Equalization of Opportunity: Definitions and Implementable Conditions.

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Abstract

Equality of opportunity defines the relevant egalitarian objective for those policies oriented toward alleviating or eliminating the effect of morally irrelevant circumstances on the distribution of a broad range of social and economic outcomes, such as education, health, wealth and, eventually, income, among individuals exerting similar effort choices. We also consider a third determinant of outcomes, luck, that gathers the morally-irrelevant random factors of inequality that do not call for explicit compensation. We propose a novel test for evaluating heterogeneity in policy impacts through the lenses of opportunity equalization. The test is validated, and opportunities equalized, whenever policy intervention fosters a reduction in the degree of consensus, among individuals exerting similar effort but heterogeneous in preferences for risky prospects, in assessing whether a circumstance provides an unambiguous advantage with respect to another. Since we consider luck, the definitions and the identification results, given data limitations, resort to inverse stochastic dominance tools.

Keywords: Equality of opportunity, policy intervention, inverse stochastic dominance, economic distance, income distribution.

JEL Codes: D63, J62, C14.

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1 Introduction

Equality of opportunity has gained popularity, in scholarly debates as well as among policymakers, for defining the relevant equalitarian objective for the distribution, among individuals, of a broad range of social and economic outcomes, such as health, wealth, income, etc. The ethical foundations of the equality of opportunity principle have been extensively discussed and are well-established (for a comprehensive discussion, see Dworkin 1981, Roemer 1998, Fleurbaey 2008). Consistent with the equality of opportunity principle, public policy has now often set as its main objective to level the playing field among citizens and to provide equality of opportunity in a variety of areas of intervention such as education, health and, eventually, income. Assessing whether policy intervention indeed succeeds at equalizing opportunities is obviously a key issue for policy evaluation.

Addressing this issue requires to draw on explicit evaluation criteria that are, to a large extent, absent from the existing literature. More specifically, while equality of opportunity has now been clearly defined in the recent literature, criteria allowing to assess the (partial) equalization of opportunity, understood as a reduction in the extent of inequality of opportunity, are so far absent from the literature.

The objective of this paper is to formally define a criterion of opportunity equalization, that would be both consistent with theoretical views of equality of opportunity and empirically implementable, to allow for the evaluation of the effect of public policy intervention.

The recent philosophical and economic literature has offered a clear characterization of the requisite of equality of opportunity. The equality of opportunity perspective amounts to draw a distinction between fair and unfair inequality of individual outcomes. This requires to take into consideration the determinants of observed outcomes and leads to single out two polar sets of determinants: on the one hand, effort gathers the legitimate source of inequality among individuals; on the other hand, circumstances corresponds to the set of morally-irrelevant factors of individual inequality that call for compensation. Define a type the set of individuals with similar circumstances. In general terms, equality of opportunity will be said to prevail, if, given effort, no set of circumstances yields an advantage over the others. This reflects what is usually referred in the literature as the compensation principle.

Provided one explicitly defines the relevant notion of notion of “advantage”, this general principle allows to assess in various empirical contexts whether equality of opportunity is satisfied or not. However, this leads to a binary criterion (equality of opportunity is satisfied or not) and it does not allow to rank, from the point of view of equality of opportunity, different situations where equality of opportunity is not satisfied. Assessing the equalizing impact of policy intervention obviously calls for such a ranking, especially when policies do not allow to reach full equality of opportunity.

The perspective of Lefranc, Pistolesi & Trannoy (2009) (henceforth denoted LPT) breaks down the dichotomy between equality and inequality of opportunity by distinguishing between a strong and a weak form of equality of opportunity. As discussed in LPT,
not all determinants of outcome fall under the two categories of effort and circumstances: there exists a third class of determinants, denoted *luck*, that gathers the morally-irrelevant factors of inequality that do not call for explicit compensation. In this context, given their level of effort and their type, individuals face a distribution of possible outcomes. Comparing the outcomes prospects offered to individuals with similar effort, when their circumstances vary, amounts to compare lotteries of outcomes.

There might be different ways of formalizing the requirement that no type is advantaged over the others. A first possibility is to require that the outcome distributions be identical across individuals with similar effort, i.e. independent of circumstances. This is the strong form of equality of opportunity analyzed in LPT. A weaker condition is to require that it is not possible to unanimously rank the outcome distributions attached to different circumstances within the class of risk-averse Von Neumann - Morgenstern (VNM) preferences under risk. This corresponds to the weak form of equality of opportunity considered in LPT. Overall, this three-tier taxonomy allows for a richer, and least partial, ranking of social states, that could be used for policy evaluation. However, the model of LPT would not allow to rank situations where the weak form of equality of opportunity is satisfied (or violated) both before and after the policy intervention.

Our analysis relies on the model of LPT and develops a definition of opportunity equalization that combines two distinct criteria.

The first criterion is an ordinal criterion that elaborates on the notion of weak equality of opportunity defined in LPT. Instead of restricting attention to the class of risk-averse preferences under risk, we consider a more general approach and assume that individual preferences over lotteries belong to a general class of preferences. For instance, we might consider the class of rank-dependent utility functions or the VNM representation. We further assume that this general class can be partitioned into nested sub-classes, according to the series of restrictions imposed on preferences. When the outcome distributions are not independent of circumstances, it might be possible to find a sub-class of individual preferences within which all preferences consistently rank the type-specific outcome distributions. This should be seen as a case of inequality of opportunity as all individuals with preference in that sub-class consistently agree on a ranking of circumstances. Our first criterion for opportunity equalization is that the class of preferences within which it is possible to unanimously rank circumstances should shrink as a result of the policy implementation. In a sense, this amounts to request that the degree of consensus on the ranking of types should fall after the implementation of the policy.

Beyond this ordinal criterion, we impose a second criterion for opportunity equalization. This second criterion is a distance criterion. It essentially requires that the cardinal advantage conferred to the “privileged” types should fall, according to all preferences in the subclass for which it is possible to unanimously rank circumstances (and inducing a form of inequality of opportunity). This criterion is, in its essence, a cardinal criterion, although the final assessment over distance reduction is more general and robust, since it holds for a general subclass of preferences.
As a result, when both criteria are satisfied, the implementation of the policy results (i) in a decrease in the degree of unanimity for the ranking of circumstances, in terms of the advantage they confer and (ii) a fall in the size of advantage of the privileged types according to all preferences in the class within which types can be ranked.

In the rest of the paper, we formalize these two criteria and try to translate them into empirically implementable conditions. Although the two criteria are not attached to a particular class of utility representation, we develop our analysis within the framework of the Yaari (1987) rank-dependent utility representation. Note that here the rank-dependent utility model is assumed to characterize the individual preferences: contrary to Aaberge (2009) and Zoli (2002) we do not assume a specific social welfare function.

Within the context of the Yaari model for representation of preferences under risk, we first show that the ordinal equalization criterion can be formalized in terms the order of inverse-stochastic dominance at which it is possible to rank the distributions of outcomes conditional on type, as in Aaberge, Havnes & Mogstad (2012). Second, we show that the economic distance criterion amounts to require that the opportunity gap between any pair of types should fall, in the sense of stochastic dominance, as a result of the policy implementation.

Our analysis is connected to several papers that have recently examined changes over time in inequality of opportunity or differences therein across various national or policy contexts. Ferreira & Gignoux (2011) (for education), Checchi & Peragine (2010) (for income) and Peragine et al. (2011) (for growth) offer some recent examples. In most cases, however, these papers rely on specific cardinal (and often ad hoc) indices of inequality of opportunity. Our combined criterion of opportunity equalization, in contrast, offers more general conditions. Van de Gaer et al. (2011) offer the only example that we are aware of, of a policy evaluation based on the equality of opportunity principle. However, their analysis is more focused on the assessment of opportunity improvement (i.e. to what extent does the opportunity set offered to any type improves as a result of the policy) rather than on the analysis of opportunity equalization (i.e. to what extent does the opportunity gap between types fall as a result of the policy).

The rest of the paper is organized as follows. We describe in section 2 the notation and the tools that we exploit to define weak and strong forms of equality and inequality of opportunity. In section 3 we combine the definitions of equality and inequality of opportunity in LPT to construct an equalization test. We show the limitations of the test by resorting to a simple framework with only two circumstances and one effort level. Within this framework, we formalize the dominance (section 4) and the distance (section 5) criteria for opportunity equalization, which we combine together in section 6 to obtain the equalization test. Then, the test is generalized to the multiple effort, multiple circumstances case. We also provide a definition of the test for the general case. Implementation issues and identification of equalization of opportunity, when the relevant determinants of outcome are only partially observable, are discussed in section 7. Finally, section 8 concludes.
2 Equality of opportunity: notations and definitions

2.1 Determinants of outcome

Our analysis builds upon the framework developed in Roemer (1998) and Lefranc et al. (2009). Individual outcome, \( y \) is determined by four types of factors. Following the terminology of Roemer, circumstances, denoted by a vector \( c \), capture the factors that are not considered a legitimate source of inequality. Effort summarized by a scalar \( e \) includes the determinants of outcome that are seen as a legitimate source of inequality. Following LPT we also consider luck, captured by a scalar \( l \), that comprises the random factors that are seen as a legitimate source of inequality as long as they affect individual outcomes in a neutral way, given circumstances and effort. Lastly, we consider that individual outcome may be affected by a policy variable, denoted \( \pi \). In the rest of the paper, the policy variable is dichotomous and takes value in \( \{0, 1\} \), thus defining two possible states of the world. These two states of the world may define two policy regimes. More generally, they may correspond to two periods or two countries, that one would like to compare.

Following Roemer, define a type as the set of individuals with similar circumstances. Following LPT, define a variety as the set of individuals with similar circumstances and effort.

These four sets of factors provide a complete partition of the determinants of individual outcome. Consequently, one may write outcome as:

\[
\begin{align*}
y = Y(c, e, l, \pi)
\end{align*}
\]

where \( Y() \) denotes the outcome function.

Alternatively, the outcome of each individual may be seen as a draw from a lottery, whenever there is uncertainty or randomness in any of the determinants of individual outcomes. We let \( F() \) denote the cumulative distribution function of outcome, which is assumed to be left-continuous. In the rest of the paper, our analysis will largely involve the comparison of conditional distribution functions. In particular, we will focus on the distribution of outcome conditional on circumstances, effort and policy: \( F(y|c, e, \pi) \). Lastly, we define \( F^{-1}(p) \) the outcome quantile distribution associated with \( F \), for all quantile \( p \) in \([0, 1]\).

