Can we Avoid (and Should we Avoid) Vote Swapping in Representative Democracies?

Sebastian Bervoets* and Vincent Merlin†

Preliminary version – June 2012
PLEASE DO NOT CIRCULATE

Abstract

It is a well-know fact that two tiers voting rules are subject to manipulation via gerrymandering by the election designers. Similarly, if there are more than three candidates, voters in distinct jurisdictions may wish swapping their votes in order to defeat the candidates they dislike most. Several contributions (Chambers (2008) and Perote-Pena (2005), Bervoets and Merlin (2011)) analyzed the problem of finding the rules which would be insensitive to gerrymandering. They found that only very undemocratic rules survive, in the sense that they do not treat the different candidates equally. In this paper we analyse the problem of vote-swapping. We first show that vote swapping is logically equivalent to gerrymandering with fixed size of jurisdictions when the voting rules are anonymous. Next, we characterise the set of voting rules which are immune to vote-swapping. We find that the results remain very pessimistic about the possibility of finding democratic swap-proof and voting rules. Last, as we cannot avoid vote-swapping manipulations, we wish to analyse the welfare issues induced by vote swapping: is vote swapping a mechanism which reduces the number of possible referendum paradoxes and therefore increases social welfare? (this section in progress)

JEL classification number: D71, D72

Keywords: Vote swapping, Gerrymandering, Manipulation, Federal Voting Rules.

*CNRS - GREQAM and IDEP - Centre de la Vieille Charité - 2, rue de la charité - 13002 Marseille, France. Email: Sebastian.Bervoets@univmed.fr

†Université de Caen and CNRS, CREM. Address: Faculté des sciences économiques et de gestion, bureau 216, 19 rue Claude Bloch, Université de Caen, 14032 Caen cedex, France. Tel: +33 (0) 231 56 62 49. Fax: +33 (0) 231 56 55 62. Email: vincent.merlin@unicaen.fr

1 Introduction

This paper is two-fold.
Part 1:
This paper examines the problem of *manipulation by vote swapping* in constitutions equipped with two step elections, such as Great-Britain, Canada or the US. In these constitutions, the country is divided into jurisdictions in which elections are held at the first step in order to choose the jurisdictional winners. In the second step, the government is appointed by an aggregation procedure over all the jurisdictional winners. It is a well-know fact that these constitutions, with usual voting rules, are sensitive to the partition of the electors.

In particular, one problem could come from vote swapping. Vote swapping is an informal agreement where two voters from different jurisdictions and parties trade votes to get representatives from their party elected while at the same time blocking out an unwanted third party from being elected to office.

During recent elections, many websites enabled electors to swap votes[1]. Some even claimed that they have reversed the result of some elections (Canada for instance). In in the US, the court of appeal has been asked to give a ruling on vote swapping. The court concluded that "vote swapping mechanisms as well as the communication and vote swaps that web sites enabled were constitutionally protected"[2]. The problem becomes real and raises the issue of manipulability of such elections. Note that manipulation by vote swapping is different from manipulation à la Gibbard-Satterthwaite which refers to a misrepresentation of preferences. The question raised in this paper is to know whether it is possible to design some electoral rules which are immune to vote swaps.

In particular, we consider that the boundaries of the jurisdiction are fixed, and that voters do not move. Thus the size of every jurisdiction is given, though they may not be of equal size. However, the voters are able to communicate, and we allow voters to swap their votes. First, votes are collected and aggregated within each jurisdiction, and local winners are designated by some local voting rule (possibly all different). Next, all the local winners are aggregated by another voting rule, so as to elect the federal winner. In this setting, we seek for voting rules which are immune manipulation by vote swapping.

When imposing anonymity on the local voting rules[3], vote swapping is logically equivalent to *gerrymandering* when the size of jurisdictions is fixed. Gerrymandering is a term that describes the deliberate rearrangement of the boundaries of electoral districts to influence the outcome of elections. The original gerrymander was created in 1812 by

[1]Several of these web sites are described in the paper by Hartvingsen (2006): Votedorset.net, helpbeathoward.org.uk, votepair.org, ditchdavis.com, voteswap.com, www.tactical-voter.org.uk. Most of them at now inactive.

[2]See http://www.ca9.uscourts.gov/ for more details

[3]We will discuss in detail the logical connections between vote swapping and gerrymandering in section 2.3.

Massachusetts governor Elbridge **Gerry**, who crafted a district for political purposes that looked like a sala-**mander**. The purpose of gerrymandering is to concentrate opposition votes into a few districts or gain more seats for the majority in surrounding districts (called packing), or to diffuse minority strength across many districts (called dilution). The literature has examined the problem of gerrymandering (see Bervoets and Merlin(2011), Chambers (2008) and Perote Peña (2005)) but assumes that the size of the jurisdictions can be changed in order to manipulate. This is an obvious weakness of the analysis as in real life elections the size of every jurisdiction is fixed. In this paper on vote swapping, this problem is fixed as the size of the jurisdictions cannot change.

Our main finding is the following. The set of voting rules which are swap-proof has three very undesirable properties: either the voters are all pivotal against unanimity (i.e. whenever everyone votes for the same candidate except for one voter, this unique voter can decide on the outcome of the election), or the local voting rules are such that candidates can be elected even though they received no votes (typically b is elected when everyone votes for a) or the voting rule is an exogenous code imposing the elected candidate regardless of the votes.

This result resonates as an impossibility to guarantee that elections can be swap-proof. If we assume furthermore that the local voting rules are anonymous, or negative results also apply if we consider gerrymander proof constitution with fixed size jurisdiction instead of swap-proofness as the key condition.

Part2:

Once we know that avoiding vote swapping manipulations is impossible, we wish to analyse its impact on social welfare. Allowing for vote swapping will often change the outcome of the election. Therefore candidates which are ideologically close from one another may wish to cooperate in order to maximise their chances of winning the election. By cooperating and implementing good vote swapping strategies, they might often reverse the so-called referendum paradox. A referendum paradox occurs whenever, in two tiers elections, the winner is not the candidate who receives the majority of votes. Without vote swapping, it is well-known that such paradoxes occur relatively often (Add some REF). By allowing for vote swapping it might be the case that the frequency of these paradoxes decrease.

In order to analyse this issue, it is necessary to find out which is the best swapping strategy for a group of candidates (say a and b for instance) when they are facing another group of candidates. We show that there is indeed an optimal strategy which guarantees that in many cases, the referendum paradox will disappear.

The rest of the paper is organized as follows. In section 2, we present the basic model, as well as the key properties we wish to impose on federal constitutions. Section 3 is devoted to the presentation of our main theorems. In section 4 we analyse the issue of strategic vote swapping and relate it to the problem of the referendum paradox. Section 5 concludes.

2 The general framework

2.1 Notations and Definitions

Let $A = \{a, b, c, \ldots\}$ (or $\{a_1, a_2, \ldots, a_v\}$ be a finite and fixed set of candidates, $N = \{1, \ldots, n\}$ the fixed set of voters, with $n \geq 3$, and $J = \{J_1, \ldots, J_m\}$ the set of jurisdictions, with $m \geq 2$. We assume that $n > m$. Let n_j be the size of jurisdiction J_j, i.e. the number of voters in J_j. Of course $\sum_j n_j = n$. The vector $\tilde{n} = (n_1, n_2, \ldots n_m)$ indicate the repartition of the population across the jurisdictions.

