Network Design under Local Complementarities*

Mohamed Belhaj[†,‡], Sebastian Bervoets[†] and Frédéric Deroïan[†]

[†] *Aix-Marseille School of Economics, France*

[‡] *Ecole Centrale Marseille, France*

WORK IN PROGRESS - THIS DRAFT IS VERY PRELIMINARY

Abstract

We consider linear games with complementarities played on a network. We address the following question: given a set of nodes and a fixed number of links, how should links be organized in order to maximize aggregate effort or the sum of utilities? The analysis reveals the key role played by Nested Split Graphs.

Keywords: Network, Complementarities, Linear Interaction, Nested Split Graphs.
JEL: C72, D85

---

* We would like to thank participants at the CTN conference (January 2012), LAGV conference (June 2012). E-mail addresses: mbelhaj@ec-marseille.fr, sebastian.bervoets@univ-amu.fr, frederic.deroian@univ-amu.fr.

1 Introduction

We investigate games played on a network, where efforts are local complements and interactions are linear. We address the following simple question: given a set of nodes and a fixed number of links, how should we organize the links in order to maximize aggregate effort or the sum of utilities?
The game has three ingredients. First, agents are embedded in a fixed network, and they interact only with their network neighbors. Second, actions are strategic complements, i.e. agents increase their action when the average action of their neighbors is higher. Last, the game exhibits linear interaction, i.e. best-replies are linear in the choice of neighbors. This setting may usefully describe some optimal network design in many economic applications, like local peer effects as in education, or in economic situations where some strategic complementarities are likely to arise, like technology adoption and R&D races (see for instance Vivès [2005]).
The linear interaction setting is compatible with many utility functions. We mainly focus on two benchmark models of linear interaction. One is the standard linear-quadratic utilities due to Ballester et al. (2006), for which the sum of equilibrium utilities is the sum of squares of efforts, the other one is a class of utilities for which the sum of equilibrium utilities is proportional to the sum of efforts. Given linear interaction, these utilities generate a unique Nash equilibrium for sufficiently low level of interaction.

Our analysis stresses three main insights. The first result is that, for high level of interaction (i.e., the highest compatible with existence of equilibrium), the link organization maximizing both aggregate efforts and aggregate utility is given by a unique architecture named the *Quasi-complete* network. This architecture is roughly built up by accumulating links around a subset of agents, and it leaves potentially many agents isolated. That is, the promotion of efforts or utilities cannot be achieved without the generation of strong inequalities in the society.
The second result is that, for low level of interaction, the link organization maximizing both aggregate efforts and aggregate utility is nearly exclusively given two architectures. For low link density, the best organization is given by the *Quasi-star* network structure. This architecture is roughly built up by forming a star network, in which a central agent is involved as links as possible, and if some links are left, it builds a second central agent, and so on. Alternatively, if link density is high, the best organization is given by the *Quasi-complete* network. Hence, for low link density, inequalities mainly arise inside the principal component, while for high link density inequalities arise between the members of the main component and outsiders.
Our third result pertains with general level of interaction (compatible with existence of equilibrium). We show that any architecture maximizing both aggregate effort and the sum of (linear-quadratic) utilities is a nested split graph. In a word, *Nested split graphs* are a hierarchical architecture, which is such that for any pair of agents, the neighborhood of one agent is a subset of the neighborhood of the other agent. Of course the Quasi-complete and Quasi-star architecture are nested split graphs. Hence, the presence of both local interaction, linearity and complementarities leads to a class of networks with sharp characteristics (i.e., nestedness of neighborhoods). To obtain our result, we use different

methods. First, we show that switching one link toward an agent with higher effort in the current network enhances aggregate effort; this precipitates the class of nested split graphs. To proceed, we intensively use the theory of M-matrices. For utilities, this kind of simple switch does not hold, and we exhibit a 25-player counter-example. We then show that an adequate multiple-link switch toward one agent with higher effort enhances the sum of linear-quadratic utilities, and the fact is that this operation also induces the class of nested split graphs.

This paper is related to the literature on network formation games (as in Jackson and Wolinsky [1996], or Bala and Goyal [2000]), in which agents' strategies concern exclusively the selection of partners, and the general issue is which network structure shall emerge. This paper is also related to games played on networks (like Ballester et al. [2006], or Bramoullé et al. [2011] - both papers consider games with linear interaction), in which agents' strategies are some action like effort, and the main question is how does the network structure affect efforts. More closely, the paper inserts in a literature which tries to put together those two strands, by making both actions and network endogenous (see for instance Goyal and Moraga [2001] and Calvo and Zénou [2004], and more recently Cabralès et al. [2011] and Galeotti and Goyal [2010]).
Our model departs from this literature in that links are not selected by agents, but by a social planner having in mind an aggregate objective. The problem addressed in our paper has been recently introduced in Calvo et al. (2006). They explore the case of high interaction and they restrict the analysis to *connected graphs* (i.e., graphs in which every agent is involved in at least one link). They mainly show that, when the intensity of synergy tends to its upper bound (above this bound there exists a network on which all efforts tend to infinity), the star is the unique network maximizing aggregate effort and sum of utilities for a number of links equal to the size of the population minus one. However, their problem is open in general. Our paper pursues their work further, by identifying, among *all* networks (both connected and disconnected), nested split graphs as key architectures, not only in the case of high level of interaction, but for general intensity of interaction.
This work also echoes a literature that emphasizes Nested split graphs. This network architecture is well-known from mathematicians (Madahev and Peled [1995] for instance). In the field of economics, König et al. (2009) find that nested split graphs emerge from a dynamic network formation game in which players get the same linear-quadratic pay-offs than in this paper. Our paper complements their findings, by showing that nested split graphs may also have desirable properties from a network design perspective.