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1So far, circumstances, effort and luck have only be defined in a formal sense, i.e. by the way they should be taken into account in equality of opportunity judgements. What precise factors should count as circumstances, effort and luck is yet another question that calls in both ethical and political value judgements, as discussed for instance in Roemer (1998) and LPT. Here we take a neutral stance on the question of what factors should count as circumstances, effort or luck.

2If the cumulative distribution function is only left continuous, we define \( F^{-1}(p) \) by the left continuous inverse distribution of \( F \):

\[
F^{-1}(p|c, e, \pi) = \inf\{y \in \mathbb{R}_+ : F(y|c, e, \pi) \geq p\}, \quad \text{with } p \in [0, 1].
\]
2.2 Definitions of equality of opportunity

Let us now define the notions of equality of opportunity that will be used in the rest of our analysis. In general terms, equality of opportunity will be said to prevail, under a given policy regime, if, given effort, no set of circumstances yields an unambiguous advantage over the others. This reflects what is usually referred in the literature as the compensation principle (e.g. Fleurbaey 2008). As discussed in LPT, this suggests at least two notions of equality of opportunity, which we now review.

2.2.1 Strong equality of opportunity

The first conception of equality of opportunity corresponds to the situation where, given effort, the distribution of outcome does not depend on circumstances. This is a strong notion of equality that requires that individuals face similar distributions of outcome, regardless of their type, once their level of effort is known. This criterion can be formalized by the following definition, adapted from LPT.

**Definition 1 (EOP-S)** For a given policy $\pi$, Strong Equality of Opportunity (EOP-S) is satisfied iff:
\[
\forall (c,c') \forall e, F(.|c,e,\pi) = F(.|c',e,\pi).
\]

Of course, this definition can be straightforwardly reformulated using quantile functions by requiring that $F^{-1}(.|c,e,\pi) = F^{-1}(.|c',e,\pi)$ for all varieties.

2.2.2 Weak equality of opportunity

The fact that two types are facing different outcome distributions does not necessarily imply that one is advantaged over the other, in terms of outcome. Furthermore, if it is not possible to unambiguously rank circumstances according to the advantage they confer, it may be argued that a weak form of equality of opportunity prevails. LPT introduce the view that a type $c$ can only be said to confer an unambiguous advantage over another type $c'$ if everyone unanimously agree to prefer the lottery $F(.|c,e,\pi)$ to the lottery $F(.|c',e,\pi)$. Of course, the implementation of this criterion requires to specify the admissible set of individual preferences under risk.

If preferences satisfy the Expected Utility Theory representation and if agents are risk averse, this view of unambiguous advantage implies comparing the outcome distribution of all varieties using the criterion of second-order stochastic dominance. This leads to the following definition of a weaker form of equality of opportunity.

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Formally, the definition of first order stochastic dominance ($F(.|c,e,\pi) \succ SD_1 F(.|c',e,\pi)$) requires that $\forall y \in \mathbb{R}_+, F(y|c,e,\pi) \leq F(y|c',e,\pi)$ and $\exists y$ for which the inequality is strict. Similarly, second order stochastic dominance ($F(.|c,e,\pi) \succ SD_2 F(.|c',e,\pi)$) is satisfied iff $\forall y \in \mathbb{R}_+, \int_y^\infty F(t|c,e,\pi)dt \leq \int_y^\infty F(t|c',e,\pi)dt$ and $\exists y$ for which the inequality is strict. See below, appendix [A.1] for additional results related to inverse stochastic dominance.
**Definition 2 (EOP-W)** For a given policy \( \pi \), Weak Equality of Opportunity (EOP-W) is satisfied iff:

(a) \( \forall c \neq c' \forall e, \ F(\cdot | c, e, \pi) \not\succ_{SD}^{2} F(\cdot | c', e, \pi) \) where \( \succ_{SD}^{2} \) denotes stochastic dominance at the order 2.

and

(b) EOP-S is not satisfied

This definition differs slightly from the one in LPT. Since equality of all distribution function is a special form of non-dominance, we have imposed that EOP-S should not be satisfied in order to get two mutually exclusive notions of equality of opportunity.

Again, given the equivalence between second-order dominance and inverse second-order dominance, the above definition can be reformulated in terms of quantile functions by requesting the absence of inverse second-order dominance.[4]

### 2.2.3 Inequality of opportunity

Lastly, we can define the case of inequality of opportunity as the complement of weak and strong inequality of opportunity.

**Definition 3 (IOP)** For a given policy \( \pi \), Inequality of Opportunity (IOP) prevails iff:

(a) EOP-S is not satisfied

and

(b) EOP-W is not satisfied

It corresponds to the case where the outcome distribution for at least two varieties with similar effort can be ordered using the criterion of strict second-order dominance. Hence all risk-averse agents will unanimously agree that one of these two varieties is advantaged over the other.

By definition, the three situations EOP-S, EOP-W and IOP offer a complete partition of all possible allocations of outcomes.

### 3 Equalization of opportunity : a simplified setting

#### 3.1 Setting

Our objective is to evaluate the efficiency of a given policy from the point of view of equality of opportunity: we want to be able to assess whether implementing policy \( \pi = 1 \) improves equality of opportunity. We refer to such an improvement as an equalization of opportunities. This requires comparing the allocation of outcome across types under both \( \pi = 0 \) and \( \pi = 1 \). The rest of the paper is devoted to defining an equalization criterion.

Before discussing possible criteria, it is worth clarifying the type of comparisons involved in an assessment of changes in equality of opportunity. To assess equality of opportunity, in the static context of a given allocation of outcomes, requires evaluating the difference in the distributions offered to the different varieties. Moving to the dynamic context of assessing equalization of opportunities requires a difference-in-difference approach in order to examine changes in the gaps in the distributions offered to different varieties.

To simplify the framework of these comparisons, we start by considering a restricted setting with only two varieties, with a common effort level $e$ and two distinct circumstances $c$ and $c'$. Both varieties are observed under the two policy regimes $\pi = 0$ and $\pi = 1$. To simplify notations, we let $F_\pi$ (resp. $F'_\pi$) denote the distribution of outcome for variety $(c, e)$ (resp. $(c', e)$), under policy regime $\pi = 0, 1$. These distributions are identical to $F(\cdot | c, e, \pi)$ and $F(\cdot | c', e, \pi)$.

### 3.2 Possible configurations

A natural first-step to assess changes in equality of opportunity is to resort to the criteria of LPT and examine which of the three situations EOP-S, EOP-W and IOP prevails, under both $\pi = 0$ and $\pi = 1$.

Possible configurations are summarized in the following table.

<table>
<thead>
<tr>
<th>$\pi = 0$</th>
<th>EOP-S</th>
<th>EOP-W</th>
<th>IOP</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi = 1$</td>
<td>A</td>
<td>C</td>
<td>C</td>
</tr>
<tr>
<td>EOP-S</td>
<td>B</td>
<td>F</td>
<td>E</td>
</tr>
<tr>
<td>IOP</td>
<td>B</td>
<td>D</td>
<td>G</td>
</tr>
</tbody>
</table>

Cell A corresponds to the case where the strong form of equality of opportunity prevails both before and after the policy: in both cases, the outcome distributions are identical across types. Hence, the policy can be considered as neutral from the point of view of equality of opportunity. Note that this does not mean that the policy has no effect: it may well affect the aggregate level of outcome or the degree of inequality within varieties or between varieties of different effort levels.

Case B and the symmetric case C are characterized by changes in the nature of “equality” of opportunity that prevails. At this point, one should remark that the three situations of EOP-S, EOP-W and IOP could reasonably be ranked in terms of how successful they are at securing equality of opportunity, as we discuss in greater details in the next section: EOP-S undeniably represents the highest form of equality of opportunity and IOP represents the worst situation. Here, cases denoted by B start with situations where the two varieties are offered different outcome prospects and end up, under $\pi = 1$ in a situation where outcome prospects are identical. Given the intuitive ranking of possible states, B
is an improvement in terms of equality of opportunity. In fact, it corresponds to a full equalization of opportunities. Symmetrically, C corresponds to a deterioration from the perspective of equality of opportunity.

To some extent, cases D and E are similar to cases B and C. EOP-W should be considered as a better situation than IOP, from the perspective of equality of opportunity: under EOP-W, there is no consensus, among all risk-averse agents, as to what type is the privileged one, while on the contrary, under IOP, there is a consensus on which type is the advantaged one. Hence, case D corresponds to a move towards equality of opportunity (although in a weak form), while cells denoted by E move away from it. Again, case D can be seen as an improvement, from the perspective of equality of opportunity, although this time, the equalization is not complete, unlike case B. Symmetrically, E corresponds to a disequalization of opportunities.

Lastly, F and G correspond to situations where, according to the notions of equality of opportunity in LPT, the policy has no effect on the nature of equality of opportunity at work. Does it necessarily imply that the policy has no effect at all, from the point of view of equality of opportunity? This answer is, obviously, no. This is a major difference with case A. In cell A, distributions are identical under both policy regimes, so that the two varieties are perfectly equal, before and after. This stands in contrast with case F : EOP-W does not require that the two distributions be identical, but simply that they cannot be ranked according to second-order stochastic dominance. However, among all the pairs of distributions that cannot be ranked according to second-order stochastic dominance, some pairs are likely to lie closer together than others. The same holds true among the pairs of distributions that can be ranked according to SD2. Hence, in cases F and G, it is possible that the degree of inequality varies before and after the policy.

The empirical relevance of cases such as F and G is demonstrated in several instances. For example, the analysis of changes over time in equality of opportunity in France, undertaken in Lefranc et al. (2009) concludes to case G : outcome distributions can almost always be ranked by the SD2 criterion, throughout the period they study, although the authors claim that the degree of dissimilarity of the outcome distribution of the different types falls over time. The same seems to hold true in cross country comparisons (Lefranc, Pistolesi & Trannoy 2008).