Let σ be a partition function from N to $\{1, \ldots, m\}$. Formally, for all $i \in N$, $\sigma(i) = j \Leftrightarrow i \in J_j$, with $\bigcup_{j=1,m} J_j = N$ and $J_j \cap J_k = \emptyset$ when $j \neq k$. In what follows, we consider partition functions in Σ, defined as the set of all partitions such that $\sigma^{-1}(j) \neq \emptyset$. There is no empty jurisdiction, there is at least one jurisdiction with strictly more than one voter. Furthermore, we define by $\Sigma^f \subseteq \Sigma$ the set of partitions such as $Card\{i; \sigma(i) = j\} = n_j$ for all $j \in \{1, \ldots, m\}$. The permutations in Σ^f keep the size of the jurisdictions fixed.

Voters vote for one unique candidate in the jurisdiction they reside in and votes are taken as given. We assume that when voters swap their votes, but there is no trust issue and voters are honest.

$\pi \in A^n$ denotes a *vote profile*. Typically, a vote profile is identified with a vector of a's, b's, ...where the i^{th} coordinate indicates voter i's vote. $\pi|_i$ denotes voter i's vote and for any subset S of N, we denote by $\pi|_S$ the restriction of π to S.

Once the vote profile is given, the winner in jurisdiction J_j is chosen via the social choice function f_j:

$$\begin{array}{rcl} f_j : \Sigma \times A^n & \rightarrow & A \\ (\sigma, \pi) & \rightarrow & z \in A \end{array}$$

We impose the following very mild condition on the functions $\{f_j\}_{j=1,m}$:

Jurisdiction Sovereignty

$$\text{If } [\sigma(i) = j \Leftrightarrow \sigma'(i) = j] \text{ and } [\pi|_{J_j} = \pi'|_{J_j}] \text{ then } f_j(\sigma, \pi) = f_j(\sigma', \pi') \text{ for all } j$$

In words, the result of an election in jurisdiction J_j is independent of what happens in other jurisdictions. The set of all social choice functions satisfying Jurisdiction Sovereignty is denoted by $\mathcal{F}$.

The m jurisdictional winners constitute a jurisdictional vote profile $\Pi \in A^m$, called a *federal profile*. The federation then appoints a federal winner using the federal social choice function g defined as follows:

$$\begin{array}{rrcl} g : & A^m & \rightarrow & A \\ & \Pi = (z_1, \ldots, z_m) & \rightarrow & z \in A \end{array}$$

Function g's domain is restricted to the set of all jurisdictional elected candidates. It does not include any other type of information such as the number of voters who elected each candidate, the margin of victory, etc.

A *federal constitution* is given by a $(m+1)$-tuple $C = (g, f_1, \ldots, f_m)$, with $f_j \in \mathcal{F}$ for all j. The federal winner of the election will be denoted by:

$$g(f_1(\sigma, \pi), \ldots, f_m(\sigma, \pi)) = g(f(\sigma, \pi))$$

Although functions f_j and g are social choice functions, the combination of these in a two steps procedure does not generate a social choice function, in the sense that the results also depends upon a non welfarist information, the partition σ.

2.2 Properties

We define properties we will impose on the different social choice functions. Most are similar to the ones that were already proposed in Bervoets and Merlin (2011).

Minimal Representativity (MinRep):

$$\text{For all } j \in J, \exists i \in N, \sigma(i) = j, \text{ such that } f_j(\sigma, \pi) = \pi|_i$$

In order to be elected in J_j, a candidate must have received at least one vote in J_j. This property implies among others the following:

Local Unanimity (LU):

$$\text{If } \pi|_i = \{z\} \text{ for all } i \in J_j \text{ then } f_j(\sigma, \pi) = z$$

A candidate should be chosen in his jurisdiction whenever he receives the votes of all voters. Note that both Minimal Representativity and Local Unanimity are silent about what happens at the federal level.

It seems reasonable to require that constitutions do not concentrate the decision in the hands of a unique voter. We express part of this idea with the following definitions, stating that one voter alone cannot overrule unanimity. Let the *Unanimous vote profile* π_z be defined as the vote profile such that $\pi|_i = z$ for all i in N, z in A.

Pivotal voter in $\boldsymbol{\pi_z}$: Voter i is called pivotal in π_z with $z \in A$ if for all σ there exists $y \in A$, $z \neq y$ and π such that $\pi|_i = y$ and $\pi|_h = z$, $h \neq i$ and $g(f(\sigma, \pi_z)) \neq g(f(\sigma, \pi))$.

Pivotal voter: Voter i is called pivotal if there exists an alternative $z \in A$ such that i is pivotal in π_z.

A pivotal voter is thus pivotal for at least one unanimous vote profile. Note that a pivotal voter i does not have the power to impose his choice on society, he can only change the winner on his own by voting for y at some unanimous profile π_z (but his choice could be a third candidate). Furthermore, being pivotal is different from having a veto power, as pivotal voters have power only against the unanimity.

The following axiom states that there should be no pivotal voters

No Pivotal Voters (NoPiv): In C there is no pivotal voter.

In social choice theory, the anonymity condition usually states that no voter enjoys a privileged status. A way to capture this idea is to set that the result of a vote depends upon the repartition of the voters among the different preferences types, regardless of the origins of the ballots. For a given partition σ and a given profile π, let $n_j(\sigma, \pi, a)$ be the number of voters who cast a vote in favor of candidate a in jurisdiction J_j. The vector $\tilde{n}_j(\sigma, \pi) = (n_j(\sigma, \pi, a_1, \ldots, n_j(\sigma, \pi, a_v))$ describes the repartition of the votes among the different candidates in jurisdiction J_j for a given profile and a given partition.

Weak Local Anonymity (WLA): Let $C = g(f(\sigma, \pi))$ be a constitution. The constitution is weakly locally anonymous if, for any jurisdiction and any given $\sigma \in \Sigma$, $\tilde{n}_j(\sigma, \pi) = \tilde{n}_j(\sigma, \pi')$ implies $f_j(\sigma, \pi) = f_j(\sigma, \pi')$.

This condition can be envisioned as a weak form of vote swapping inside the jurisdiction J_j. Exchanging votes within a jurisdiction does not change the outcome of f_j, as the repartition of the votes among the candidate is still the same. However, this condition considers that identity of the voters in J_j is perfectly known, and does not change.

Local Anonymity (LA): Let $C = g(f(\sigma, \pi))$ be a constitution. The constitution is locally anonymous if, for any jurisdiction and any given $\sigma \in \Sigma$, $\tilde{n}_j(\sigma, \pi) = \tilde{n}_j(\sigma', \pi')$ implies $f_j(\sigma, \pi) = f_j(\sigma', \pi')$.

On the top of being silent about the origins of the votes in J_j, the rule f_j doe not take into consideration the origins of the voters either.

2.3 When are Swap-proofness and Constrained Gerrymander-proofness equivalent ?

Bervoets and Merlin (2011) defined the Gerry-mander profness as a very strong property: no single voter and no group of voters can affect the outcome by moving from one jurisdiction to another while voting for the same candidate.