This paper is organized as follows. Section 2 introduces the model, section 3 addresses the problem in the context of limit cases of high and low interaction. The general issue is examined in section 4, and section 5 makes some concluding remarks. A last section gathers all proofs.

2 The model

We consider the problem of a planner who wishes to identify, among all networks with a given number of links, the subset of networks maximizing either aggregate effort or aggregate utilities.

Consider a finite set of agents $N = \{1, 2, \cdots, n\}$. This set will be fixed throughout the paper. Each agent is inserted in a social network. Formally, let the exogenous matrix $G = [g_{ij}]$ represent the adjacency matrix of a symmetric and binary network, in which $g_{ij} = g_{ji} = 1$ if and only if agents i and j know each other. By convention G has null diagonal. We shall abuse the notations for convenience: we may indifferently describe G as a network or its adjacency matrix, we shall refer to link ij and we may write $ij \in G$ or $ij \notin G$, as well as $G + ij$ or $G - ij$. For $L \in \{0, 1, \cdots, \frac{n(n-1)}{2}\}$, we define $\mathcal{G}(L)$ as the set of all networks L links.
The profile $X = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+$ represents the vector of individual efforts. We consider linear best-responses of the form:

$$x_i - \delta \sum_{j=1}^{n} g_{ij} x_j = 1 \qquad \text{for all } i \tag{1}$$

where parameter δ, which is assumed to be positive throughout the paper, measures the strength of synergies between neighbours. Since $\delta > 0$, this means that neighbours' efforts are strategic complements.
The system (1) can represent some strategic interaction derived from individual optimization in the following games. In perhaps the most standard game with linear interaction, each agent i derives from the effort profile the following linear-quadratic utility augmented with local multiplicative synergies (Ballester et al. [2006]):

$$u_i(x_i, x_{-i}; G, \delta) = x_i - \frac{1}{2} x_i^2 + \delta \sum_{j=1}^{n} g_{ij} x_i x_j \tag{2}$$

With the utility function as in equation (5), the sum of utilities at equilibrium X^* (satisfying the system (1)) is such that

$$\sum_{i=1}^{n} u_i(X^*; G, \delta) = \frac{1}{2} \sum_{i=1}^{n} x_i^{*2} \tag{3}$$

The system (1) can be generated by the following alternative pay-off function:

$$w_i(X^*; G) = a \cdot x_i - h(x_i - \delta \sum_{j=1}^{n} g_{ij} x_j) \tag{4}$$

with $h()$ increasing and convex, $a > 0$, $h'(0) < a$, $h(0) = 0$ and $h'^{-1}(a) = 1$. With the utility function as in equation (4), the sum of utilities at equilibrium X^* is such that

$$\sum_{i=1}^{n} w_i(X^*; G) = a \sum_{i=1}^{n} x_i^* - h(1) \tag{5}$$

That is, the sum of utilities given by equation (4) is proportional to the aggregate equilibrium effort.

To guarantee that (pure Nash) equilibrium efforts exist (i.e., to ensure that no effort would be infinite) on network G, we assume that $\delta\mu(G) < 1$, where $\mu(G)$ represents the largest eigenvalue of matrix G. Fixing the number of links, we will have to compare different network configurations on which Nash equilibria exist. Define $\mu_{max}(L) = \max_{G\in\mathcal{G}(L)} \mu(G)$, i.e. the maximal index among all networks with L links. We will therefore restrict attention to $\delta < \delta_{max}(L) = \frac{1}{\mu_{max}(L)}$. We will see thereafter that $\mu_{max}(L)$ is known and attained for a unique network architecture.
A (pure) Nash equilibrium X^* satisfies the first order conditions represented in system 1, i.e. $(I - \delta G)X^* = \mathbf{1}$. As $\delta\mu(G) < 1$, the invert matrix $M = (I - \delta G)^{-1}$ is non-negative and this linear system admits a unique solution. Individual efforts are thus written as Bonacich centralities, i.e. $x_i^* = b_i(G, \delta)$ for all i, where profile $B(G, \delta) = \sum_{k=0}^{+\infty} \delta^k G^k \mathbf{1}$. We shall instead refer to b_i and to B when there is no confusion.

To finish, we present the class of nested split graphs, which will play a prominent role in our analysis.

Definition 1 (Nested split graph). *A graph $G = (N, L)$ is called a* **nested split graph** *if [$ij \in g$ and $deg(k, G) \geq deg(j, G)$] $\Longrightarrow ik \in g$.*
In words, this architecture can be ordered by classes of degrees. If there are $2k$ classes, the set of agents belonging to the union of the first k classes form a complete subgraph (each pair of agents is linked to each other); then the rest of links is given as follows. All agents in class $k + 1$ form links with all agents in classes 1 to k, agents in class $k + 2$ form links with agents in classes 1 to $k - 1$, etc, and in final agents in class $2k$ form links with agents in class 1. If there are $2k + 1$ classes, this is the same logic, except that the complete subgraph is formed by agents in classes 1 to $k + 1$. Note that a nested split graph has diameter two.
In the class of nested split graphs, some specific network architectures deserve special attention. Let K_p denote a complete subgraph with p agents.