One limitation of existing characterizations of equality of opportunity is that they provide little guidance regarding the criteria that should be used to assert changes in equality of opportunity, in cases such as F and G. To some extent, this critique extends to studies that attempt to measure the degree of inequality of opportunity using scalar indices (Bourguignon, Ferreira & Walton 2007). Our objective is precisely to offer a formal analysis of possible criteria of opportunity equalization. As we will now discuss, two distinct approaches can be taken. The first one is directly inspired by the definitions of LPT and is based on the order of stochastic dominance at which the distributions of the different varieties can be compared. The second criterion investigates the use of distance measures to assess the degree of inequality of opportunity.
4 Equalization of opportunity: a dominance-order criterion

4.1 Introductory example

A case such as G might look as a status quo situation, from the perspective of the taxonomy of equality of opportunity that underlie table [1]. However, it should be noted that case G still mixes together situations that differ in terms of the nature of the dominance relationships that prevail. IOP is defined by the occurrence of second-order dominance. Hence it incorporates both first- and second-order stochastic dominance. The criterion of LPT can thus be refined in case G in order to achieve a least partial criterion for asserting equalization of opportunity.

This refinement amounts to separate first-order from second-order dominance in cases such as G. This allows to achieve a least partial ranking of situations in terms of opportunity equalization, by splitting cell G in four sub-cases: Cases G1 and G4 are two situations

Table 2: Equality of opportunity configurations under \( \pi = 0 \) and \( \pi = 1 \) - refinement

<table>
<thead>
<tr>
<th>( \pi = 0 )</th>
<th>( \pi = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \succ_2 )</td>
<td>G1, G3</td>
</tr>
<tr>
<td>( \succ_1 )</td>
<td>G2, G4</td>
</tr>
</tbody>
</table>

where the nature of the dominance relationships at work are similar under \( \pi = 0 \) and \( \pi = 1 \). But this does not occur in cases G2 and G3. In case G2, the nature of stochastic dominance relationships changes from first-order dominance under \( \pi = 0 \) to second-order \( \pi = 1 \). As discussed below, this can be interpreted as a weakening of the (nature of the) advantage conferred to the dominant type over the dominated type. For this reason, one might argue that case G2 corresponds to a partial equalization of opportunities. Conversely, G3 could be considered as a deterioration of equality of opportunity.

This reasoning can be extended to provide a more complete ranking criterion in case F. We now consider this extension and provide a formal definition of our criterion of opportunity equalization.

4.2 Criterion

The taxonomy of equality of opportunity situations introduced in LPT rests on an ordinal criterion of stochastic dominance between the income distributions of different varieties. As already mentioned, it distinguishes between three situations: strict equality of the income distributions, stochastic dominance of order 2, and an intermediate situation where neither second-order stochastic dominance, nor equality prevail.

The discussion in the previous paragraph amounts to distinguish first-order from second-order stochastic dominance in the IOP case. A straightforward generalization of
this idea is to consider higher orders of stochastic dominance. This allows for an extension of the taxonomy of LPT based on the minimum order of stochastic dominance, $\kappa$, at which the outcome distributions of two varieties can be ranked. This extension amounts to claim that the higher the value of $\kappa$, the closer the situation is from a case of strong equality of opportunity. Under this view, we then derive a criterion of opportunity equalization: we say that situation $\pi = 1$ brings partial opportunity equalization compared to $\pi = 0$ if the minimum order of stochastic dominance at which the relevant outcome distributions can be ranked is higher under $\pi = 1$ than under $\pi = 0$.

We now formally define our criterion of opportunity equalization and the notions of inequality and equality of opportunity it rests upon. The rest of the analysis relies on the notions of inverse-stochastic dominance, introduced by (Muliere & Scarsini 1989) and further studied by Aaberge (2009), Maccheroni, Muliere & Zoli (2005), and Zoli (1999, 2002). As discussed in the next sub-section, a parallel analysis could be undertaken using the criterion of standard (direct) stochastic dominance.

4.2.1 Order-$k$ equality and inequality of opportunity

We let $F \succ_{ISDk} F'$ and $F \succeq_{ISDk} F'$ denote respectively strict and weak inverse stochastic dominance at order $k$ of the distribution with cdf $F$ over the distribution with cdf $F'$. We refer the reader to A.1 for a formal definition. First- and second-order inverse stochastic dominance are equivalent to their direct dominance counterparts. Higher-order inverse dominance is defined on the basis of integrals of the Generalized Lorenz curve.

We define $\kappa(c, c', e, \pi)$ the minimum order of stochastic dominance according to which $F(c, e, \pi)$ and $F(c', e, \pi)$ can be ranked. For short, since in this section we only consider two possible circumstances, $c$ and $c'$ and one effort level $e$, and concentrate on policy change, we let $\kappa_{\pi}$ denote $\kappa(c, c', e, \pi)$. In the case where one distribution dominates the other, we let, by convention, $c$ denote the dominant circumstance. If we let $c$ be the dominant circumstance at order $\kappa_{\pi}$, it is well know that $c$ will also be the dominant circumstance at all orders greater than $\kappa_{\pi}$. Hence, we have, by definition for all $k \geq \kappa_{\pi}$ $F_{\pi} \succ_{ISDk} F'_{\pi}$.

In order to characterize inequality of opportunity on the basis of $\kappa_{\pi}$, we need to be sure that for all pairs of distributions, $\kappa$ can be defined. The following proposition establishes that for all pairs of distributions satisfying a mild condition, there always exists a minimum order of stochastic dominance $\kappa$.

**Proposition 1** For any pair of distributions with bounded support, with inverse cumulative distribution functions denoted by $F^{-1}()$ and $F'^{-1}()$ satisfying:

$\exists p_{\beta} > 0 \forall p \in [0, p_{\beta}) F^{-1}(p) \geq F'^{-1}(p) \text{ and the strict inequality holds on a positive mass interval } [p_{\beta} - \epsilon, p_{\beta}] \text{ with } \epsilon > 0$, we have:

$\exists k \in \mathbb{R}_+ \text{ and finite such that } F \succ_{ISDk} F' \forall k \in \mathbb{N}_+ \text{ such that } k > \kappa$.

**Proof.** See appendix A.2 \[\blacksquare\]
In the rest of the paper we restrict our attention to the set of distribution functions satisfying the conditions in proposition 1. For instance the class of cumulative distribution function with finite domain or the class of step functions continuous on the right which represent empirical distributions obtained from finite size samples accommodate such requirements.

For any pair-wise comparison of outcome distributions of varieties with similar effort, the taxonomy of LPT can be extended on the basis of the minimal order of stochastic dominance $κ$. This leads to define two notions of inequality of opportunity and weak equality of opportunity for $k$th-order stochastic dominance.

**Definition 4 (IOP-$k$)** For a policy $π$, Inequality of Opportunity at order $k$ (IOP-$k$) prevails between two varieties $(c,e)$ and $(c',e')$ with outcome distributions $F_π$ and $F'_π$ iff:

$$ F_π \succ_{ISDk} F'_π \text{ or } F'_π \succ_{ISDk} F_π $$

**Definition 5 (EOP-W$k$)** For a policy $π$, Weak Equality of Opportunity at order $k$ (EOP-W$k$) prevails between two varieties $(c,e)$ and $(c',e')$ with outcome distributions $F_π$ and $F'_π$ iff:

$$ \forall l \leq k \quad F_π \nexists_{ISDk} F'_π \text{ and } F'_π \nexists_{ISDk} F_π $$

The relationships between inequality and equality of opportunity at different orders follow from the properties of stochastic dominance at different orders. It is straightforward to establish the following proposition:

**Proposition 2** For all $l > k$ : IOP-$k$ $\Rightarrow$ IOP-$l$ and EOP-W$l$ $\Rightarrow$ EOP-W$k$

Definitions 4 and 5, together with the definition of $κ$, imply that the pair $(F_π, F'_π)$ satisfies IOP-$k$, for all $k \geq κ_π$ and satisfies EOP-W$k$ for all $k < κ_π$.

These definitions replicate the definitions of LPT and simply extend them by considering different forms of individual preferences in the ranking of circumstances. The link between these definitions and definitions 1, 2 and 3 is obvious. LPT’s notion of IOP corresponds to the IOP-$k$ for $k = 2$; EOP-W$k$ for $k = 2$ gathers LPT’s notions of EOP-W and EOP-S. It is also worth stressing that LPT’s notion of EOP-W gathers both IOP-$k$, for $k \geq 3$, and EOP-W$k$ for $k \geq 2$. This makes it clear that EOP-W is an intermediate situation, the labeling of which is somehow misleading: to some extent, EOP-W, in the definition of LPT, could also be seen as weak form of Inequality of opportunity.

### 4.2.2 Dominance-order equalization of opportunity

These definitions allow for a refinement of the partition of the configurations that may occur when moving from $π = 0$ to $π = 1$. This partition is based on the pair $(κ_0, κ_1)$, which summarizes all the relevant information in the perspective of a dominance approach to equality of opportunity. By definition of $κ_π$, under under policy $π$, IOP-$k$ will be satisfied for all $k \geq κ_π$ and EOP-W$k$ will be satisfied for all $k < κ_π$. When $κ_1$ is greater than $κ_0$, moving from $π = 0$ to $π = 1$ leads to satisfy a more stringent form of (weak) equality of
opportunity, as established by proposition 2. This leads us to define the following criterion of partial equalization of opportunities, based on the order of dominance:

**Criterion 1 (Dominance-order criterion of opportunity equalization - O-ezOP)**

Moving from $\pi = 0$ to $\pi = 1$ equalizes opportunities between varieties $(c,e)$ and $(c',e)$ according to the dominance-order criterion iff $\kappa(c,c',e,1) > \kappa(c,c',e,0)$.

### 4.3 Interpretation

To understand the foundation for dominance-order criterion of opportunity equalization, one first need to analyze the content of definitions 4 and 5. Conceptions IOP-$k$ and EOP-W$k$ define intermediate cases between two polar situations. The first polar case is the EOP-S situation emphasized by LPT. In this situation, every agent, regardless of her preferences will be indifferent between the two varieties $(c,e)$ and $(c',e)$. The second polar case is the IOP-1 situation. In this situation, since the outcome distribution for $(c,e)$ strictly dominates that of $(c',e)$ at the first order, every agent will prefer variety $(c,e)$ over $(c',e)$ regardless of her preferences, as soon as these preferences are increasing in outcome. Hence, there is unanimity across all agents on the evaluation of EOP-S and IOP-1, regardless of agent’s preferences.

On the contrary, agents’ judgement on all intermediate configurations between EOP-S and IOP-1 will never be unanimous and will be contingent on their preferences. Of course, in this broad set of intermediate cases, all configurations are not identical. Some may lie closer to one of the two polar cases than others. And it is crucial to be able to differentiate among these intermediate cases. One way to differentiate between these cases is by considering the restrictions that need to be placed on individual preferences for agents to concur with the view that Inequality of Opportunity does prevail or not.