Gerrymander-proofness (G-P)

$$\text{For all } \pi \in A^n,\ g(f(\sigma, \pi)) = g(f(\sigma', \pi)) \ \text{ for all } \sigma,\ \sigma' \in \Sigma$$

This condition is very strong in the sense that no constraint exists on the size of the jurisdiction: we do not care if millions of voters leaves jurisdiction 1, leaving a single voter alone! The possibilities for gerrymandering are thus extremely large. This is why we consider in this article more restrictive forms of movements of the electors.

When vote swapping between two individuals i and j occurs, we consider the vote profile π_{ij} such that $\pi_{ij}|_{N\setminus\{i,j\}} = \pi|_{N\setminus\{i,j\}}$ and $\pi_{ij}|_i = \pi|_j$, $\pi_{ij}|_j = \pi|_i$.

Swap-proofness (SwPr):

$$\text{For all } \pi \in A^n, \text{all } \sigma \in \Sigma^f,$$

$$g(f(\sigma, \pi)) = g(f(\sigma, \pi_{ij})) \text{ for all } i, j \in N$$

Note that in the previous definition, each voter stays in his home state. But if we consider vote swapping between two voters as equivalent to home swapping, we need to define $\sigma_{ij} \in \Sigma^f$ as the partition of society such that $\sigma_{ij}(j) = \sigma(i)$ and $\sigma_{ij}(i) = \sigma(j)$ while $\sigma_{ij}(k) = \sigma(k)$ otherwise. We thus define

Gerrymander-proofness with fixed jurisdiction size (GPf):

$$\text{For all } \pi \in A^n,\ g(f(\sigma, \pi)) = g(f(\sigma', \pi)) \text{ for all } \sigma,\, \sigma' \in \Sigma^f$$

Proposition 1 *Assume that the consitution C is locally anonymous. Then gerrymander-proofness with fixed jurisdiction size and swap-proofness are equivalent.*

First, notice that there exist gerrymander-proof constitutions with fixed jurisdiction size that are not swap-proof, and vice-versa.

Example 1. Let $A = \{a, b\}$. Consider a constitution $C^1 = (g^1, f_1^1, \ldots f_m^1)$, where $f_j^1(\sigma, \pi) = a$ whenever a gets at least one vote in J_j and $g^1(\Pi) = a$ whenever at least one jurisdiction supports a. This Constitution is called a Priority Rule (see Chambers (2008) and Bervoets and Merlin (2011)). Let C^2 be the constitution such that the f_j's and g are the priority rule for a if voter 1 belongs to jurisdiction 1, and the priority rule for b otherwise. For a given partition σ, C^2 is swap-proof, but it is not gerrymander-proof.

Example 2. Let $A = \{a, b\}$. We define the dictatorship of voter 1 by $f_j(\sigma, \pi) = \pi|_1$ if $\sigma(1) = j$ and $g = \Pi|_j$ if $\sigma(1) = j$. By construction, the rule is gerrymander proof, but it is not swap proof. If voter 1 exchanges his vote for the (different) vote of voter 2, the result is different.

Proof of Proposition 1:

First part. Assume that C is swap-proof and locally anonymous, but does not satisfy constrained gerrymander-proofness. Thus, there exist two voters, $i, j \in N$, and two partitions $\sigma, \sigma' \in \Sigma^f$ such as $\sigma(k) = \sigma'(k) \forall k \neq i, j$, $\sigma(i) = p = \sigma'(j)$, $\sigma(j) = q = \sigma'(i)$, and $g(f(\sigma, \pi)) \neq g(f(\sigma', \pi))$. Note that otherwise, it would be impossible to violate gerrymander-proofness. Consider now the permutation π_{ij}, which permutes the votes of i and j in profile π. Hence, by Swap-proofness, $g(f(\sigma', \pi)) = g(f(\sigma', \pi_{ij}))$. As $n_p(\sigma', \pi_{ij}, a) = n_p(\sigma, \pi, a) \, \forall a \in A$ by construction, Local Anonymity implies that $g(f(\sigma', \pi)) = g(f(\sigma', \pi_{ij})) = g(f(\sigma, \pi))$, a contradiction.

Second part. Assume that C is gerrymander-proof, satisfies Local Anonymity, but is not Swap-proof. Thus, there exists a profile π_{ij} such that $g(f(\sigma, \pi)) = g(f(\sigma, \pi^{ij}))$. Note that $\sigma(i) = p \neq q = \sigma(j)$. Otherwise, by LA, we would have a contradiction. Consider the permutation σ' such that $\sigma'(i) = q$, $\sigma'(j) = p$, and $\sigma(k) = \sigma'(k) \forall k \neq i, j$. Thus, by constrained gerrymander-proofness, $g(f(\sigma, \pi_{ij})) = g(f(\sigma', \pi_{ij}))$. By construction, $n_p(\sigma, \pi, a) = n_p(\sigma', \pi_{ij}, a) \ \forall a \in A$, and $n_q(\sigma, \pi, a) = n_q(\sigma', \pi_{ij}, a) \ \forall a \in A$. Hence, LA implies that $g(f(\sigma, \pi_{ij})) = g(f(\sigma', \pi_{ij})) = g(f(\sigma, \pi))$, a contradiction. □

3 The possibility of swap-proof constitutions

3.1 A result for two alternatives.

Exogenous Code. A constitution C is the exogenous code if and only if $g(f(\sigma, \pi)) = z \ \forall \sigma \in \Sigma, \ \forall \pi \in A^n$.

Theorem 1 *Let $A = \{a, b\}$. Then C satisfies NoPiv, MinRep and SwPr if and only if it is the Exogenous Code.*

Proof: the Exogenous Code trivially satisfies NoPiv, MinRep and SwPr.

Consider any partition σ and the individual profile $\pi_0 = a...a$. By MinRep, $f_j(\pi_0, \sigma) = a$ for all j. Call z the federal winner associated to the federal profile $\Pi = a...a$ (i.e. $g(f(\pi_0, \sigma)) = z$).

Let $t_j(\sigma, a, b)$ be the minimal number of votes for b in J_j such that the winner in J_j is b, whenever all the other votes are for candidate a. Of course, as the f_j's might not be anonymous, we could have $t_j(\sigma, a, b) \neq t_j(\sigma', a, b)$. Given MinRep, $0 < t_j(\sigma, a, b) \leq n_j$. Let $t_j^*(a, b)$ be the minimal value of $t_j(\sigma, a, b)$ for any σ. Then let S_1 be the set of j's such that $t_j^*(a, b) = 1$ and S_+ be the set of all j's such that $2 \leq t_j^*(a, b) \leq n_j$.

For any federal profile Π, let $P(\Pi)$ be the number of jurisdictions J_j which have elected b. Obviously, $P(\Pi)$ ranges from 0 to m.

We need to show that if C satisfies NoPiv, MinRep and SwPr then $g(\Pi) = z$ for any Π such that $0 \leq P(\Pi) \leq m$. We do this by induction over natural numbers, by first showing that $[g(\Pi) = z$ when $P(\Pi) = 0]$ implies that $[g(\Pi) = z$ for all Π such that $P(\Pi) = 1]$. We will next assume that $[g(\Pi) = z$ whenever $P(\Pi) \leq k]$ and show that this implies that $[g(\Pi) = z$ whenever $P(\Pi) = k + 1]$.