Definition 2 (Quasi-complete graph). *A graph $G = (N, L)$ is called a* **Quasi-complete graph** *$QC(N, L)$ if it is built by forming the largest possible complete subgraph K_p (thus $\frac{p(p-1)}{2} \leq L < \frac{p(p+1)}{2}$), and by forming $L - p$ edges involving agents in K_p and a unique additional agent* $(1, p + 1), (2, p + 1, \cdots, (L - p, p + 1)$.
Then, a Quasi-complete graph contains a unique non trivial component, plus in general some isolated agents. Note that the non trivial component contains either symmetric agents (when it is a complete component), or three classes of agents. Figure 1 presents some Quasi-complete graphs.

Definition 3 (Quasi-star graph). *A graph $G = (N, L)$ is called a* **Quasi-star graph** *$QS(N, L)$ if it is built by forming all possible links with a same agent, then adding all possible residual links to another same agent, then to a third same agent, etc.*

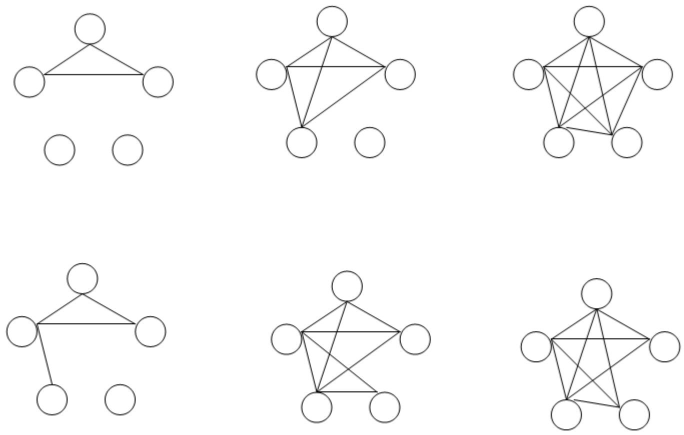

Figure 1: Some Quasi-complete graphs ($N = 5$)

Then, a Quasi-star graph contains a unique non trivial component. The non trivial component contains up to four classes of agents. Figure 2 presents some Quasi-star graphs. Note that $QS(N, L)$ is the graph complement of $QC(N, \frac{N(N-1}{2} - L)$.

3 Limit cases

In our setting, the intensity of synergies is bounded below by 0, i.e. we only explore complementarities, and it is also bounded above by $\delta_{max}(l)$ to guarantee the existence of an equilibrium in all compared networks. In this section, we fully address the issue for both high interaction and low interaction.

High interaction

The following proposition solves the problem for high interaction, from the set of all networks (connected or not).

Proposition 1 *When δ tends to $\delta_{max}(l)$, the network in $\mathcal{G}(l)$ which maximizes both aggregate effort and the sum of utilities given by (5) and (4) is the Quasi-Complete network.*

This result has a strong policy implication. If a social planner wanted to design the network appropriately, many individuals would be excluded from the principal component. Indeed, the main component

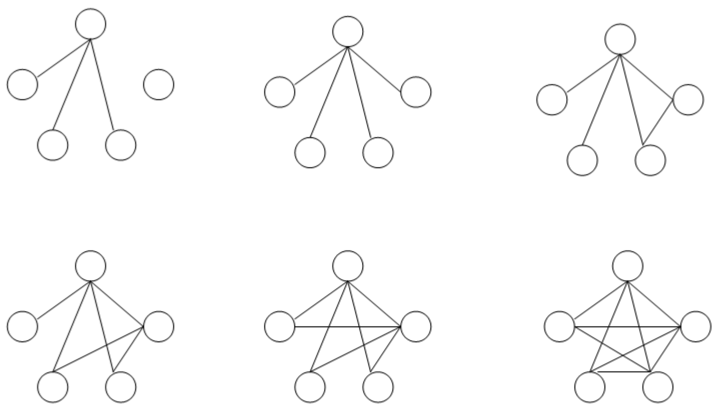

Figure 2: Some Quasi-star graphs ($N = 5$)

has one or three classes of agents, which generates some inequalities. But the main inequalities in this society arise from the presence of isolated agents.

Low interaction

Under sufficiently low decay, we show that the problem turns out to search for the networks generating the largest number of paths of length 2. In particular, we show that the sum of paths of length three or more is negligible with respect to the sum of paths of length two. This observation greatly simplifies the analysis. Indeed, the number of paths of length 2 is actually equal to the sum of squares of degrees. Finding the networks generating the maximal sum of squares of degrees, for a fixed number of edges, is a solved problem in mathematics. The exhaustive solution is given in Abrego et al. (2009). We find:

Proposition 2 *When δ tends to 0, the network maximizing both aggregate effort and the sum of utilities given by (5) and (4) is one of six structures. In particular, for all L, the targeted set contains either the Quasi-star network, or the Quasi-complete network, and the set is, for most of the cases, restricted to the Quasi-star network and/or the Quasi-complete network. The Quasi-star network dominates if L is small, while the Quasi-complete network dominates if L is big.*

In contrast with the case of high interaction, the policy implication of proposition 2 depend on the density of links. For low link density, the Quasi-star should be designed. This architecture mainly generates inequalities inside the principal component (except $L < n-1$, there are no isolated agents). Alternatively, for high level of interaction, the Quasi-complete network should be designed, which, as said earlier, originates inequalities between members of the principal component and isolated agents.

4 General case

The preceding section has shown that, in the limit cases of high and low intensity of interaction, two polar network architectures should be selected to maximize aggregate effort and utilities. We address now the general case, i.e. we consider any intensity of interaction compatible with existence of equilibrium.

