the case in a more informal network like acquaintances of brokers. So the principal might have some probabilistic knowledge of the network.

A different type of idea can also be modeled with random networks. Consumers might have different a-prioris depending on which state of the nature they are in. For instance, consumers might have a perfect neighborhood foresight with some probability, expansive ($\lambda > 0$) a-prioris with another probability, and contracting a-prioris with the remaining probability. This could help understand the dynamics of business cycles, for instance.

In both cases, this means there is a (finite) vector of probabilities $p_1, \ldots, p_n$ with an associated group of networks $G_1(V, E_1), \ldots, G_n(V, E_n)$. If the principal cannot observe the states of the world, the optimal strategy is then the weighted sum of all strategies.

If the principal can however observe the states of nature, there is an additional difficulty of choosing a reaction function that will embed the future changes of regime. This might require more thinking as this is dynamic programming on a network of consumers already using the max operator.

4.4 Reaction of Consumers

It might also seem unrealistic to assume fixed a-prioris from the consumers, as if the only maximizing choice they have is on the signal they can pick. This case can be modeled in part with the random network case, but there is still the problem of finding a policy function that will embed the dynamic changes of the consumers. I will not discuss this issue here, but rather illustrate the interest in exploring it by looking at the behavior of the network with very simple changes.

I will simply assume that a^s is an autoregressive process of the form : $B_{ijt} = A_{ij} - 1.8 \times (s_{i(t-1)})^{0.2}) + \epsilon$ where ϵ is some uniformly distributed shock. This stabilizing function is an approximation of the standard concave value functions one would get through dynamic programming. This behavior can accommodate a positive spin on the network (with $\lambda = 5$). If we apply this to the network in figure 1, we get interesting fluctuations of the sum of signals in the network. In figure 4, I graphed the sum of signals against time after a single unit shock. This simple graph suggests that the way people adapt their behaviors regarding a signal in the economy might generate cyclical and persistent fluctuations.

4.5 Max and Min Behaviors

When I defined the consumer's behavior, I mentioned that the sole max operator could not always be an adequate representation of consumers, especially in a political context. In a setup with two candidates, say A and B, a faction supporting A might have a max behavior regarding signals coming from supporters of A but have a min behavior with signals coming from supporters of B. Such kind of spin is proeminent from political staffers.

Figure 4: **An Impulse Response Function of the Network in Figure 2**

In order to do this in a linear algebra framework, the tropical algebra needs to be extended into an algebra that accepts both the max and min operators. This is called the symmetrized algebra (of the tropical algebra) and is discussed extensively in Bacelli, Cohen, Olsder and Quadrat. The analysis is not more complicated in principle, but is further away from the standard algebra and is thus less accessible to new readers.

5 Conclusion

In this paper, I characterized the optimal solution to a marketing campaign. Given a principal that seeks to extract the highest possible value of a signal shared over a network of agents, what strategy should this principal adopt? In short, she should use broad marketing when she is impatient and viral marketing when she is patient. The optimal solution will be an "in-between" solution depending on her degree of patience.

I have showed that when agents in the networks can perfectly detect spins, that is an increase or a decrease of the signal done by their sources in a way to sway the signal in one direction, the structure of the network is unimportant. In this case, the optimal solution is broad marketing. This is what I call a perfect neighborhood foresight equilibrium. Outside such equilibrium, there exists an optimal strategy that will use the network to build momentum and amplify the signal through the network. This higher value of the signal comes at the cost of waiting further in order to diffuse it through all agents. Hence the trade-off between the number of nodes reached in a given time and the strength of the signal.

I also show that after a transition period, consumers coordinate themselves in such a fashion that the network becomes the most efficient structure possible given the possible sources of information available to consumers. This occurs even if they are not aware of the whole structure of the network.

Finally, I show some possible extensions to the model so that results can be generalized to many different type of networks or many different type of behaviors. One interesting avenue is to use this framework as a new ground to explain fluctuations in an economy.

A A Primer On Tropical Algebras

In this appendix, I introduce various definitions of the mathematical language I use, namely a tropical algebra. This algebra is useful as it allows to describe the whole problem in a linear space, which reveals a tractable analytic solution.