Starting the induction: assume $g(\Pi) = z$ when $P(\Pi) = 0$. For any jurisdiction J_j such that $j \in S_1$ the condition NoPiv implies that $g(a, ..., b, ..., a) = z$. This is enough to conclude for the jurisdictions in S_1, but not for the others as g is not necessarily anonymous. Now, for jurisdictions belonging to S_+ we consider two cases:

Case 1: $Card(S_+) = 1$ and $Card(S_1) \geq 1$. Let j belong to S_+ and k belong to S_1. Then the individual vote profile such that everyone votes for a except for one individual who votes for b in J_k. That individual is chosen such that $f_k(a, ..., b, a, ...) = b$. Then $g(a, ..., \underbrace{a}_{J_j}, ..., \underbrace{b}_{J_k}, ..., a)$ gives z as the winner by using NoPiv. Changing $\pi|_{J_j}$ to $(a...a \underbrace{b...b}_{t_j(\sigma,a,b)-1})$ does not change the federal winner as the winner in J_j is still a and by swapping the b vote in J_k for the appropriate vote for a in J_j we get $g(a, ..., \underbrace{b}_{J_j}, ..., \underbrace{a}_{J_k}, ..., a) = z$ by SwPr.

Case 2: $Card(S_+) \geq 2$. Let j and k belong to S_+, consider the partition σ and the vote profile π such that $\pi|_{J_l} = a...a$ for all $l \neq j, k$, $\pi|_{J_j} = (a...a \underbrace{b...b}_{t_j(\sigma,a,b)-1})$ and $\pi|_{J_k} = (a...a \underbrace{b...b}_{t_k(\sigma,a,b)-1})$. Both J_j and J_k elect a and every J_l elects a as well. Hence the winner is z. By swapping one appropriate vote for b from J_k with one appropriate vote for a from J_j the federal profile becomes $g(a, ..., \underbrace{b}_{J_j}, ..., a)$ and by SwPr the winner is still z.

Hence whenever the federal profile contains exactly one b the winner is always z. We can turn to the induction hypothesis: assume $[g(\Pi) = z$ whenever $P(\Pi) \leq k$, $k < m]$. We will consider without loss of generality that it is the k first jurisdictions (J_1 to J_k) which have switched their vote to b. Hence $g(\underbrace{b, ..., b}_{J_1 \text{ to } J_k}, a, ..., a) = z$ and this can be obtained by the vote profile π such that $\pi|_{J_j} = \underbrace{a...a}_{n_j - t_j}\underbrace{b...b}_{t_j}$ for $1 \leq j \leq k$ and $\pi|_{J_j} = \underbrace{a...a}_{n_j}$ for all $j \geq k + 1$.
Again we have two cases.

Case 1: there is an l, $1 \leq l \leq k$, such that $t_l < n_l$. Consider then the profile π such that

$\pi|_{J_j} = \underbrace{a...a}_{n_j - t_j}\underbrace{b...b}_{t_j}$ for all $j \leq k$, $j \neq l$, and $\pi|_{J_l} = \underbrace{a...a}_{n_l - t_l - 1}\ \underbrace{b...b}_{t_l + 1}$ and $\pi|_{J_{k'}} = \underbrace{a...a}_{n_{k'} - t_{k'} + 1}\ \underbrace{b...b}_{t_{k'} - 1}$ for all $k' > k$. This profile generates either the federal profile $(b, ..., \underbrace{a}_{J_l}, ..., b, a, ..., a)$ or the federal profile $(b, ..., \underbrace{b}_{J_l}, ..., b, a, ..., a)$ depending on the winner in J_l. But in both cases the federal profile is such that $P(\Pi) \leq k$ so that the winner is z. Consider now swapping a vote for b in J_l with a vote for a in $J_{k'}$ for any $k' > k$. This gives the federal profile $(b, ...b, a, ..., \underbrace{b}_{J_{k'}}, ...a)$ and by SwPr this gives z as the winner, although Π is such that $P(\Pi) = k + 1$.

Case 2: for all l, $1 \leq l \leq k$, $t_l = n_l$. We need to distinguish three subcases.
Case 2a: if $t_j < n_j$ for a $j > k$. Then consider π such that $\pi|_{J_l} = \underbrace{b...b}_{n_l = t_l}$ for all $l \leq k - 1$, and $\pi|_{J_k} = a\underbrace{b...b}_{n_k - 1}$ and $\pi|_{J_j} = \underbrace{a...a}_{n_j - t_j - 1}\ \underbrace{b...b}_{t_j + 1}$. This profile generates either the federal profile $(\underbrace{b, ..., b}_{J_1 \text{ to } J_{k-1}}, \underbrace{a}_{J_k}, ..., \underbrace{a}_{J_j}, ..., a)$ or the federal profile $(\underbrace{b, ..., b}_{J_1 \text{ to } J_{k-1}}, \underbrace{a}_{J_k}, ..., \underbrace{b}_{J_j}, ..., a)$ depending on the winner in J_j. But in both cases the federal profile is such that $P(\Pi) \leq k$ so that the winner is z.

Now switch one b in J_j with the unique a in J_k in order to get the federal profile $(\underbrace{b, ..., b}_{J_1 \text{ to } J_k}, a..., \underbrace{b}_{J_j}, ..., a)$. By SwPr the winner is z.

Case 2b: if $t_j = n_j$ for a $j > k$ and $k < m - 1$, i.e. we are not at the last step of the induction. Consider the profile such that $\pi|_{J_l} = b...b$ for $1 \leq l \leq k$, $\pi|_{J_j} = a\underbrace{b...b}_{n_j - 1}$ and $\pi|_{J_{j'}} = \underbrace{a...a}_{n_{j'} - 1}b$ for a $j' > k, j' \neq j$. Then the federal profile is either $(\underbrace{b, ...b}_{J_1 \text{ to } J_k}, a, ..., \underbrace{a}_{J_j}, ..., \underbrace{a}_{J_{j'}}, ..., a)$ and the winner is z or it is $(\underbrace{b, ...b}_{J_1 \text{ to } J_k}, a, ..., \underbrace{a}_{J_j}, ..., \underbrace{b}_{J_{j'}}, ..., a)$ if $t_{j'} = 1$ and then the winner is also z according to case 3a. Swapping the b in $J_{j'}$ with an a in J_j gives the federal profile $(\underbrace{b, ...b}_{J_1 \text{ to } J_k}, a, ..., \underbrace{b}_{J_j}, ..., \underbrace{a}_{J_{j'}}, ..., a)$ and by SwPr the winner is z.

Case 2c: if $t_j = n_j$ for a $j > k$ and $k = m - 1$. In that case we know that the profile $\pi|_{J_l} = b...b$ for all $1 \leq l \leq m - 1$ and $\pi|_{J_j} = a\underbrace{b...b}_{n_j - 1}$ generates the federal profile $(\underbrace{b, ...b}_{J_1 \text{ to } J_{m-1}}, a)$ and this gives z as the winner (as $m - 1 = k$). By NoPiv, changing the unique a vote into a b vote cannot change the result so that $(b, ..., b)$ gives also z as the winner. Q.E.D.