As a first step, we examine if targeted networks have more than one non-trivial component (a component is said non-trivial if it contains at least two agents). The following lemma states a remarkably simple result:

Lemma 1 *For all $\delta \in [0, \delta_{max}(L)]$, a network maximizing both aggregate effort and the sum of utilities given by (5) and (4) has at most one non-trivial component.*

Lemma 1 implies, for all intensity of interaction, that inequalities cannot arise in-between different components. Put differently, inequalities result exclusively from differences inside the non-trivial component, or from differences between members of the component and isolated agents.

Which architecture should have the non-trivial component? We pursue the analysis further, by identifying a strong restriction in the architecture of the non-trivial component.

4.1 Maximising aggregate efforts: single-link switch

From the above analysis, we know that an optimal network has a unique non-trivial component. We would like to precise further some structural features of this non-trivial component:

Lemma 2 *Suppose that $\delta \in [0, \delta_{max}(L)]$. Suppose that $ij \in G, ik \notin G$, and $B_j(G,\delta) \leq B_k(G,\delta)$. Then, both aggregate effort and the sum of utilities given by (4) are increased in the network $G - ij + ik$.*

Lemma 2 gives a simple condition under which a one-link switch enhances aggregate effort. The relevant switch involves only three players, and is such that a given agent switches a current connection toward an agent with higher effort. Figure 3 illustrates lemma 2.
This lemma precipitates the structure of desirable networks:

Theorem 1 *For all $\delta \in [0, \delta_{max}(L)]$, a network maximizing both aggregate effort and the sum of utilities given by (4) is a nested split graph.*

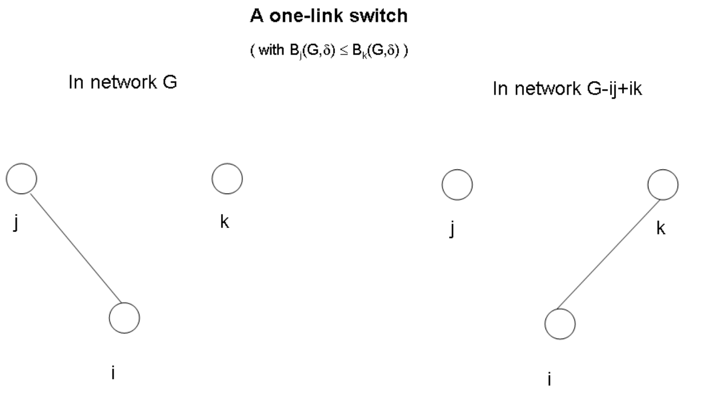

Figure 3: Graphical illustration of lemma 2

Remark. The optimal networks for the polar cases given in the preceding section are nested-split graphs.

A conjecture. At this moment, the study lets open an important issue. We conjecture with the help of simulations that, among nested split graphs, the networks maximizing the sum of efforts is either a Quasi-star, or a Quasi-complete network. To give an insight of the difficulty, figure 4 presents a nested split graphs with five classes, thus distinct from the two latter, which maximizes effort locally, in the sense that any one-link deviation reduces (strictly) aggregate effort. Thus, the problem is intrinsically more difficult than for instance the problem of finding the maximal index among networks with a fixed number of links, for which from any network, it is possible to build an improving path with one-link deviation step by step (see Cvetković et al. [2008]).

4.2 Maximising linear-quadratic utilities: multiple-link switch

By lemma 2 we know that, from any initial network, any single-link three-player switch toward an agent with higher effort enhances aggregate effort. This property inevitably forbids from optimality any network which is not a nested split graph. However, lemma 2 does not hold regarding the sum of linear-quadratic utilities given by (5), as illustrated by the following example. Consider $N = 25, L = 59, \delta = .03$, and suppose that there are two separate components, a complete component K with 10 agent, and a star component S with 14 agents. A last agent i has to allocate one link. Should he choose to form a link with one agent, say j, in the complete component, or with the center, say k, of the star? Let $G = K + S + ij$, that is, i is connected to the complete component. Let $G' = G_ij + ik$.

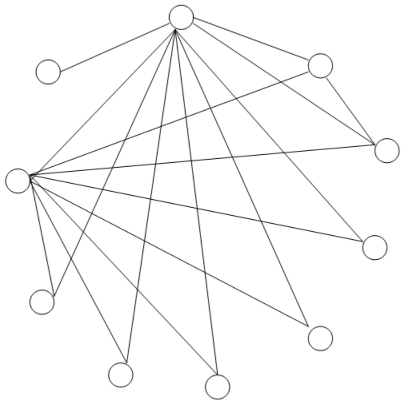

Figure 4: A nested split graph which maximizes aggregate effort locally

At this value of δ, $B_k(G,\delta) - B_j(G,\delta) = .005$. Thus, aggregate effort increases when passing from G to G'. However, the sum of linear-quadratic utilities is decreased when switching the link ij to ik (from 36.0677 to 36.0675). The point is that agent j's partners exert a much higher is higher effort than that of agent k. Even if the cumulative loss of agent j's partners is weaker than that of agent k's partners, the distribution of efforts is more concentrated around j than k. This distributional consideration, which matter as equilibrium utilities are given by the square of effort, prevails and explains the result.

To address aggregate linear-quadratic utilities, we need a more sophisticated method. The next lemma provides a profitable switch in that context. For that purpose, we need to define, for all j,k, the set $N_{j\setminus k} = \{p/ij \in G, ik \notin G\}$ of neighbors of j who are not neighbors of k.