When two distributions cannot be unanimously ranked, it is of course always possible to rank them by placing further restrictions on the preferences used to compare such distributions. One may for instance assume a specific individual utility function. In this case, the achieved ranking will of course lack robustness. Alternatively, for the comparison of two varieties $(c,e)$ and $(c',e)$, under a given policy $\pi$, one may try to isolate a class of preferences within which $(c,e)$ is unanimously preferred to $(c',e)$. Let $C_\pi$ denote such a class of preferences. All individuals with preferences in $C_\pi$ will concur with the view that $(c,e)$ is advantaged over $(c',e)$, and that consequently equality of opportunity does not prevail between $(c,e)$ and $(c',e)$. Now consider a policy change from $\pi = 0$ to $\pi = 1$. Assume that the class of preferences within which $(c,e)$ is unanimously preferred to $(c,e)$ under $\pi = 1$ is a strict subset of the class of preferences within which $(c,e)$ is unanimously preferred to $(c',e)$ under $\pi = 0$: $C_1 \subset C_0$. In this case, all preferences according to which equality of opportunity is violated under $\pi = 1$ will also concur with the view that equality of opportunity is violated under $\pi = 0$. But the reverse is not true. For some preferences, equality of opportunity prevails under $\pi = 1$ but not under $\pi = 0$. This leads to conclude that a more encompassing form of inequality of opportunity prevails.
under \( \pi = 0 \) as compared to \( \pi = 1 \), or equivalently that a less weak form of equality of opportunity is satisfied under \( \pi = 1 \).

Of course, implementing this approach requires to resort to a partitioning of the set of admissible individual preferences. The analysis of this paper rests upon the class \( \mathcal{R} \) of rank dependent utility functions (Yaari 1988, Yaari 1987). A utility function \( W \in \mathcal{R} \) can be written as

\[
W(F(y|c,e,\pi)) = \int_{0}^{1} w(p)F^{-1}(p|c,e,\pi)dp
\]

, where \( w(p) \geq 0 \ \forall p \in [0,1] \) and \( \int_{0}^{1} w(p)dp = 1 \). The function \( w(p) \) is interpreted as a weight assigned to rank \( p \) in the outcome conditional distribution (e.g. Zoli 2002). Restrictions on the sign of the derivatives of \( w() \) define the restriction on the set \( \mathcal{R} \). For instance \( \mathcal{R}^2 \) is the class of utility functions where \( w(p) \) is positive, decreasing over the domain of \( p \) and such that \( w'(1) = 0 \). More generally, let \( \mathcal{R}^k \subseteq \mathcal{R} \) define the subset of rank dependent utility functions \( W() \) that satisfy:

\[
\mathcal{R}^k = \left\{ W \in \mathcal{R} : (-1)^i \cdot \frac{d^i w(p)}{dp^i} \geq 0, \quad \frac{d^i w(1)}{dp^i} = 0 \ \forall p \in [0,1] \ \text{and} \ i = 2,3,\ldots,k \right\}.
\]

Obviously, these subsets are nested : \( \mathcal{R}^k \subset \mathcal{R}^{k-1} \subset \ldots \subset \mathcal{R}^1 \). Imposing restrictions on higher-order derivatives of the weight function implies putting more weight on least favorable outcomes.

Properties of the class \( \mathcal{R} \) have been investigated, amongst others, in several papers (e.g. Zoli 1999, Zoli 2002). In particular Aaberge (2009) has established the logical equivalence between inverse stochastic dominance at order \( k \) and unanimity in ranking the pair of distribution functions within the class \( \mathcal{R}^k \). This results established the link between EOP-Wk and IOP-\( k-1 \), on the one hand, and the class of individual preferences under which individual agree in their equality of opportunity judgement. For instance, if IOP-\( k \) is satisfied, all agents with preferences in the class \( \mathcal{R}^k \) will agree that variety \((c,e)\) will be advantaged over \((c',e)\).

Let us now consider the question of changes in equality of opportunity, as we move from \( \pi = 0 \) to \( \pi = 1 \). A case where \( \kappa_1 > \kappa_0 \) corresponds to a situation where one need to put more restrictions on individual preferences under \( \pi = 1 \) than under \( \pi = 0 \) to be able to reach the conclusion that inequality of opportunity prevails. In other terms, a more restrictive form of inequality of opportunity - or equivalently a more general form of weak equality of opportunity - prevails under \( \pi = 1 \). This point can be easily established by noting the following relationships between IOP-\( k \) and EOP-Wk:

\[
\text{IOP-} k \Leftrightarrow F \succ ISD_k F' ; \quad \text{EOP-Wk} = \bigcap_{l=1}^{k} \text{IOP-} l ; \quad \text{EOP-S} = \bigcap_{l=1}^{\infty} \text{IOP-} l
\]

This leads to conclude, on the base of dominance-order, that moving from \( \pi = 0 \) to \( \pi = 1 \)
improves equality of opportunity.

Of course this approach could rest on a different class of individual preferences and
different partitioning thereof. For instance, within the framework of the expected utility
theory, one could define subsets of preferences based on nested restrictions on the Von
Neumann - Morgenstern utility function.

5 Equalization of opportunity: a distance criterion

The dominance-order criterion offers only a partial criterion for assessing opportunity
equalization. In particular, it is unable to assess opportunity equalization or disequalization
when the degree of dominance $\kappa$ remains the same before and after the implementation
of the policy. In such situations, the class of preferences according to which $(c, e)$ is
advantaged over $(c', e)$ is not affected by the policy. However, it might be the case that
the distance between the distributions for the two varieties decreases as a result of the
implementation of the policy. Provided that all agents agree with this view, we would like,
in this case, to conclude that opportunities have been equalized by the policy. We now
formalize this criterion.

5.1 Distance measures

To assess the distance between the distributions attached to two different varieties, one
may use the notions of economic distance between two distributions developed in particular
in Shorrocks (1982) and Chakravarty & Dutta (1987). These contributions suggest to
characterize each distribution by its certainty equivalent and to measure the economic
distance between distributions by the gap in their certainty equivalent.

Let $W(F)$ denote the expected utility derived from a distribution with cdf $F$, where
$W()$ defines individual preferences under risk. Let $D(y)$ define the egalitarian distribution
in which each percentile receives income $y$. For preferences under risk $W()$, we define
$CE_W(F)$, the certainty equivalent of distribution $F$. It is defined by:

$$W(D(CE_W(F))) = W(F).$$

For a pair of distributions $F$ and $F'$, Chakravarty & Dutta (1987) define $\Delta_W(F,F')$
the distance between these two distributions as :

$$\Delta_W(F,F') := |CE_W(F) - CE_W(F')|.$$

When the two distributions are equal, their distance is obviously zero. Otherwise, the
measure of distance depends upon the degree of dissimilarity of the two distribution but
also on the individual preferences under risk, as captured by $W()$. 
5.2 Criterion

Opportunity equalization can be assessed using measures of economic distance by comparing the distance between the outcome distribution of the two varieties before and after the implementation of the policy. If it the case that \( \Delta W(F_0, F_0') > \Delta W(F_1, F_1') \), it can be argued that the implementation has equalized opportunities between the two varieties.

Of course, we would like this judgement to be robust to the utility function used to evaluate the opportunities. A very strong case for opportunity equalization would be if all utility functions under risk would agree with the view that the distance between the two distributions have fallen after the implementation of the policy. A less general requirement is that all utility functions in a particular class \( C \) conclude that the distance has fallen. This is summarized by the following definition.

**Criterion 2 (Distance criterion of opportunity equalization - D-ezOP)** Moving from \( \pi = 0 \) to \( \pi = 1 \) equalizes opportunity between varieties \( (c, e) \) and \( (c', e) \) according to the distance criterion on the set of preferences \( C \) iff \( \forall W \in C : \Delta W(F_0, F_0') > \Delta W(F_1, F_1') \) or at most equal.

5.3 Characterization

The condition defining the distance criterion is not easily implementable in practice, since it is not directly expressed as a restriction on the objects of the comparison, \( F_\pi \) and \( F'_\pi \), and requires computing distance measures over a set of utility functions. Hence, it may be useful to identify some conditions on the quadruple \( (F_0, F_0', F_1, F_1') \) under which distance falls as a result of the policy. We work out such conditions under some restrictions on the individual preferences.

5.3.1 The rank dependent utility function under first- and second-order stochastic dominance

We consider first the distance measures based on the class of Yaari-type rank dependent utility functions. Define \( G(F_\pi, F'_\pi, p) \) the income gap between distributions \( F_\pi \) and \( F'_\pi \) at each quantile:

\[
G(F_\pi, F'_\pi, p) = F_\pi^{-1}(p) - F'_\pi^{-1}(p).
\]

For \( W \in R \), the certainty equivalent is defined by:

\[
\int_{0}^{1} w(p)CE_W(F_\pi)dp = CE_W(F_\pi) = \int_{0}^{1} w(p)F_\pi^{-1}(p)dp
\]

An advantage of the Yaari-type rank dependent utility function is that the certain equivalent is linear in its argument and therefore *distributionally homogeneous*: if \( F_\pi^{-1}(p) = \alpha F'_\pi^{-1}(p) + \beta \), \( \forall p \in [0, 1] \), then \( CE_W(F_\pi) = \alpha CE_W(F_\pi) + \beta \). Chakravarty & Dutta (1987) showed that this is a necessary and sufficient condition for a distance measure based on certain equivalents to satisfy the homogeneity (from the previous example, \( \Delta W(F_\pi, F'_\pi) = \alpha \Delta W(F_\pi, F'_\pi) + \beta \)) and translation invariance (\( \Delta W(F_\pi, F'_\pi) = \Delta W(F_\pi, F'_\pi) \)) when \( \alpha = 1 \) properties in Ebert (1984).
and the distance can be written as:

$$\Delta_W(F_\pi, F'_\pi) = \left| \int_0^1 w(p)[F_\pi^{-1}(p) - F'_\pi^{-1}(p)]dp \right| = \left| \int_0^1 w(p)G(F_\pi, F'_\pi, p)dp \right|$$

Under first-order stochastic dominance, proposition 3 establishes a necessary and sufficient condition for D-ezOP:

**Proposition 3** Let $F_\pi \succ SD_1 F'_\pi \forall \pi$, then $\Delta_W(F_\pi, F'_\pi) \geq 0$ for all $W \in \mathcal{R}$ iff $G(F_0, F'_0, p) - G(F_1, F'_1, p) \geq 0$ for all $p \in [0, 1]$.