3.2 The general case.

Theorem 2 *Let A be a finite set of candidates (with possibly $f_j \neq f_k$ and $n_j \neq n_k$). Then C satisfies NoPiv, MinRep and SwPr if and only if it is the Exogenous Code.*

Proof 1 *This proof uses lemma 1 and 2 stated and proved below.*

We proceed by induction over natural numbers. Let $X(\Pi)$ be the set of all candidates which are represented in the federal profile Π. First we show that for any federal profile Π such that $Card(X(\Pi)) = 2$, the winner is always the same. Theorem 1bis tells us that when $X(\Pi) = \{a, b\}$ the winner is some option z, whatever Π. We need to show that the winner is the same z when $Card(X(\Pi)) = 2$, and for instance $X(\Pi) = \{c, d\}$.

As z is the winner in case of an unanimous profile for a, generating the federal profile $(a...a)$, whether the second candidate is b or c does not change the logic of Theorem 1bis. Therefore z wins any election when $X(\Pi) = \{a, c\}$. But this implies that the unanimous profile for c, generating the federal profile $(c...c)$ gives z as the winner. Therefore, z must also be the winner when $X(\Pi) = \{c, d\}$ by applying the reasoning of Theorem 1bis with the pair $\{c, d\}$ instead of $\{a, b\}$.

Next we use the following induction hypothesis: the winner is z whenever $Card(X(\Pi)) < s$. We will show that this implies that the winner is still z whenever $Card(X(\Pi)) = s$, with $s < m$ (the case $s = m$ will be treated separately).
Let the individual vote profile π and the partition σ be such that $\pi|_{J_1} = \underbrace{a_1...a_1}_{n_1-1} a_2$, $\pi|_{J_2} = a_1 \underbrace{a_2...a_2}_{n_2-1}$, while all other jurisdictions are unanimous for candidates a_3 to a_s. By MinRep, $f_1(\pi, \sigma) = a_1$ or a_2 and $f_2(\pi, \sigma) = a_1$ or a_2. There are now three cases.

Case 1: $f_1(\pi, \sigma) = f_2(\pi, \sigma)$. In that case the federal profile Π associated to π and σ contains only $s - 1$ different alternatives and by the induction hypothesis the winner has to be z. Now by swapping the a_2 vote in J_1 for the a_1 vote in J_2, MinRep implies that the winner in J_1 is a_1 and the winner in J_2 is a_2, hence generating a federal profile Π such that $Card(X(\Pi)) = s$, while SwPr implies that the winner is z.

Case 2: $f_1(\pi, \sigma) = a_1$ and $f_2(\pi, \sigma) = a_2$. Then consider the vote profile π' such that $\pi|_{J_j} = \pi'|_{J_j}$ for all $j \neq 1$ and $\pi'|_{J_1} = \underbrace{a_1...a_1}_{n_1-t_1} \underbrace{a_2...a_2}_{t_1}$. Thus $f_1(\pi', \sigma) = a_2$ and $f_2(\pi', \sigma) = a_2$, hence generating a federal profile Π such that $Card(X(\Pi)) = s - 1$. The winner is z by the induction hypothesis. Now, by swapping one vote for a_2 in J_1 with the unique vote for a_1 in J_2 we get a_1 as the winner in J_1 and a_2 as the winner in J_2

and hence a federal profile Π such that $Card(X(\Pi)) = s$, while SwPr implies that the winner is z.

Case 3: $f_1(\pi,\sigma) = a_2$ and $f_2(\pi,\sigma) = a_1$. Consider in J_1 and J_2 the sequence of vote profiles, starting from $a_1...a_1$, and switching individuals one by one to a_2 until reaching $a_2...a_2$. There are two subcases. Along this sequence, either J_1 or J_2 are such that the winner is the same for two consecutive steps of the sequence, i.e. for $j = 1$ $j = 2$, $f_j(\underbrace{a_1...a_1}_{k}\underbrace{a_2...a_2}_{n_j-k},\sigma) = f_j(\underbrace{a_1...a_1}_{k-1}\underbrace{a_2...a_2}_{n_j-k+1},\sigma)$ or the winner in both J_1 and J_2 changes every time one individual switches his vote from a_1 to a_2.

Case 3a: Assume that there is such a sequence in J_1 (or in J_2), i.e. there is a k such that $f_1(\underbrace{a_1...a_1}_{k}\underbrace{a_2...a_2}_{n_1-k},\sigma) = f_1(\underbrace{a_1...a_1}_{k-1}\underbrace{a_2...a_2}_{n_1-k+1},\sigma)$. If the local winner in that case is a_1, then consider π' such that $\pi'|_{J_1} = \underbrace{a_1...a_1}_{k-1}\underbrace{a_2...a_2}_{n_1-k+1}$, and $\pi'|_{J_j} = \pi|_{J_j}$ when $j \neq 1$. The local winners in J_1 and J_2 are a_1 and hence the federal profile contains only $s-1$ candidates. The federal winner is thus z. By swapping one vote for a_2 in J_1 for the unique vote for a_1 in J_2, we get respectively a_1 in J_1 (by hypothesis) and a_2 in J_2 (by MinRep) and hence a federal profile with s candidates. The winner is still z by SwPr. If the local winner in that case is a_2, then consider π'' such that $\pi''|_{J_1} = \underbrace{a_1...a_1}_{k}\underbrace{a_2...a_2}_{n_1-k}$, $\pi''|_{J_2} = \underbrace{a_2...a_2}_{n_2}$ and $\pi''|_{J_j} = \pi|_{J_j}$ for $j \neq 1, 2$. The local winners are respectively a_2 and a_2 and hence there are only $s-1$ different local winners. The federal winner is thus z. By swapping one vote for a_1 in J_1 with one vote for a_2 in J_2 we get respectively $\pi''|_{J_1} = \underbrace{a_1...a_1}_{k-1}\underbrace{a_2...a_2}_{n_1-k+1}$ which gives a_2 as the local winner and $\pi''|_{J_2} = \underbrace{a_2...a_2}_{n_2-1}b = \pi|_{J_2}$ which gives a_1 as the local winner. By SwPr the winner is z.

Case 3b: the winner in both J_1 and J_2 changes every time one individual switches his vote from a_1 to a_2. By MinRep, this is only possible if the size of the jurisdictions J_1 and J_2 are odd as $f_j(a_1...a_1,\sigma) = a_1$ and $f_j(a_2...a_2) = a_2$ and if all jurisdictions elect the candidate who receives an odd number of votes, whenever the vote profile contains only a_1 and a_2. Consider then the profile such that $\pi^*|_{J_1} = \underbrace{a_1...a_1}_{k\ odd}\underbrace{a_2...a_2}_{n_1-k}$, $\pi^*|_{J_2} = \underbrace{a_1...a_1}_{k'odd}\underbrace{a_2...a_2}_{n_2-k'}$, $\pi^*|_{J_3} = a_1...a_1$ and $\pi^*|_{J_j} = \pi|_{J_j}$ for all $j > 3$. The associated federal profile is thus given by $\Pi^* = a_1, a_1, a_1, a_3, ..., a_s$ which contains $s-1$ alternatives and thus gives z as the winner.
By swapping an a_2 in J_1 for an a_1 in J_2 changes both parities and produces the federal profile $\Pi' = a_2, a_2, a_1, a_3, ..., a_s$ which contains exactly s alternatives. By SwPr the win-

ner should be z.