Lemma 3 *Consider j,k such that $B_j(G,\delta) \leq B_k(G,\delta)$. Then, the sum of utilities given by (5) is increased in the network $G' = G + \sum_{l \in N_{j\setminus k}}(lk - lj)$.*

Lemma 3 states that, considering all neighbors of agent j who are not neighbors of k, if we switch their connection from j to k simultaneously, then aggregate utilities increases. Figure 6 illustrates lemma 3.

The collective reallocation induced by lemma 3 also leads to nested split graphs:

Theorem 2 *For all $\delta \in [0, \delta_{max}(L)]$, a network maximizing both aggregate effort and the sum of utilities given by (5) is a nested split graph.*

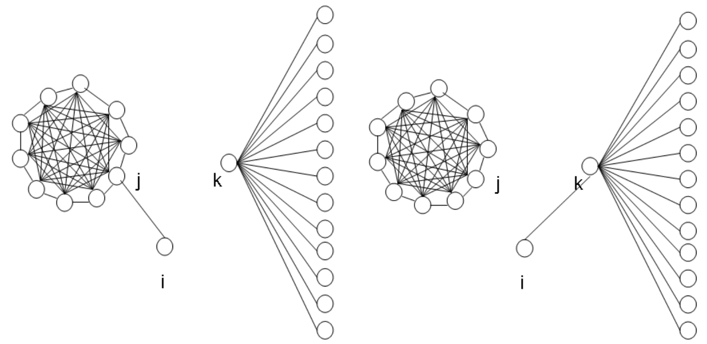

Figure 5: $N = 25$, $L = 59$; lemma 2 does not apply to the sum of linear-quadratic utilities

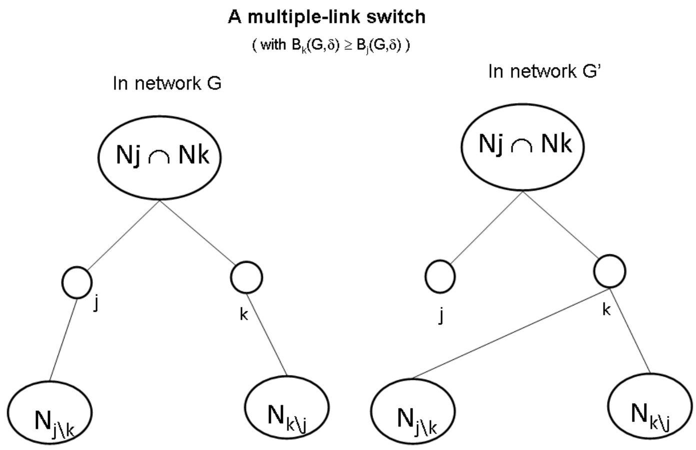

Figure 6: Graphical illustration of lemma 3

5 Concluding remarks

The present paper has considered a network game with linear interaction and local strategic complementarities. Using the theory of M-matrices, we have shown that nested split graphs play a key role is the design of networks with a fixed number of links.

At least two directions of research may deserve interest. First, our model fits with situations in which interaction is linear, while in many applications the level of complementarity dries up with link accumulation. It would be challenging to address this issue. Second, for some economic applications, it would be interesting to examine the dual problem of minimizing the sum of efforts, and intuition suggests that the pattern of targeted network architectures should be drastically different. Possible applications concern the field of crime economics (Ballester et al. [2006]), or some fields in finance where some strategic complementarities have been identified (attack on money, foreclosing a loan to a firm, bank run; see Vivès [2011] and references therein).

6 Proofs (Sketch for this preliminary version)

Proof of Proposition 1. Enhancing the intensity of interaction, an equilibrium exists until the level of interaction such that some effort goes to infinity. Efforts being proportional to Bonacich centralities, the problem reduces to find the network having maximal index, i.e. $G \in Argmax\{\mu(G); G \in \mathcal{G}(L)\}$. Indeed, among all netorks with L links, such a network has the lowest threshold value of δ at which some effort is infinite. Using Rowlinson (1988), the result is immediate: the Quasi-complete network is the unique one. ■

Proof of Proposition 2. We show that targeted networks $G \in ArgMaxP_2(G)$. Check indeed that the residual series is a little o of the order 2 term. Basically,

$$\sum_i B_i = n + \delta 2L + \delta^2 \sum_{i,j} G_{ij}^{[2]} + \sum_{k=3}^{+\infty} \delta^k \sum_{i,j} G_{ij}^{[k]} \tag{6}$$

Given that L is the same for all G, the point is to show that $\sum_{k=3}^{+\infty} \delta^k \sum_{i,j} G_{ij}^{[k]} = o(\delta^2 \sum_{i,j} G_{ij}^{[2]})$ when δ tends to 0; that is, denoting

$$R = \frac{\sum_{k=3}^{+\infty} \delta^k \sum_{i,j} G_{ij}^{[k]}}{\delta^2 \sum_{i,j} G_{ij}^{[2]}} \tag{7}$$

we have to show that

$$\lim_{\delta \to 0} R = 0 \tag{8}$$

Now, the numerator of R is bounded above by the series in the complete network, the denominator is bounded below by δ^2. The number of walks of length k is the complete network with n agents is $n(n-1)^k$. Thus,

$$R \leq R' \tag{9}$$

with

$$R' = \frac{\delta^3 n(n-1)^3 [\sum_{j=0}^{+\infty} (\delta(n-1))^j]}{\delta^2} \tag{10}$$

That is, given that $\sum_{j=0}^{+\infty}(\delta(n-1))^j = \frac{1}{1-\delta(n-1)}$,

$$R' = \frac{\delta n(n-1)^3}{1-\delta(n-1)} \tag{11}$$

Then,

$$\lim_{\delta \to 0} R' = 0 \tag{12}$$

and we are done.