**Proof.** See Appendix A.3

In other words, there is unanimity within the largest class of utility functions $\mathcal{R}$ in saying that the distance between the income prospects of two comparable varieties has decreased by effect of policy intervention if and only if the income gap between the distributions at each quantile of luck is smaller in the treated pair of distributions than it is in the non treated one. This induces a form of dominance at the first order of the gap distributions, once that gaps have been ordered according to their percentile (and not according to the gap magnitude itself).

One can also note that $G(F_0, F'_0, p) - G(F_1, F'_1, p) = G(F_1, F_0, p) - G(F'_1, F'_0, p)$. Requiring that $G(F_1, F_0, p) - G(F'_1, F'_0, p) \geq 0$ for all $p$ is requiring that the income gains due to policy treatment be greater at all percentiles of luck for the most disadvantaged type $c'$.

Consider now the case in which distributions within each pari can only be ordered at second-order stochastic dominance. It is now interesting to evaluate if, within the class of utility displaying risk aversion, it is still possible to rank the pair of distributions according to the distance criterion. Under second-order stochastic dominance, proposition 4 establishes a necessary and sufficient condition for D-ezOP:

**Proposition 4** Let $F_\pi \succ SD_2 F'_\pi \forall \pi$, then $\Delta_W(F_\pi, F'_\pi) \geq 0$ for all $W \in \mathcal{R}^2$ iff $\int_0^1 G(F_0, F'_0, t)dt - \int_0^1 G(F_1, F'_1, t)dt \geq 0$ for all $p \in [0, 1]$.

**Proof.** See appendix A.4

In other words, there is unanimity within the largest class of utility functions $\mathcal{R}^2$ in saying that the distance between the income prospects of two comparable varieties has decreased by effect of policy intervention if and only if the income *cumulated* gap between the distributions at each quantile of luck is smaller in the treated pair of distributions than it is in the non treated one. This induces a form of dominance at the second order of the gap distributions, once that gaps have been ordered according to their percentile (and not according to the gap magnitude itself).

Consider writing the gap difference as $G(F_1, F_0, p) - G(F'_1, F'_0, p)$. Requiring that $\int_0^p [G(F'_1, F'_0, t) - G(F_1, F_0, t)]dt \geq 0$ for all $p$ is requiring that the cumulated income gain due to policy treatment be greater at all percentiles for the most disadvantaged type $c'$.
Following a similar scheme, it is possible to derive a dominance condition for all classes of utility functions, provided that outcomes prospects can be ranked according to the corresponding order of inverse stochastic dominance.

5.3.2 The rank dependent utility function under $k^{th}$ order stochastic dominance

Consider $k$-th integral of the quantile functions as defined in Appendix A.1. Under $k$-th order inverse stochastic dominance, proposition 5 establishes a necessary and sufficient condition for D-ezOP:

**Proposition 5** Let $F_{\pi} \succeq ISD^k_{\pi}$, then $\Delta_u(F_0, F'_0) \geq \Delta_u(F_1, F'_1)$ for all $W \in R^k$ iff $G(\Lambda^k_0, \Lambda^k_0, p) \geq G(\Lambda^k_1, \Lambda^k_1, p)$ for all $p \in [0, 1]$.

**Proof.** See appendix A.5

In other words, there is unanimity within the largest class of utility functions $R^k$ in saying that the distance between the income prospects of two comparable varieties has decreased by effect of policy intervention if and only if the income *cumulated* gap at the $k$-th order between the distributions at each percentile of luck is smaller in the treated pair of distributions than it is in the non treated one. This induces a form of dominance at the $k$-th order of the gap distributions, once that gaps have been ordered according to their percentile (and not according to the gap magnitude itself).

One can also note that $G(\Lambda^k_0, \Lambda^k_0, p) - G(\Lambda^k_1, \Lambda^k_1, p) = G(\Lambda^k_1, \Lambda^k_0, p) - G(\Lambda^k_1, \Lambda^k_0, p)$. Requiring that $G(\Lambda^k_1, \Lambda^k_0, p) - G(\Lambda^k_0, \Lambda^k_1, p) \geq 0$ for all $p$ is requiring that the gains by reduction of inequality at order $k$ when moving from $\pi = 0$ to $\pi = 1$ should be greater at all percentiles for the more disadvantaged type.

5.3.3 The Von Neumann-Morgenstern utility function

As we have seen in the previous section, when considering rank-dependent utility functions, the Shorrocks measure of economic distance can be easily expressed as a function of the income gap at different quantiles of the income distribution. The obvious reason is that in the class of rank dependent utility functions, social welfare is a linear function of the income level at each quantile.

We now consider the class $U$ of additive utility functions, and the related distance measure. Integrating by parts and assuming that the support for the outcomes distributions is bounded by $y \in [\underline{y}, \overline{y}]$, than the additive utility function $W(.) \in U$ admits the following representation:

$$W(F(y|c, e, \pi)) = \int_{\underline{y}}^{\overline{y}} u(y)dF(y|c, e, \pi) = (u(\overline{y}) - u(\underline{y})) - \int_{\underline{y}}^{\overline{y}} u'(y)F(y|c, e, \pi)dy.$$ (1)

It turns out that, in the general case, the gap in utility levels cannot be expressed as a function of the income gap at the different levels of the underlying distributions. Results
can only be obtained in special cases. The reason for this problem originates in the fact that the distance measure is no longer invariant to a translation of all incomes in the compared distribution. And as a consequence, the welfare gap cannot only be expressed as function of the income gaps, independently of the income levels.

More specifically, for \( W \in \mathcal{U} \), we have:

\[
CE_W(F) = u^{-1} \left( \int_0^\infty u(x)dF(x) \right)
\]

The problem for expressing the distance condition as a restriction on the cdf is that contrary to the rank-dependent case, \( \Delta_W(F_0, F_0') \) is no longer linear in \( W(F) - W(G) \). In the general case, \( u() \) is simply an increasing function and so is \( u^{-1} \). Hence the economic distance condition implies no specific restrictions regarding the gap in aggregate welfare between the two distributions.

Some conditions can be expressed in the special case where \( u() \) is a concave function. In this case, \( u^{-1} \) is a convex function. Next assume stochastic dominance of order 2 of \( F_i \) over \( F_i' \), for \( i = 0, 1 \). As a result of dominance, we can get rid of the absolute values in the expression of \( \Delta_W(F_\pi, F'_\pi) \).

Under these assumptions, we have:

\[
\Delta_W(F_0, F_0') \geq \Delta_W(F_1, F_1') \iff u^{-1}(W(F_0)) - u^{-1}(W(F_0')) \geq u^{-1}(W(F_1)) - u^{-1}(W(F_1'))
\]

The last condition is a condition on the gap in certain equivalent incomes. This condition does not imply any restriction on the sign of \( [W(F_0) - W(F_0')] - [W(F_1) - W(F_1')] \). In other words, it does not imply any condition on the gap in social welfare between types before or after the policy. Indeed, because of the convexity of \( u^{-1} \), the gap in social welfare depends not only on the gap in certain equivalents, but also on their magnitude.

Imposing additional conditions on level of welfare of both types before and after the policy, helps remove this indeterminacy. A relatively clear case occurs when the situation of the worst type is improved as a result of the policy: \( W(F_1') \geq W(F_0') \). In this case, using the convexity of the \( u^{-1} \) function, one can show that the economic distance between the distributions decreases for all utilities in \( \mathcal{U} \) if the gap in expected utility falls as a result of the policy. This is summarized by the following proposal:

**Proposition 6** Under the following assumptions: (i) \( F_\pi \succ_2 F'_\pi, \forall \pi \) and (ii) \( W(F_0') \leq W(F_1') \) we have:

\[
\forall W \in \mathcal{U}^2 \Delta_W(F_0, F_0') \geq \Delta_W(F_1, F_1') \Rightarrow \forall W \in \mathcal{U}^2, W(F_0) - W(F_0') \geq W(F_1) - W(F_1')
\]

**Proof.** See appendix [A.6] ■

The proposition shows that the right direction in the variation in welfare differential associated to policy intervention is only a necessary condition for decreasing economic distance between outcome prospects. The convexity of the utility function plays the central
role in showing that one gets at least this result by assuming only if one is also willing to accept that $F_1 \succ_{SD_2} F_0'$, that is by effect of the policy the situation of the worse off is ameliorated by effect of policy intervention.

Consider firstly a very restrictive case of second order stochastic dominance, the one in which $F_\pi \succ_{SD_1} F'_\pi$. Using (1), one can equivalently check that the necessary condition in proposition 6 is satisfied by looking at the gap between cdf, $(F_0(y) - F'_0(y)) - (F_1(y) - F'_1(y))$, is less than zero at any quantile $y \in [\underline{y}, \overline{y}]$. Another way to see the relation, is that $(F_0(y) - F_1(y)) - (F'_0(y) - F'_1(y)) \leq 0$, that is the change in the percentage of risk of receiving an income lower than $y$ due to policy change is larger for the more disadvantaged variety $(c', e)$ compared to the more advantaged variety $(c, e)$.

A similar argument can be used to determine a dominance condition for the case when only second order stochastic dominance is verified. The equivalence between $\succ_{SD_2}$ and $\succ_{ISD_2}$, or alternatively Generalized Lorenz Dominance (Shorrocks 1983), gives the following proposition, which can be easily demonstrated making use of integration by part of (1):

**Proposition 7** Under the following assumptions : (i) $F_\pi \succ_2 F'_\pi$, $\forall \pi$ and (ii) $W(F_0') \leq W(F_1')$, $\forall W \in U^2$, $W(F_0) - W(F'_0) \geq W(F_1) - W(F'_1)$ iff

$$\int_y [(F_0(y) - F'_0(y)) - (F_1(y) - F'_1(y))] \, dy \leq 0 \quad \forall y \in [\underline{y}, \overline{y}].$$

The condition provides an alternative, equivalent but empirically attractive necessary condition for the distance dominance in proposition 6. Nevertheless, if one is not willing to go beyond in restricting the class of preferences $U^2$, it is not possible to restore the equivalence in proposition 6. A meaningful restriction would consists in selecting families of additive utility functions by according to the properties of the risk aversion coefficient. Such restrictions allows to keep the risk attitude as constant within a class of utility functions, thus providing more intuitions on the distortion in the certain equivalent caused by the evaluation $u^{-1}(\cdot)$.