Intuitively, this algebra is almost identical to the classical algebra teached in high-school except for one property: there is no unique inverse under addition. Hence $a \oplus x = b$ has no unique solution for x. Otherwise, properties are similar.

Definition 12 (Tropical Algebra)**.** *A tropical algebra is a semiring over a set* $\mathbb{R}_{\max} \equiv \mathbb{R} \cup \{\infty\}$ *with an addition operator and a multiplication operator defined by:*

$$a \oplus b \equiv \max\{a, b\} \;\; \forall a, b \in \mathbb{R}_{\max}$$
$$a \otimes b \equiv a + b \; \forall a, b \in \mathbb{R}_{\max}$$

With such an algebra, the multiplicative and additive identities are given by $e = 0$ and $\epsilon = -\infty$. One can easily check that some usual properties of standard algebra and some others are met:

$$x \oplus (y \oplus z) = (x \oplus y) \oplus z, \qquad x \otimes (y \otimes z) = (x \otimes y) \otimes z \qquad \text{(associativity)}$$
$$x \oplus y = y \oplus x \qquad x \otimes y, = y \otimes x \qquad \text{(commutativity)}$$
$$x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z) \qquad \text{(distributivity)}$$
$$x \oplus \epsilon = \epsilon \oplus x = x \qquad \text{(zero element)}$$
$$x \otimes e = e \otimes x = x \qquad \text{(unit element)}$$
$$x \otimes \epsilon = \epsilon \otimes x = \epsilon \qquad \text{(absorbing element)}$$
$$\forall x \in \mathbb{R}, \exists! y : x \otimes y = e \qquad \text{(unique multiplicative inverse)}$$
$$x \oplus x = x \qquad \text{(idempotency of addition)}$$

Unacoutumed readers might want to check that the following equations are true:

$$\begin{aligned} 10^2 &\equiv 10 \otimes 10 = 20 & (-0.5)^2 &= -1 \\ 6 &= 2 \otimes 4 \oplus 5 & \sqrt{-1} &= -0.5 \\ 7 &= 3 \otimes 2^2 \oplus 1 \otimes 2 \oplus 1 & 4^5 &= 5^4 = 20 \\ \begin{bmatrix} 7 \\ 6 \\ 11 \end{bmatrix} &= \begin{bmatrix} 1 & 2 & 3 \\ e & \epsilon & 2 \\ 5 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} & \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} &= \underbrace{\begin{bmatrix} e & \epsilon & \epsilon \\ \epsilon & e & \epsilon \\ \epsilon & \epsilon & e \end{bmatrix}}_{\equiv I} \cdot \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} \end{aligned}$$

Definition 13 (Eigenvalues and Eigenvectors)**.** *Let $A \in \mathbb{R}_{\max}^{|V|\times|V|}$. An eigenvalue λ of such matrix is a solution to the equation*

$$A \cdot v = \lambda \otimes v.$$

A particular vector v satisfying this equation for a given λ is called an eigenvector.

Definition 14 (Cycles)**.** *A simple cycle of length n in a graph $G(V, E)$ is a set of edges $\{a_{12}, a_{23}, \ldots, a_{n-1n}\} \subseteq E$ such that nodes $1 = n$, that no nodes is repeated and that $a_{ij} \neq \epsilon \forall a_{ij}$.*

Definition 15 (Maximal Average Weight)**.** *Let $G(V, E)$ be a graph defined on a tropical algebra and denote C_G the set of all cycles . The maximal weight of a cycle of length is given by :*

$$\lambda = \max_{c \in C_G} \frac{\bigotimes_1^{n-1} c_{i,i+1}}{|c|} = \max_{c \in C_G} \frac{\sum_{i=1}^{n-1} c_{i,i+1}}{|c|}$$

Definition 16 (Irreducible Matrix)**.** *Let $A \in \mathbb{R}_{\max}^{|V|\times|V|}$, the A is irreducible if it spans the full $|V|$-dimensional vector space or equivalently, if its determinant $|A|$ is not equal to ϵ.*

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