Now, a direct calculation shows that the sum of paths of length two is equal to the sum of squares of degrees, i.e. $G \in ArgMaxP_2(G) \equiv ArgMax \sum_i d_i^2$. We thus refer to Abrego et al (JIPAM 2009), who solve the problem of finding the network, with fixed number of links, which maximizes the sum of squares of degrees. ■

Proof of Lemma 1.
Maximizing aggregate effort. We will show that, by joining adequately two separate components, we have more paths of any length, thus the junction will increase aggregate effort.
Precisely, suppose that an optimal network g contains two non degenerated components C_1, C_2. Consider any pair of agents i, j such that $i \in C_1$, $j \in C_2$. We build network $\tilde{g}$ as follows. We start from the network g. Then for every neighbour of agent j in network g, we delete the link jk and we set up the connection ik. All other links are unchanged. The network $\tilde{g}$ is more optimal than the network g. Indeed, since the structure of the two components C_1, C_2 are unchanged, all paths in g are also in $\tilde{g}$. Further, $\tilde{g}$ possesses other links, those passing through agent i and making the bridge between the two component C_1, C_2 of g.

Maximizing the sum of utilities. Everyone is better of except j. But i wins more than what j looses, so the sum of utilities increases. □

Proof of Theorem 1. We will show that if $B_3 \geq B_2$ and $12 \in g$ but $13 \notin g$, then the aggregate effort in $G + 13 - 12$ is larger than in G. Second, we observe that this condition implies the class of nested-split graphs.

PART I: *we show that if $B_3 \geq B_2$ and $12 \in g$ but $13 \notin g$, then the aggregate effort in $G + 13 - 12$ is larger than in G.*

The Bonacich vector in networks G and G' satisfies respectively $(I - \delta G)B = \mathbf{1}$ and $(I - \delta G')B' = \mathbf{1}$. Writing $A = G' - G$ and $M = (I - \delta G)^{-1}$ we get:

$$B' - B = \delta MAB' \tag{13}$$

and

$$B' = (I - \delta MA)^{-1} B \tag{14}$$

Considering

$$A = \begin{pmatrix} 0 & -1 & +1 & 0 & \dots & 0 \\ -1 & 0 & 0 & 0 & \dots & 0 \\ +1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

we get

$$B' - B = \delta MAB' = \delta \begin{pmatrix} \vdots \\ m_{i1}(B'_3 - B'_2) + B'1(m_{i3} - m_{i2}) \\ \vdots \end{pmatrix}$$

and

$$\sum_i B'_i - \sum_i B_i = \delta[B_1(B'_3 - B'_2) + B'1(B_3 - B_2)]$$

We get

$$B' - B = \delta MAB' = \delta \begin{pmatrix} \vdots \\ m_{i1}(B'_3 - B'_2) + B'1(m_{i3} - m_{i2}) \\ \vdots \end{pmatrix}$$

and

$$\sum_i B'_i - \sum_i B_i = \delta[B_1 \underbrace{(B'_3 - B'_2)}_{?} + B'1 \underbrace{(B_3 - B_2)}_{\geq 0}]$$

Given that $B_3 - B_2 \geq 0$, it is sufficient to show that $B'_3 - B'_2 \geq 0$. But, $B' = (I - \delta MA)^{-1}B$ gives

$$B'_3 - B'_2 = \frac{B_3 - B_2 + \delta B'_1 \overbrace{(m_{33} + m_{22} - 2m_{23})}^{?}}{\underbrace{1 - \delta(m_{13} - m_{12})}_{?}}$$

We show the positivity of both terms using three points:

STEP 1. We bound below both terms by functions of principal minors of $M = (I - \delta G)^{-1}$:

- $m_{33} + m_{22} - 2m_{23} > 0$ ***whenever*** $m_{22}m_{33} - m_{23}^2 > 0$. Indeed,

⋄ $m_{33} + m_{22} > 2m_{23}$ meaning $(m_{33} + m_{22})^2 - 4m_{23}^2 > 0$, that is $m_{33}^2 + m_{22}^2 + 2m_{33}m_{22} - 4m_{23}^2 > 0$

⋄ As $m_{22}m_{33} > m_{23}^2$, we find $4m_{22}m_{33} > 4m_{23}^2$ and $2m_{33}m_{22} - 4m_{23}^2 > -2m_{33}m_{22}$.

⋄ Thus $m_{33}^2 + m_{22}^2 + 2m_{33}m_{22} - 4m_{23}^2 > (m_{33} - m_{22})^2 > 0$.

- $1 - \delta(m_{13} - m_{12}) > \frac{1}{2}\delta^2[m_{11}m_{22} - m_{12}^2 + m_{11}m_{33} - m_{13}^2]$. Indeed,

⋄ $(I - \delta MA) = (I - \delta G)^{-1}(I - \delta G')$

$\Rightarrow$ $Det(I - \delta MA) > 0$ as product of determinants of one M-matrix and one inverse M-matrix (which is also positive)

◇ Take $G" = G + 13$; writing $Det(I - \delta MA")$ we obtain $(1 - \delta m_{13})^2 - \delta^2 m_{11} m_{33} > 0$
◇ Take $G'' = G - 12$; we find $(1 + \delta m_{12})^2 - \delta^2 m_{11} m_{22} > 0$
◇ To replace link 12 with 13, we add 13 and substract 12; from the two inequalities we deduce: $1 - \delta(m_{13} - m_{12}) > \frac{1}{2}\delta^2[m_{11}m_{22} - m_{12}^2 + m_{11}m_{33} - m_{13}^2]$

STEP 2. The sign of principal minors of M and $(I - \delta G)$ are the same. Indeed, the Jacobi Identity gives the sign of the principal minors of $(I - \delta G)^{-1} = M$

$$\det\ A^{-1}[\beta, \alpha] = (-1)^{s(\alpha)+s(\beta)}\ \frac{\det\ A[\alpha', \beta']}{\det\ A}$$

STEP 3. $I - \delta G$ is an M-Matrix. This implies that it is a P-Matrix, whose principal minors are all positive.
It follows that $B'_3 - B'_2 > 0$, and thus $\sum B'_i - \sum B_i > 0$.