6 **Equalization of opportunity: a definition**

The two criteria defined in the previous sections, emphasize two distinct facets of opportunity inequality: the first one is the degree of unanimity with the assessment of the advantage enjoyed by one variety over another variety; the second one is the measure of the extent of the advantage enjoyed by one variety over another variety. In this section, we provide an encompassing definition of opportunity equalization that combines both criteria. We first do so within the restricted setting where we compare only two varieties but extend this definition to the general case where we have multiple circumstances and multiple effort levels.
6.1 Definition

**Definition 6 (ezOP for two varieties)** Moving from $\pi = 0$ to $\pi = 1$ equalizes opportunities between varieties $(c,e)$ and $(c',e)$ iff criteria O-ezOP is satisfied and D-ezOP is satisfied on the set of preferences $R^{c,c',e,1}$.

Possible configurations are illustrated in the table 3.

6.2 Discussion

Under the ordinal view, inequality of opportunity is considered to decrease if the class of preferences according to which it is possible to rank circumstances unanimously becomes smaller. Under the cardinal view, inequality of opportunity decreases if the extent of the advantage enjoyed by one class diminishes.

In principle, these two criteria might contradict each other: for instance the dominance-order might be satisfied and at the same time the distance order might be violated or indicate a deterioration of equality of opportunity. The above definition requires that both criteria be satisfied. More precisely, we require that the class of preferences according to which circumstances can be unanimously ranked becomes smaller and that within the class of preferences according to which inequality of opportunity unanimously prevails before and after the policy implementation, individuals agree that the extent of the advantage of the dominant circumstances has fallen.

6.3 Extension: multiple circumstances and effort levels

We now examine the case where there are multiple circumstances and effort levels. Consider first the situation where there is only one effort level and multiple circumstances. Equality of opportunity requires that none of the vector of circumstance should yield an advantage over any over any alternative vector of circumstances. Now assume we start from an initial situation $\pi = 0$ in which EOP-S does not prevail and consider a policy change to $\pi = 1$. For each pair of circumstances, it is possible to state, based on definition 6 whether opportunities have been equalized within this specific pair. The overall judgment on the equalizing impact of the policy will depend on the series of pair-wise equalization judgements. There might be several ways to combine these pair-wise judgements. For instance one may be willing to trade off a mild disequalization of opportunities for some pair of circumstances against a strong equalization for some other. Equality of opportunity indices implicitly undertake this form of aggregation. The grounds for such an aggregation should probably be clarified. If we rule it out, it is reasonable that a situation of overall opportunity equalization be a situation where opportunities are equalized for all pair-wise comparisons.

Now consider a situation with multiple levels of effort. Since individuals with different effort exert different responsibility levels they should not be compared. This leads
Table 3: Test for Equalization of Opportunity: two types and one responsibility level

<table>
<thead>
<tr>
<th>( \pi = 0 )</th>
<th>EOP-S ( \cong_{ISDk} )</th>
<th>LPT’s EOP-W</th>
<th>LPT’s IOP</th>
</tr>
</thead>
<tbody>
<tr>
<td>EOP-S</td>
<td>( \cong_{ISD\infty} )</td>
<td>No effect</td>
<td>( \succeq_{SD} )</td>
</tr>
<tr>
<td>LPT’s EOP-W</td>
<td>( \cong_{ISDk} )</td>
<td>Disequalization</td>
<td>Disequalization</td>
</tr>
<tr>
<td>( \cong_{ISDh} )</td>
<td>Disequalization</td>
<td>Disequalization</td>
<td>Disequalization</td>
</tr>
<tr>
<td>( \cong_{SD2} )</td>
<td>Disequalization</td>
<td>Disequalization</td>
<td>Disequalization</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \pi = 1 )</th>
<th>EOP-S ( \cong_{ISD\infty} )</th>
<th>LPT’s EOP-W</th>
<th>LPT’s IOP</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \cong_{ISDk} )</td>
<td>ezOP-S ( \forall W \in \mathbb{R}^k )</td>
<td>Disequalization</td>
<td>Disequalization</td>
</tr>
<tr>
<td>( \cong_{ISDh} )</td>
<td>Disequalization</td>
<td>Disequalization</td>
<td>Disequalization</td>
</tr>
<tr>
<td>( \cong_{SD2} )</td>
<td>Disequalization</td>
<td>Disequalization</td>
<td>Disequalization</td>
</tr>
</tbody>
</table>

Notation: LPT’s EOP-S (def. 1), EOP-W (def. 2), IOP (def. 3); ezOP is partial equalization of opportunity; ezOP-S is strong equalization of opportunity and D-ezOP is the distance criterion associated to the class of preferences \( C \).
us to require that opportunities be equalized for all pair-wise comparisons of varieties with similar effort.

This is formalized in the following definition:

**Definition 7 (ezOP)** Moving from $\pi = 0$ to $\pi = 1$ equalizes opportunities if ezOP is satisfied for all varieties $(c,e)$ and $(c',e)$ with $c,c' \in C$ and $e \in E$.

### 7 Implementation

The empirical implementation of the criteria of opportunity equalization defined in the previous sections requires two main ingredients. The first one is a testing procedure. The second one is a framework that allows for the fact that in actual applications, some of the main variables of interest are likely to be only partially observable.

#### 7.1 Implementation algorithm

We propose a procedure for testing equalization of opportunity based on the ezOP concept. We first assume that individual outcome, circumstances and effort are observed for a representative sample of the population. We construct an algorithm for comparing the outcome prospects for all pairs of circumstances in $C$, evaluated at all possible effort levels in $E$.

Admittedly, assuming that all circumstances and effort are observable is a strong requirement that may be met easily. We leave for the next section the discussion of implementation when effort is not observable.

We construct the algorithm based on D-ezOP and O-ezOP. For each criterion we also provide the corresponding empirical analog instrument, as demonstrated by the sections throughout the paper. For inverse stochastic dominance, we use integrals of the generalized Lorenz curve, for distance dominance we use the cumulative gain function.

**Algorithm 1 (Implementable ezOP)** The following sequence of estimations and tests implements ezOP

$F(\cdot|c,e,\pi)$ For any pair $(c,c')$, any policy level $\pi$, and any effort level, estimate the distributions $F(y|c,e,\pi)$ and $F(y|c',e,\pi)$.

$\kappa(c,c',e,\pi)$ For each $(c,c',e,\pi)$ compute $\kappa(c,c',e,\pi)$ as follows:

- For $k = 1, \ldots$ define the following pair of tests:
  
  \[ H_0 : F(y|c,e,\pi) \succ_{ISDk} F(y|c',e,\pi) \text{ vs. } H_a : F(y|c,e,\pi) \not\succ_{ISDk} F(y|c',e,\pi) \]
  
  Define $I_k = (u,v)$ the result of this pair of tests, where $u,v$ is equal to 1 if the null hypothesis is rejected and 0 otherwise, respectively for both null hypothesis.

- Compute $I_k$ for $k = 1$:
– if \( I_k = (0, 0) \) : \( \kappa(c, c', e, \pi) = \infty \) - stop
– if \( I_k = (0, 1) \) or if \( I_k = (1, 0) \) : \( \kappa(c, c', e, \pi) = k \) - stop
– if \( I_k = (1, 1) \) : let \( k = k + 1 \) and iterate.

**O-ezOP** Verify that \( \kappa(c, c', e, 1) \geq \kappa(c, c', e, 0) \), for all \((c, c')\) and all \(e\).

**D-ezOP** Verify that distance in outcome prospects is reduced by the policy within the class \( \mathcal{R}^{\kappa(c, c', e, 1)} \), :

for this, compute the gap function at order \( \kappa(c, c', e, 1) \) (= \( \kappa \)) and test

\[
G(\Lambda^{\kappa}(p|c, e, 0), \Lambda^{\kappa}(p|c, e, 1)) \geq G(\Lambda^{\kappa}(p|c, e, 1), \Lambda^{\kappa}(p|c, e, 1)) \quad \forall p \in [0, 1].
\]

If steps O-ezOP and D-ezOP are verified for all \((c, c', e)\), ezOP is verified.

The definitions of the null hypothesis, the test statistics and their asymptotic behavior are provided in section ??.

### 7.2 Unobservable effort

Assume circumstances are observable but effort is not. Hence, we only observe the distribution of outcome conditional on circumstances under both policy regimes. This distribution is obtained by averaging quantiles of the outcomes distribution according to the effort distribution:

\[
F^{-1}(p|c, \pi) = \int_{e} F^{-1}(p|c, e, \pi) dG(e|c, \pi).
\]  

(2)

We now turn to the following question: is it possible to assess ezOP under these observational constraints?

#### 7.2.1 Ex ante perspective

Under these constraints, it is first possible to assess a restricted form of equalization, namely the one that corresponds to an ex ante conception of equality of opportunity. In a nutshell, the ex ante approach amounts to assume that individuals do not know their final level of effort when making equality of opportunity judgments. Hence, these judgements are only based on the distribution of outcome conditional on circumstances. All effort levels are aggregated and this is formally equivalent to assuming a single effort level. This leads to the **Opportunity Dominance** concept (Fleurbaey 2008, Ch. 9). The criterion requires that the distance between types in the outcome space be reduced as a result of the policy, regardless of their responsibility.

It is straightforward to adapt the ezOP test to the ex ante perspective of equality of opportunity. It suffices to replace the distributions \( F(.|c, e, \pi) \) with the ex ante counterpart
for all pair of circumstances $c \neq c'$ and policy in the definitions of O-ezOP, D-ezOP and ezOP. Moreover, is ex ante ezOP holds for all pairs of circumstances, than ex ante ezOP is verified. This is summarized in the following definition:

**Definition 8 (ex ante ezOP)** Define $F(y|c,\pi)$ and $F(y|c',\pi)$ the outcome conditional distributions that can be estimated when effort is not observable. For $c \neq c'$, define $\kappa(c,c',\pi)$ as the minimum order for which $F(y|c,\pi)$ and $F(y|c',\pi)$ can be ordered according to inverse stochastic dominance. Moving from $\pi = 0$ to $\pi = 1$ equalizes opportunities in the ex ante perspective if for all pairs of types $c,c' \in C$ holds that $\kappa(c,c',1) \geq \kappa(c,c',0)$ and $\Delta_W(F(y|c,0),F(y|c',0)) - \Delta_W(F(y|c,1),F(y|c',1)) \geq 0$ for all $W \in \mathcal{R}^{\kappa(c,c',1)}$.