What remains to be shown is that the induction is also correct when $s = m$. All the previous cases (case 1, case 2 and case 3a) still apply, it is only the case 3b which cannot be solved. Case 3b when $s = m$ implies that in every J_j and for any two candidates a and b, then $f_j(\underbrace{a...a}_{k}\underbrace{b...b}_{n_j-k}, \sigma) \neq f_j(\underbrace{a...a}_{k-1}\ \underbrace{b...b}_{n_j-k+1}, \sigma)$ for all k, i.e. the winner changes every time one individual switches his vote from one candidate to the other whenever the profile in any jurisdiction contains two candidates. Moreover, by MinRep, this implies that every n_j is odd and that the winner in every jurisdiction is the candidate receiving an odd number of votes. Then, consider the following individual profile π^+ such that $\pi^+|_{J_1} = a_1...a_1a_2a_3$, $\pi^+|_{J_2} = a_2...a_2a_1$, $\pi^+|_{J_3} = a_3...a_3a_1$, and $\pi^+|_{J_j} = \pi|_{J_j}$ if $j > 3$.

The local winner is J_1 is unknown, call him f_1, but a_1 is the winner in both J_2 and J_3 as he is the candidate who receives an odd number of votes. In the other jurisdictions the winners are a_4 to a_m. This implies that the federal profile contains $s-2$ or $s-1$ different alternatives, according to the value of f_1. The winner is thus z. Now, swap the unique vote for a_1 in J_3 for the unique vote for a_3 in J_1. In this new profile, the winner in J_1 is a_2 because there are only two candidates represented and a_2 receives an odd number of votes. In J_2 the winner is still a_1 while now in J_3 the winner is a_3 by MinRep. We thus have a federal profile with exactly $s(= m)$ alternatives and by SwPr the winner is z.

Q.E.D

Lemma 1 *Let A be a finite set of candidates. If C satisfies NoPiv, MinRep and SwPr then g satisfies Federal Anonymity*

Consider any federal profile Π and another profile Π' such that $\Pi'|_k = \Pi|_k$ for $k \neq i, j$ and $\Pi'|_i = \Pi|_j$, $\Pi'|_j = \Pi|_i$. Assume without loss of generality that $\Pi|_i = a_i$ and $\Pi|_j = a_j$. Denote by $t_l(\sigma, a_i, a_j)$ (CHANGER AILLEURS !!) the minimal number of votes for a_j needed in J_l such that the winner in J_l is a_j when the only candidates represented in the individual vote profile $\pi|_{J_l}$ are a_i and a_j. By MinRep, $n_l \geq t_l^{a_ia_j} \geq 1$. The federal profile Π can be obtained with an individual vote profile π such that $\pi|_{J_k} = \underbrace{\Pi|_k...\Pi|_k}_{n_k}$ for all $k \neq i, j$ and $\pi|_{J_i} = \underbrace{a_i...a_i}_{n_i-t_i^{a_ia_j}+1}\ \underbrace{a_j...a_j}_{t_i^{a_ia_j}-1}$, $\pi|_{J_j} = \underbrace{a_j...a_j}_{n_j-t_j^{a_ja_i}+1}\ \underbrace{a_i...a_i}_{t_j^{a_ja_i}-1}$.

Indeed, with this profile π, the winner in any J_k, $k \neq i, j$ is $\Pi|_k$ by MinRep, while the winner in J_i is a_i and the winner in J_j is a_j. Now, by swapping a vote for a_i in J_i with a vote for a_j in J_j, we get the federal profile Π'. By SwPr, it must be the case that $g(\Pi) = g(\Pi')$.

Q.E.D

Lemma 2 *Let A be a finite set of candidates. If C satisfies NoPiv, MinRep and SwPr then for any two federal profiles Π and $\Pi' \in A^m$ such that $X(\Pi) = X(\Pi')$, we have $g(\Pi) = g(\Pi')$.*

If $Card(X(\Pi)) = m$ then the result is straightforward using the previous lemma. Now consider that $Card(X(\Pi)) = s < m$. Then it must be the case that at least one candidate at the federal level is present more than once. We will show that if a local winner which is present more than once in a federal profile is replaced by another candidate in $X(\Pi)$ then the outcome of the election will be unchanged.

By noticing that any two federal profiles Π and Π' such that $X(\Pi) = X(\Pi')$ and $Card(X(\Pi)) = s < m$ can be obtained one from the other by a succession of such changes, the proof will be complete.

Assume candidate a_j is present more than once in the federal profile Π and that $g(\Pi) = z$. We show that if one a_j in Π is changed for an a_i, with $a_i \in X(\Pi)$, then the winner will not change. By Federal Anonymity, if the profile Π elects z, any permutation of this profile over the jurisdictions also elects z. There are two cases.

Case 1: If there is a jurisdiction J_l and a number k such that $f_l(\underbrace{a_i...a_i}_{k} a_j...a_j) = f_l(\underbrace{a_i...a_i}_{k+1} a_j...a_j) = a_i$ (resp. such that $f_l(\underbrace{a_i...a_i}_{k} a_j...a_j) = f_l(\underbrace{a_i...a_i}_{k+1} a_j...a_j) = a_j$. Then consider the partition σ and the individual profile π such that $\pi|_{J_l} = \underbrace{a_i...a_i}_{k+1} a_j...a_j$ and $\pi|_{J_m} = \underbrace{a_i...a_i}_{t_m(\sigma,a_j,a_i)-1} a_j...a_j$. We get $f_l(\pi|_{J_l}) = a_i$ and $f_m(\pi|_{J_m}) = a_j$ (resp. a_j and a_j). Now by swapping an a_j in J_m for an a_i in J_l we get $\pi|_{J_l} = \underbrace{a_i...a_i}_{k} a_j...a_j$ and $\pi|_{J_m} = \underbrace{a_i...a_i}_{t_m(\sigma,a_j,a_i)} a_j...a_j$ which yields $f_l(\pi|_{J_l}) = a_i$ and $f_m(\pi|_{J_m}) = a_i$ (resp. a_j and a_i). Hence at the federal level one a_j has been replaced by an a_i and by SwPr the federal winner is the same.

Case 3: this is the complementary case, i.e. in every J_l we have that for any k, $f_l(\underbrace{a_i...a_i}_{k} a_j...a_j) \neq f_l(\underbrace{a_i...a_i}_{k+1} a_j...a_j)$. By MinRep, this implies that in every J_l, whenever the individual profile contains only a_i and a_j, then the winner is the candidate receiving an odd number of votes and n_l is odd.
To be completed

3.3 Independence of the Axioms.

We need to prove that there exist voting rules that satisfy two out of the three axioms, but are different from the Exegenous code.

Example 3: A constitution that meets MinRep, SwPr, but fails to satisfy NoPiv. The definition of the Priority Rule for two candiddates can be extended for any number of alternative. Let $P = a_1 \succ a_2 \succ \ldots \succ a_m$ be a linear ordering over A. Hence, the priority rule for the ordering P his defined by:

- $\forall J_j,\ f_j(\sigma, \pi) = a_t$ if and only if $n_j(\sigma, \pi, a_s) = 0\ \forall s < t$.
- $g(\Pi) = a_t$ if and only if there are no j such that $f_j(\sigma, \pi) = a_s$, with $s < t$.