PART II: *we check that this condition implies the class of nested-split graphs.* (trivial) ■

Proof of lemma 3. We show that, following the switch, we can find a sequence of myopic best-replies which leads to an equilibrium with higher aggregate utilities. In particular, we show that all utilities increase except possibly agent j.

. Case 1: the link $jk \notin G$. Then, a simultaneous best-reply algorithm (SBRA) applied to all agents on the network is sufficient. Indeed, we fix the initial effort profile at $X^0 = X(G)$, and start a simple SBRA on the network $G' = G + \sum_{l \in N_{j \setminus k}} (lk - lj)$. The SBRA $\{X^t\}_{t \geq 0}$ is such that for all $i \in N$, all $t \geq 0$, $x_i^{t+1} = 1 + \delta \sum_{j/ij \in G' g'_{ij}} x_j^t$.
The initial configuration X^0 (defined on network G) satisfies:

$$\begin{cases} x_j^0 = 1 + \delta \sum_{c \in N_{j \cap k}(G)} x_c^0 + \delta \sum_{l \in N_{j \setminus k}(G)} x_l^0 + \delta x_k^0 \\ x_k^0 = 1 + \delta \sum_{c \in N_{j \cap k}(G)} x_c^0 + \sum_{p \in N_{k \setminus j}(G)} x_p^0 + \delta x_j^0 \end{cases} \tag{15}$$

At step 1 of the process, we obtain:

$$\begin{cases} x_j^1 = x_j^0 - \delta \sum_{l \in N_{j \setminus k}(G)} x_l^0 \\ x_k^1 = x_k^0 + \delta \sum_{l \in N_{j \setminus k}(G)} x_l^0 \\ x_l^1 = x_l^0 + \delta(x_k^0 - x_j^0) \text{ for all } l \in N_{j \setminus k}(G) \\ x_q^1 = x_q^0 \text{ for all } q \neq j, k, l \end{cases} \tag{16}$$

Thus, since $x_k^0 \geq x_j^0$ by hypothesis, all efforts, except that of agent j, increase (weakly) at step 1. More, utilities are quadratic in effort, thus the increase in utility of agent k is larger than the loss of agent j by convexity of utility (given that $x_k^0 \geq x_j^0$). It follows that the sum of utilities at X^1 exceeds the sum of utilities at X^0.

At step 2, we get:

$$\begin{cases} x_j^2 = x_j^1 \\ x_k^2 = x_k^1 + \delta \sum\limits_{l \in N_{j \setminus k}(G)} (x_l^1 - x_l^0) \\ x_l^2 = x_l^1 + \delta(x_k^1 - x_k^0) \\ x_q^2 \geq x_q^1 \text{ for all } q \neq j,k,l \end{cases} \tag{17}$$

Noting that $x_k^2 > x_k^1$, we obtain in total that $X^2 \geq X^1$. Now, because of complementarities, the SBRA still increases at each step, and thus all utilities at convergent state X^∞ are increased compared to utilities in X^1 (convergence is guaranteed by contraction property). As the sum of utilities at X^1 exceeds the sum of utilities at X^0, we conclude that the sum of utilities at X^∞ exceeds the sum of utilities at X^0.

. Case 2: the link $jk \in G$. As in case 1, we present a sequence of best-reply revisions guaranteeing that all efforts increase at the end of the process, except possibly agent j's effort, and that the eventual loss in agent j's effort is over-compensated by the increase of agent k's effort. The process is decomposed into two sequential SBRA on the network G'. In a first step, we apply on G' a SBRA restricted to both agent j and k (that is, keeping all other efforts fixed throughout the process) and with $X^0 = X(G)$ as initial condition. This process converges to a configuration X^1, where agent j's and agent k's efforts, only, have new values. In a second step, we apply, on G' again, a SBRA over all agents, with X^1 as initial condition. This process converges toward a configuration X^2, where all efforts on the network are modified.

The first SBRA, restricted to agents j and k, leads to the following result:

$$\begin{cases} x_j^1 + x_k^1 = x_j^0 + x_k^0 \\ x_k^1 > x_k^0 \end{cases} \tag{18}$$

To see these points, we note that the configuration X^1 (defined on network G'), which represents the profile toward which the first SBRA converges to, is such that:

$$\begin{cases} x_j^1 = 1 + \delta \sum\limits_{c \in N_{j \cap k}(G)} x_c^0 + \delta x_k^1 \\ x_k^1 = 1 + \delta \sum\limits_{c \in N_{j \cap k}(G)} x_c^0 + \delta \sum\limits_{l \in N_{j \setminus k}(G)} x_l^0 + \delta \sum\limits_{p \in N_{k \setminus j}(G)} x_p^0 + \delta x_j^1 \end{cases} \tag{19}$$