The ex ante ezOP criterion can be empirically tested by resorting to the sequence proposed in algorithm 1 while replacing the estimates of the outcome conditional distribution functions $F(.|c,e,\pi)$ with $F(.|c,\pi)$ at any effort level in the first step of the algorithm (F(.|c,e,\pi))

The ex ante ezOP procedure defines the implementation criterion for the ezOP test, based upon an ex post perspective. We discuss in the following section the relation between the two criteria.

**7.2.2 Ex post perspective**

TO BE COMPLETED.

**7.3 Partially observable circumstances**

TO BE COMPLETED

**8 Conclusions**

This work contributes in three directions. Firstly, it proposes a new definition for equality of opportunity that encompasses others models proposed in the literature. The advantage of our model is that we are able to assign a degree of equality of opportunity. Moreover, the test is enough general to take into considerations other components of outcome achievement such as luck. We employ preference under risk as a tool to evaluate the risk associated to prospects for similar varieties. Our main normative principle is based on robustness: the smaller is the class of preferences that unanimously agree upon ordering the prospects associated to two any comparable varieties, the weaker the contribution of these two varieties to the overall level of inequality of opportunity. Hence, the higher is the equality of opportunity granted.

Secondly, we employ the rank comparison, along with a distance comparison, to gain robustness in the evaluation of equalization of opportunity. This dynamic component of the test allows to compare, for instance, the distributional effect of different policies, when the evaluation criterion is the one of equality of opportunity. A natural extension
consists in applying the method to the real world. One would need to find a policy which
rises concerns about its effectiveness in equalizing opportunity. Our method requires the
estimation of actual and counterfactual outcome distribution functions. It is worth noting
that we leave aside any concern about inequality that should arise from the scheme of
effort reward that is embodied in the policy.

We finally contribute to the empirical literature. This paper is extremely empirically
minded in covering situations where effort is not observable or circumstances are only
partially observable, as it is often the case in the vast majority of microdata available. We
are able to weaken enough, and in a meaningful economic interpretation, our definition
of equalization of opportunity such that it can be identified through common survey data
available. We also discuss the stochastic dominance test and provide their asymptotic
distributions.

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A Definitions and Proofs

A.1 Notions of stochastic dominance

Following Gastwirth (1971), the integral function

\[ GL(p|c,e,\pi) = \int_0^p F^{-1}(t|c,e,\pi) \, dt \]

defines the Generalized Lorenz curve \( GL \) of the distribution \( F(y|c,e,\pi) \). The integral condition of order \( k \) of the \( GL \) curve, \( \Lambda^k(p|c,e,\pi) \) is defined in a recursive way by the following relations:

\[
\Lambda^k(p|c,e,\pi) = \int_0^p \Lambda^{k-1}(t|c,e,\pi) \, dt, \quad p \in [0,1]
\]

\[ \Lambda^2(p|c,e,\pi) = GL(p|c,e,\pi). \]

Muliere & Scarsini (1989) introduced the inverse stochastic order \( \succsim_{ISDk} \) as a criterion to rank outcome distributions. The order is defined implicitly by the following inequalities. Let \( F(y|c,e,\pi) \) and \( F(y|c',e,\pi) \) be two distribution functions:

\[ F(y|c,e,\pi) \succsim_k F(y|c',e,\pi) \iff \Lambda^k(p|c,e,\pi) \geq \Lambda^k(p|c',e,\pi) \quad \forall p \in [0,1]. \]

Note that \( \succsim_{ISD1} \) denotes rank dominance and \( \succsim_{ISD2} \) denotes \( GL \) dominance. Moreover, by proposition 1 in Maccheroni et al. (2005), if \( F(y|c,e,\pi) \succsim_{ISDk} F(y|c',e,\pi) \) then \( F(y|c,e,\pi) \succsim_{ISDl} F(y|c',e,\pi) \), for all \( l > k \). It follow that \( GL \) dominance is sufficient for any other inverse dominance comparison.

Along with restrictions on the order of integration, one can introduce restrictions on the set of rank-dependent utilities. This approach will lead to define general subsets of \( R \) by imposing restrictions over the derivatives of the weighting function, which amount to specifying that the utility must assign increasing weight to the lowest ranks of the outcome distributions.

Let \( R_k \subseteq R \) define the set of rank dependent utilities \( W(\cdot) \) giving more weight to the more disadvantaged percentiles, where \( k \) is the the number of assumption imposed on the signs of the derivatives of \( W(\cdot) \), such that:

\[ R_k = \left\{ W \in R : \begin{array}{l} (-1)^i \cdot \frac{d^iw(p)}{dp^i} \geq 0, \quad \frac{d^iw(1)}{dp^i} = 0 \forall p \in [0,1] \quad \text{and} \quad i = 2, 3, \ldots, k \end{array} \right\}. \]

\[ It \ is \ well \ known \ (e.g. \ Muliere \ & \ Scarsini \ 1989) \ that \ first \ and \ second \ order \ inverse \ stochastic \ dominance \ are \ equivalent \ to \ direct \ first \ and \ second \ order \ stochastic \ dominance, \ which \ is \ in \ turn \ equivalent \ to \ generalized \ Lorenz \ dominance \ for \ incomes \ distributions \ with \ different \ means \ (Shorrocks \ 1983). \ Atkinson \ (1970) \ showed \ the \ logical \ relation \ between \ GL \ dominance \ with \ fixed \ means \ and \ an \ utilitarian \ type \ social \ welfare \ function, \ later \ generalized \ to \ all \ \( S \)-concave social welfare functions and to income distributions with different means.

\[ When \ GL \ curves \ cross, \ a \ different \ dominance \ approach \ would \ consists \ in \ taking \ on \ with \ additive \ Von-Neuman \ Morgenstern \ utility \ functions \ representation, \ while \ introducing \ additional \ restrictions \ on \ he \ higher \ order \ derivatives \ of \ the \ evaluation \ function. \ This \ approach \ would \ lead \ to \ the \ definition \ of \ direct \ stochastic \ dominance \ for \ orders \ larger \ than \ two. \ Although \ the \ approach \ is \ appealing \ for \ the \ direct \ connection \ with \ choice \ under \ risk, \ the \ economic \ interpretation \ of \ the \ conditions \ on \ higher \ order \ derivatives \ remains \ obscure. \ Along \ this \ line, \ Zoli \ (2002) \ and \ Aaberge \ (2009) \ have \ pushed \ the \ argument \ of \ inverse \ stochastic \ dominance \ comparisons. \]
It is not difficult to show that for a class of outcome distribution functions that can be ranked at a finite integration order the following proposition holds:

**Proposition 8** For any \( F(y|c,e,\pi) \) and \( F(y|c',e,\pi) \):

\[
F(y|c,e,\pi) \succeq_k F(y|c',e,\pi) \iff CE_W(F(y|c,e,\pi)) \geq CE_W(F(y|c',e,\pi)), \quad \forall W \in \mathcal{R}^k.
\]

**Proof.** See Aaberge (2009). ■

The equivalence proposed in the previous proposition can always be checked, as there always exists a finite order of integration for which it is possible to verify whether one element of the pair \( (F(y|c,e,\pi), F(y|c',e,\pi)) \) dominates the other according to inverse stochastic dominance. The main technical advantage of inverse dominance is that the support of integration is finite.

### A.2 Proof of Proposition 1

**Proof.** The proof consists in showing that if \( F(y|c,e,\pi) \) inverse stochastically dominates \( F(y|c',e,\pi) \) at the first order for some positive percentiles between 0 and \( p_\beta > 0 \), then we have a sufficient condition for the two distribution to be comparable at a finite degree of integration.

Define \( \Delta F^{-1}(p) := F^{-1}(p|c,e,\pi) - F^{-1}(p|c',e,\pi) \) and \( \Delta \Lambda^k(p) := \Lambda^k(p|c,e,\pi) - \Lambda^k(p|c',e,\pi) \) at any \( p \in [0,1] \). Integrate by part up to \( k-2 \) times the function \( \Delta \Lambda^k(p) \) to obtain the following:

\[
\Delta \Lambda^k(p) = \int_0^p \Delta \Lambda^{k-1}(t)dt = -\int_0^p t \cdot \Delta \Lambda^{k-2}(t)dt + \left[t\Delta \Lambda^{k-1}(t)\right]_0^p
\]

\[
= \int_0^p (p-t)\Delta \Lambda^{k-2}(t)dt
\]

\[
= \int_0^p \frac{1}{2}(p-t)^2\Delta \Lambda^{k-3}(t)dt + \left[\frac{1}{2}(p-t)^2\Lambda^{k-2}(t)\right]_0^p
\]

\[
= \int_0^p \frac{1}{(k-2)!}(p-t)^{k-2}\Delta F^{-1}(t)dt \quad (3)
\]

To see the result in (3), it is sufficient to note that \( \Lambda^0(0) = 0 \) and therefore \( \Delta \Lambda^k(0) = 0 \) for any \( k \), and that \( \Delta \Lambda^k(p) = \int_0^p \Delta F^{-1}(t)dt \).

The sufficient conditions of the proposition states that \( \Delta F^{-1}(p) \geq 0 \) for all \( p \in [0,p_\beta] \) and there exists a \( p \) such that the strong inequality holds. As long as we use continuous or at most left inverse cumulative distribution functions, we make sure that the function \( \Delta F^{-1}(p) \) is well behaved on the whole percentile domain. Moreover, the function takes only finite values even in \( p = 1 \) or \( p = 0 \). As a consequence the value \( p_\beta \) exists.

Moreover, define \( \bar{\pi} := \sup\{\Delta F^{-1}(p) : p \in [0,p_\beta]\} > 0 \) and \( -\beta := \inf\{\Delta F^{-1}(p) : p \in [p_\beta,1]\} < 0 \), which exists finite and the signs are given by construction of the sufficient conditions\footnote{If the conditions do not hold we have either that type \( c' \) dominates type \( c \) or type \( c \) dominates on the first order type \( c' \)}.

The differential curve and the infimum and supremum levels are marked with solid lines on the graph in figure 1.