The priority rules give to every voter the power to overcome the unanimous decision of the other $n - 1$ voters for a_t if he chooses to report a_s, with $s < t$.

Example 4: A constitution that meets MinRep, NoPiv, but fails to satisfy SwPr. Let us assume that the $f_j's$ as well as g are defined by the plurality rule, with a lexicographic tie breaking rule. We denote by $n(\Pi, a_t)$ the number of jurisdictions that chose a_t in the federal profile Π. Thus, the plurality constitution is defined by.

- $f_j(\sigma, \pi) = a_t$ iff $n_j(\sigma, \pi, a_t) \geq n_j(\sigma, \pi, a_s) \forall a_s \in A$ and $n_j(\sigma, \pi, a_t) > n_j(\sigma, \pi, a_s)$ for all $s < t$.
- $g(\sigma, \Pi) = a_t$ iff $n(\Pi, a_t) \geq n(\Pi, a_s) \forall a_s \in A$ and $n(\Pi, a_t) > n(\Pi, a_s)$ for all $s < t$.

It is easy to find examples where SwPr is not met.

Example 5: A constitution that meets SwPr, NoPiv, but fails to satisfy MinRep. Remember that MinRep is only defined at the f_j's level.

3.4 Can we avoid gerrymandering by fixing the population sizes ?

Using a very similar framework, Bervoets and Merlin (2011) proved the following result:

Theorem 3 *Let A contain any finite number p of alternatives. If a constitution $C = (g, f_1, \ldots, f_m)$ satisfies G-P and Local Unanimity then C is either the Exogenous Code or in C every individual is pivotal.*

As said previously, gerrymander proofness is a very strong requirement, as the manipulator can change as he wishes the sizes of the jurisdictions. Does the result still hold if we restrict ourselves to constrained gerrymandering ?

Theorem 4 *Let A contain any finite number p of alternatives. If a constitution $C = (g, f_1, \ldots, f_m)$ satisfies Constrained Gerrymander Proofness, Minimal Representativity, Local Anonimity if and only if C is either the Exogenous Code or in C every individual is pivotal.*

The proof is just a corollary of Theorem 1, which states that Constrained Gerrymander Proofness and Swap-proofness are equivalent when C is locally anonymous.

4 Should we allow for vote swapping? - IN PROGRESS

We only analyse the case of plurality rule

We first show that the case in which candidates $a_1, ..., a_l$ form a coalition to beat candidates $b_1, ..., b_p$ can be analysed through the simplest case a, b against c (the proof is available but has to be written down correctly). This simplifies a lot the problem because c alone cannot engage in vote swaps while a and b can. We thus just need to focus on the optimal strategy for a and b without looking at possible counter deviations from the opponents.

Example: 3 jurisdictions with 100 voters each.

J_1	J_2	J_3
$30a$	$45a$	$20a$
$24b$	$8b$	$31b$
$46c$	$47c$	$49c$

Here, candidate c wins in every jurisdiction under plurality rule and therefore wins the election. If a and b voters agree to swap, they could achieve the following situation

J_1	J_2	J_3
$47a$	$48a$	$0a$
$7b$	$5b$	$51b$
$46c$	$47c$	$49c$

The winners would therefore be a in J_1 and J_2 and b in J_3. Candidate a will win with the help of b while candidate c looses in every jurisdiction.

Obviously in this example c voters cannot swap votes with anyone. Therefore the situation is easier to analyse than if a and b were competing against c and d for instance. But our first result shows that we can focus on a and b against c.

Assume m, the number of jurisdictions, is odd. Let every jurisdiction be of equal size, i.e. n. There are thus $n.m$ voters. We denote by n_i^j the number of votes for candidate j in jurisdiction i, and $n_i = (n_i^a, n_i^b, n_i^c)$ is the vote profile in jurisdiction i, while $n^j = \sum_{i=1,m} n_i^j$ is the total number of votes for candidate j.
Let $S_j = \{i; n_i^j > n_i^c\}$ for $j \in \{a, b\}$ be the set of all jurisdictions in which a (or b) beats c. Finally, let $S_{ab} = \{i; n_i^a + n_i^b > n_i^c\}$ be the set of jurisdictions such that a and b taken together could win (but still, c could be the winner using plurality rule).

Assume that card$(S_{ab}) = \frac{m+1}{2} + k$ with $k > 0$. It is then always possible to swap votes between a and b in order to form at least $\frac{m+1}{2}$ jurisdictions in S_{ab} in which, apart from votes for c, only votes for a or for b are present. We call these jurisdictions unanimous (in a or b). In these $\frac{m+1}{2}$ jurisdictions, we know that either a or b will win (but never c). Therefore, put together, a and b are guaranteed to win the election.

Hence we have shown that if initially (before swaps take place) a and b, taken together, are the winners in strictly more than half of the jurisdictions, but loose the election, then the group (a, b) can always engage into swaps in order to reverse the result and win the election. (THIS CASE IS EASY BUT THE STATEMENT SHOULD BE PRECISE AND THE PROOF SHOULD BE CORRECTLY WRITTEN).

Now assume that $\text{card}(S_{ab}) = \frac{m+1}{2}$. In that case, the coalition (a, b) is not guaranteed to win the election, because they can secure only $\frac{m+1}{2} - 1$ unanimous jurisdictions and loose the $\frac{m+1}{2}$-th one, together with the jurisdictions which are not in S_{ab}.

Example:

J_1	J_2	J_3	J_4	J_5
$51a$	$0a$	$25a$	$0a$	$0a$
$0b$	$51b$	$26b$	$0b$	$0b$
$50c$	$50c$	$50c$	$101c$	$101c$

Here, c wins in 3 jurisdictions and thus gets elected and there is no swapping strategy which allows to reverse the outcome, although $\text{card}(S_{ab}) = 3 > \text{card}(S_c) = 2$.

When $\text{card}(S_{ab}) = \frac{m+1}{2}$, what is the optimal swapping strategy for a and b? Assuming that $n^a > n^b$, one can think that a best strategy would be to swap votes in a way that harms b less, because b has to win some jurisdictions. Therefore the intuition says that the best strategy is to form unanimous jurisdictions with votes for a in order to maximise the number of b in the jurisdictions that b has to win. However, this intuition is wrong. Indeed, the following example illustrates this.

Example:

J_1	J_2	J_3	J_4	J_5
$46a$	$47a$	$52a$	$0a$	$0a$
$8b$	$6b$	$0b$	$0b$	$0b$
$47c$	$48c$	$48c$	$101c$	$101c$

Here, c wins in 4 jurisdictions but $\text{card}(S_{ab}) = \frac{m+1}{2} = 3$. Forming unanimous jurisdictions with votes for a would lead to

J_1	J_2	J_3	J_4	J_5
$54a$	$53a$	$39a$	$0a$	$0a$
$0b$	$0b$	$14b$	$0b$	$0b$
$47c$	$48c$	$48c$	$101c$	$101c$

which also gives c elected (c wins in 3 jurisdictions). A correct swapping strategy in this case would have been

J_1	J_2	J_3	J_4	J_5
$48a$	$49a$	$49a$	$0a$	$0a$
$6b$	$4b$	$4b$	$0b$	$0b$
$47c$	$48c$	$48c$	$101c$	$101c$

which would have elected a. Observe that here the best strategy is to spread out the votes for b, weakening his power at the most.