Reminding that the initial configuration satisfies the equations (15), we find:

$$x_j^0 + x_k^0 = x_j^1 + x_k^1 = \frac{2\left[1 + \delta \sum\limits_{c \in N_{j \cap k}(G)} x_c^0\right] + \delta \sum\limits_{p \in N_{k \setminus j}(G)} x_p^0 + \delta \sum\limits_{l \in N_{j \setminus k}(G)} x_l^0}{1 - \delta} \tag{20}$$

Moreover, $x_k^1 - x_k^0 = \delta \sum\limits_{l \in N_{j \setminus k}(G)} x_l^0 + \delta(x_j^1 - x_j^0)$. If $x_j^1 \geq x_j^0$, then $x_k^1 > x_k^0$, which contradicts that $x_j^0 + x_k^0 = x_j^1 + x_k^1$. Therefore, we have $x_j^1 < x_j^0$ (and $x_k^1 > x_k^0$, with $x_k^1 - x_k^0 = x_j^0 - x_j^1$).

We turn to the second SBRA on network G'. This process is applied to all agents, and we set X^1 as initial condition.

Actually, this process is such that $X^2 \geq X^1$. Indeed, after the first revision, the neighbors who have switched (those with label l in our convention) increase their effort as $x_k^1 > x_j^0$. Also, common neighbors to j and k (those with label c in our convention) do not change their effort level as $x_k^1 - x_k^0 = x_j^0 - x_j^1$. Further, agents j and k, as well as all other agents on the network, do not modify their effort level. Hence, at the end of this first revision, the new effort profile is dominates X^1. By complementarity, and given this initial increase, any successive step therefore dominates the preceding one, which guarantees that $X^2 \geq X^1$. □

Proof of Theorem 2. This is a direct consequence of lemma 3. ■

References

[1] Abrego, B., S. Fernandez-Merchant, M. Neubauer, and W. Watkins, 2009, Sum of Squares of Degrees in a Graph, Journal of Inequalities in Pure and Applied Mathematics, vol. 10(3), 1-34.

[2] Bala, V. and S. Goyal, 2000, A noncooperative model of network formation, Econometrica, vol. 68(5), 11811230.

[3] Ballester, C. and A. Calvó-Armengol, 2010, Interactions with hidden complementarities, Regional Science and Urban Economics, vol. 40(6), 397-406.

[4] Ballester, C., A. Calvó-Armengol and Y. Zénou, 2006, Who's who in networks. Wanted: the key player, Econometrica, vol. 74(5), 1403-1417.

[5] Belhaj, M. and F. Deroïan, 2010, Endogenous effort in communication networks under strategic complementarity, 2010, International Journal of Game Theory, vol. 39, 391-408.

[6] Belhaj, M. and F. Deroïan, 2010, Asymmetric Interaction and Aggregate Incentive, GREQAM Working Paper.

[7] Belhaj, M. and F. Deroïan, 2010, Competing activities in social networks, GREQAM Working Paper.

[8] Belhaj, M. and F. Deroïan, 2011, Risk taking under heterogenous revenue sharing, forthcoming in the Journal of Development Economics.

[9] Bolton, P. and M. Dewatripont, 1994, The Firm as a Communication Network, Quarterly Journal of Economics, vol. 109(4), 809-839.

[10] Bramoullé, Y. and R. Kranton, 2007, Public goods in networks, Journal of Economic Theory, vol. 135, 478-494.

[11] Bramoullé, Y., R. Kranton and M. D'Amours, 2011, Strategic interaction and networks, mimeo.

[12] Cabrales, A., A. Calvó-Armengol, and Y. Zénou, 2011, Social interactions and spillovers, Games and Economic Behavior, vol. 72, 339360.

[13] Calvó-Armengol, A., Zénou, Y., 2004, Social networks and crime decisions: the role of social structure in facilitating delinquent behavior, International Economic Review, vol. 45(3), 939958.

[14] Corbo, J., A. Calvo and D. Parkes, 2006, A study of Nash equilibrium in contribution games for peer-to-peer networks, in SIGOPS Oper. Syst. Rev., vol. 40(3), 61-66.

[15] Cvetković, D., P. Rowlinson and S. Simić, 2008, Eigenspaces of Graphs, Cambridge University Press.

[16] Galeotti, A. and S. Goyal, 2010, The law of the few, American Economic Review, vol. 100(4), 14681492.

[17] Goyal, S. and J. Moraga-González, 2001, R&D networks, Rand Journal of Economics, vol. 32, 686-707.

[18] Hendricks, K., M. Piccione and G. Tan, 1995, The Economics of Hubs: The Case of Monopoly, Review of Economic Studies, vol. 62(1), 83-99.

[19] Jackson, M. and A. Wolinsky, 1996, A strategic model of social and economic networks, Journal of Economic Theory, vol. 71(1), 4474.

[20] König, M., C. Tessone and Y. Zénou, 2009, A dynamic model of network formation with strategic interactions, CCSS Working Paper Series CCSS-09-00006.

[21] Mahadev, N. and U. Peled, 1995, Threshold Graphs and Related Topics, North Holland.

[22] Radner, R., 1993, The Organization of Decentralized Information Processing, Econometrica, vol. 61 (5), 1109-1146.

[23] Rowlinson, P., 1988, On the maximal index of graphs with a prescribed number of edges, Linear Algebra and its Applications, vol. 110, 43-53.

[24] Vivès, X., 2005, Complementarities and Games: New Developments, Journal of Economic Literature, vol. 43(2), 437-379.

[25] Vivès, X., 2011, Strategic complementarity, fragility, and regulation, IESE Working Paper 928, CESifo Working Paper 3507, CEPR Discussion Paper 6743.