We show that the best strategy for coalition (a, b) consists of swapping votes such as to maximise the number of jurisdictions that one candidate (say a) can win, regardless of what votes remain for the other candidate (say b). This is not intuitive but can be proved.

Proposition 2 *Assume card(S_a) $= k$ and that a wins the k jurisdictions but still, the group (a, b) looses the election (i.e. b does not win $\frac{m+1}{2} - k$ jurisdictions). Then there is no swapping strategy such that b could win $\frac{m+1}{2} - k + 1$ and the group (a, b) would win.*

In words, if a wins the greatest possible number of jurisdictions, and however looses the election, it is not because the votes for b are incorrectly placed. If they cannot win with that strategy, then they can never win even if another swapping strategy increases the number of jurisdiction that b wins.

Proof: The proof uses the following lemma:

Lemma 3 *If card(S_a) $= k$ and a and b use a swapping strategy such that a wins the k jurisdictions, but the group (a, b) looses the election, it must be the case that the k jurisdictions won by a are unanimous in a.*

Proof of lemma: assume at least one among the k jurisdictions is not unanimous (i.e. a's and b's are present). Call it J_l. We say that J_l is mixed. Two cases.
Case 1: the other $\frac{m+1}{2} - k$ are unanimous in b. In that case we have a contradiction because b would be winning these jurisdictions and the group (a, b) would have won the election.
Case 2: there is at least a mixed jurisdiction among the other $\frac{m+1}{2} - k$. Call it J_p. As J_l is also a mixed jurisdiction, a and b could have swapped the votes for b from J_l with the votes for a in J_p and would have done better.

This lemma guarantees that the conditions of proposition 3 hold only if the k jurisdictions won by a are unanimous in a. Therefore, there are no votes for b in the k jurisdictions won by a. Let $n^b(S)$ denote the total number of votes for b restricted to the jurisdictions in S. We have $n^b(1, ..., \frac{m+1}{2}) = n^b_{k+1} + ... + n^b_{\frac{m+1}{2}}$ because $n^b_i = 0 \forall i \in \{1, ..., k\}$.

As b does not win the $\frac{m+1}{2} - k$ jurisdictions, it must be the case that $n^c_{k+1} + ... + n^c_{\frac{m+1}{2}} > n^b(1, ..., \frac{m+1}{2})$, and therefore $(\frac{m+1}{2} - k) * \frac{n}{2} > n^b(1, ..., \frac{m+1}{2})$ because $n^c_i < \frac{n}{2} \forall i \in \{1, ..., \frac{m+1}{2}\}$.

Now assume, like in the statement of the proposition, that another swapping strategy would have allowed b to win in $\frac{m+1}{2} - k + 1$ jurisdiction. Notations referring to this new partition of votes are denoted with a $\tilde{}$. Then, $n^b(1, ..., \frac{m+1}{2}) > \tilde{n}^c_1 + ... + \tilde{n}^c_{\frac{m+1}{2}-k+1}$

Of course, candidate a receives every vote except those for a and those for c:

$$n^a(1, ..., \frac{m+1}{2}) = \frac{m+1}{2}.n - n^b(1, ..., \frac{m+1}{2}) - \sum_{j=1}^{\frac{m+1}{2}} n^c_j \tag{1}$$

Thus

$$n^a(1, ..., \frac{m+1}{2}) > (\frac{m+1}{2}.n - \frac{m+1}{2} - k) * \frac{n}{2}) - \sum_{j=1}^{\frac{m+1}{2}} n^c_j = (\frac{m+1}{2} + k)\frac{n}{2} - \sum_{j=1}^{\frac{m+1}{2}} n^c_j \tag{2}$$

However, we can use the fact that the number of votes for c is constant:

$$\sum_{j=1}^{\frac{m+1}{2}} n^c_j = \sum_{j=1}^{\frac{m+1}{2}} \tilde{n}^c_j = \tilde{n}^c_1 + ... + \tilde{n}^c_{\frac{m+1}{2}-k+1} + \sum_{j=\frac{m+1}{2}-k+2}^{\frac{m+1}{2}} \tilde{n}^c_j \tag{3}$$

Therefore,

$$n^a(1, ..., \frac{m+1}{2}) > (\frac{m+1}{2} + k - (\frac{m+1}{2} - k))\frac{n}{2} - \sum_{j=\frac{m+1}{2}-k+2}^{\frac{m+1}{2}} \tilde{n}^c_j \tag{4}$$

$$n^a(1, ..., \frac{m+1}{2}) > k.n - \sum_{j=\frac{m+1}{2}-k+2}^{\frac{m+1}{2}} \tilde{n}^c_j \tag{5}$$

But again, $\tilde{n}^c_j < \frac{n}{2} \forall j \in \{1, ..., \frac{m+1}{2}\}$ so that

$$n^a(1, ..., \frac{m+1}{2}) > (k+1)\frac{n}{2} \tag{6}$$

which is a contradiction, given that otherwise $\text{card}(S_a) \geq k+1$, when we assumed that $\text{card}(S_a) = k$. Q.E.D.

This proposition provides us with an optimal swapping algorithm for candidates a and b:
Step 1: Find out the greatest number k of jurisdictions that a could win alone. Cover

them with the maximal amount of votes for a such that a wins. In case of multiple possibilities, choose the set of k jurisdictions which "consumes" the greatest number of votes for a

Step 2: If some votes for a are still available, then put them into any of these k jurisdictions, so that the maximal number of votes for b remain available.

Step 3: Put all the remaining votes for b in the other $\frac{m+1}{2} - k$ jurisdictions and check whether b wins them or not.

If this algorithm gives c as the winner, then there is no other way that a and b could have won by swapping votes.

TO DO: whenever $\text{card}(S_{ab}) > \frac{m+1}{2}$ then vote swapping guarantees the victory of a and b. Only in the case when $\text{card}(S_{ab}) = \frac{m+1}{2}$ it is not clear.

In the first case, we need to determine how many times vote swapping allows to reverse the referendum paradox. In the second case we need to investigate more in order to determine who the winner will be. Then we can determine how many times vote swapping has allowed to reverse the referendum paradox.

5 Conclusion

The conclusion of our study is that avoiding vote swapping is not possible. No acceptable voting rule that is swap-proof will exist. So, if vote swapping is a manipulation possibility, we are interested in understanding if it is a good or a bad practice, in terms of social welfare (measured by the satisfaction of the majority). For that part, our main conclusion is that a lot of work remains to be done!

References

[1] Bervoets S and Merlin V (2011) Gerrymander-proof representative democracies, International Journal of Game Theory - forthcoming.

[2] Chambers C (2008) Consistent Representative Democracy. Games and Economic Behavior 62:348–363.

[3] Gibbard A (1973) Manipulation of Voting Schemes: A General Result. Econometrica, 41:587-601.

[4] Hartvigsen D (2006) Vote Trading in Public Elections. Mathematical Social Sciences 52:31-48.

[5] Perote Peña J (2006) Gerrymendering Proof Social Welfare Functions. Mimeo, Universidad de Zaragoza.

[6] Satterthwaite M (1975) Strategy Proofness and Arrow's Conditions. Journal of Economic Theory, 10:187-217.