Let 0 < \( \alpha \leq \bar{\pi} \) such that it is possible to define at least two points \( p_\alpha, p'_\alpha \in [0,p_\beta] \), such that for \( p_\alpha \leq p \leq p'_\alpha \), \( \Delta F^{-1}(p) > 0 \) holds. Consequently, we define the new differences...
curve \( \Delta F^{-1}(p) \) in the following way:

\[
\Delta F^{-1}(p) := \begin{cases} 
0 & \text{if } p \in [0,p_\alpha), \\
\alpha & \text{if } p \in [p_\alpha, p'_\alpha], \\
0 & \text{if } p \in (p'_\alpha, p_\beta), \\
-\beta & \text{if } p \in [p_\beta, 1]. 
\end{cases}
\]

The curve is represented by the dashed line in figure 1. It is not difficult to see that \( \tilde{\alpha} \) and \( -\beta \) are defined by the distribution functions, while it always hold that \( \Delta F^{-1}(p) \leq \Delta F^{-1}(p) \) for all \( p \). The function reduces the positive domain of the difference \( \Delta F^{-1}(p) \) for percentiles in the lower side of the domain, while it magnify the negative effect of the difference for the percentiles in the remaining side of the domain. Therefore, making use of (3), if it is possible to find a value of \( \tilde{k}^* \) such that \( \forall k > \tilde{k}^* \):

\[
\int_0^p \frac{1}{(k-2)!} (p-t)^{k-2} \Delta F^{-1}(t) dt \geq 0 \quad \forall p \in [0,1],
\]

then there must exists also a value \( k^* \) satisfying our proposition (that is inverse stochastic dominance at a finite order is always granted).

Not that in the interval \([0,p_\alpha)\) and \( (p'_\alpha,p_\beta) \) the expression (4) is always zero. Moreover, (4) is always strictly positive on the interval domain \([p_\alpha,p'_\alpha)\). It remains to check the condition for any \( p \geq p_\beta \).

\[
\int_0^p \frac{(p-t)^{k-2}}{(k-2)!} \Delta F^{-1}(t) dt = \frac{1}{(k-2)!} \left[ \int_{p_\alpha}^{p'} (p-t)^{k-2} \alpha dt + \int_{p'}^{p_\beta} (p-t)^{k-2} (-\beta) dt \right] = \frac{\alpha \left[ (p-p_\alpha)^{k-1} - (p-p'_\alpha)^{k-1} - \beta (p-p_\beta)^{k-1} \right]}{(k-2)!} \geq 0 \quad \forall p \geq p_\beta.
\]

To check the solution it suffice that there exists a \( \tilde{k}^* \) such that:

\[
\frac{(p-p_\alpha)^{k-1} - (p-p'_\alpha)^{k-1}}{(p-p_\beta)^{k-1}} \geq \frac{\beta}{\alpha}, \quad \forall p \geq p_\beta.
\]

By construction of \( \Delta F^{-1}(p) \), if the condition holds for \( p = 1 \), then it must hold for all \( p < 1 \), because the differential takes only negative values for \( p \geq p_\beta \). Note that the numerator and denominator of the left hand side of (5) are positive, but the ration is not said to be greater than one. Nevertheless, one can always pick up a value of \( \alpha < \pi \) such that \( (p-p'_\alpha) \approx (p-p_\beta) \) and (5) is therefore satisfied if and only if the following holds:

\[
\left( \frac{1-p_\alpha}{1-p_\beta} \right)^{k-1} \geq 1 + \frac{\beta}{\alpha}.
\]

Both sides of (6) are positives and greater than one. Thus, by taking logs on the left and right side, it is easy to show that the integral condition in (3) is satisfied if and only if the integration order \( \tilde{k}^* \) is large enough to verify:

\[
\tilde{k}^* \geq 1 + \frac{\ln(1 + \beta/\alpha)}{\ln(1-p_\alpha) - \ln(1-p_\beta)}.
\]

Note that \( \tilde{k}^* \) is positive and greater than one and it always exists finite for any \( 0 <
\[ p_\alpha < p_\beta < 1 \text{ and for } \alpha, \beta > 0. \text{ Therefore the value } k^* \text{ exists as well, which concludes the proof.} \]

A.3 Proof of Proposition 3

Proof. As a consequence of the dominance hypothesis, we have:

\[ \forall W \in R, \forall \pi \int_0^1 w(p)F^{-1}_\pi(p)dp > \int_0^1 w(p)F'^{-1}_\pi(p)dp \]

Consequently, for all \( W \in R \), we can write:

\[ \Delta W(F_\pi, F'_\pi) = \int_0^1 w(p)G(F_\pi, F'_\pi, p)dp \]

Hence, we have:

\[ \Delta W(F_0, F'_0) - \Delta W(F_1, F'_1) = \int_0^1 w(p)[G(F_0, F'_0, p) - G(F_1, F'_1, p)]dp \tag{7} \]

If \( [G(F_0, F'_0, p) - G(F_1, F'_1, p)] \geq 0 \) for all \( p \), since the weights \( w(p) \) are non-negative, the integrand in equation (7) is positive for all \( p \) and the integral is positive.

If on the contrary \( [G(F_0, F'_0, p) - G(F_1, F'_1, p)] \) is negative in the neighborhood of a quantile \( p_0 \), we can find a weight profile \( w(p) \) that is arbitrarily small outside this neighborhood and make the integral in negative.

A.4 Proof of Proposition 4

Proof. We use the same type of proof argument as in Aaberge (2009). As a consequence of the dominance hypothesis, we have:

\[ \forall W \in R^2, \forall \pi \int_0^1 w(p)F^{-1}_\pi(p)dp > \int_0^1 w(p)F'^{-1}_\pi(p)dp \]

Consequently, for all \( W \in R^2 \), we can write:

\[ \Delta W(F_\pi, F'_\pi) = \int_0^1 w(p)G(F_\pi, F'_\pi, p)dp \]

Hence, \( \forall W \in R^2 \) we have:

\[ \Delta W(F_0, F'_0) - \Delta W(F_1, F'_1) = \int_0^1 w(p)[G(F_0, F'_0, p) - G(F_1, F'_1, p)]dp \tag{8} \]

It is possible to integrate (8) by parts once,

\[ \Delta W(F_0, F'_0) - \Delta W(F_1, F'_1) = w(1) \int_0^1 \left[G(F_0, F'_0, p) - G(F_1, F'_1, p)\right]dp \]

\[ + \int_0^1 (-1)w'(p) \int_0^p \left[G(F_0, F'_0, t) - G(F_1, F'_1, t)\right] dt dp \]

By \( W \in R^2 \) then \( w(1) = 0 \) and the first term disappears. By \( w'(p) \leq 0 \) for all \( p \) makes
\[ \int_0^p [G(F_0, F'_0, t) - G(F_1, F'_1, t)] dt \] sufficient for (8). Moreover, Lemma 1 in Aaberge (2009) gives the necessary part.

A.5 Proof of Proposition 5

Proof. We use the same type of proof argument as in Aaberge (2009). As a consequence of the dominance hypothesis, we have:

\[ \forall W \in \mathcal{R}^k, \forall \pi \int_0^1 w(p)F^{-1}_\pi(p)dp > \int_0^1 w(p)F'^{-1}_\pi(p)dp \]

Consequently, for all \( W \in \mathcal{R}^k \), we can write by proposition 8:

\[ \Delta_W(F_\pi, F'_\pi) = \int_0^1 w(p)G(F_\pi, F'_\pi, p)dp \]

Hence, \( \forall W \in \mathcal{R}^k \) we have:

\[ \Delta_W(F_0, F'_0) - \Delta_W(F_1, F'_1) = \int_0^1 w(p)[G(F_0, F'_0, p) - G(F_1, F'_1, p)]dp \quad (9) \]

It is possible to integrate (9) by parts \( k \) times,

\[
\begin{align*}
\Delta_W(F_0, F'_0) - \Delta_W(F_1, F'_1) &= w(1) \int_0^1 [G(F_0, F'_0, p) - G(F_1, F'_1, p)] \\
&\quad + \sum_{j=1}^i (-1)^j \frac{d^j w(1)}{dp^j} \left[ G(\Lambda^k_0, \Lambda^k_0, 1) - G(\Lambda^k_1, \Lambda^k_1, 1) \right] \\
&\quad + (-1)^i \int_0^1 \frac{d^i w(p)}{dp^i} \left[ G(\Lambda^k_0, \Lambda^k_0, p) - G(\Lambda^k_1, \Lambda^k_1, p) \right] dp
\end{align*}
\]

By \( W \in \mathcal{R}^k \) then \( w(1) = 0 \) and \( \frac{d^i w(1)}{dp^i} = 0 \) for all \( j \leq i \) and the first term disappears. Thus follows that the conditions for \( W \in \mathcal{R}^k \) make \( G(\Lambda^k_0, \Lambda^k_0, 1) - G(\Lambda^k_1, \Lambda^k_1, 1) \geq 0 \) sufficient for (9). Moreover, Lemma 1 in Aaberge (2009) gives the necessary part.

A.6 Proof of Proposition 6

Proof. As already noted:

\[ \Delta_W(F_0, F'_0) \geq \Delta_W(F_1, F'_1) \iff u^{-1}(W(F_0)) - u^{-1}(W(F'_0)) \geq u^{-1}(W(F_1)) - u^{-1}(W(F'_1)) \]

We first prove that the condition cannot be satisfied if \( W(F_1) - W(F'_1) \geq \int_0^p [W(F_0) - W(F'_0)] dt \).

For any function \( \phi(x) \) defined on \( \mathbb{R} \) with \( \phi' > 0 \) and \( \phi'' > 0 \), and for all \( d > 0 \), the function \( \psi(x) \) defined by \( \psi(x) = \phi(x + d) - \phi(x) \) is an increasing function of \( x \).

\( u^{-1} \) is increasing and convex. We have \( W(F'_1) \geq W(F'_0) \). Let \( dW = W(F_0) - W(F'_0) \). We have:

\[ u^{-1}(W(F'_0)+dW) - u^{-1}(W(F'_1)) > u^{-1}(W(F'_0)) - u^{-1}(W(F'_1)) \]

Hence, \( W(F_1) - W(F'_1) \geq W(F_0) - W(F'_0) \Rightarrow u^{-1}(W(F'_1) + dW) - u^{-1}(W(F'_1)) > \]

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\( u^{-1}(W(F_0)) - u^{-1}(W(F_0')) \).

The reciprocal is however not true as illustrated by the graphical argument in Figure 2.

B  Hypothesis, test statistics and limiting distributions

This section develops the distribution theory which is used to test for inverse stochastic dominance and gap dominance, as prescribed by the ezOP algorithm. We firstly derive the asymptotic properties distributions and asymptotic variance covariance matrices for empirical cumulative distribution functions, empirical generalized Lorenz curves and their integrals up to order \( k \). We then exploit the asymptotic results to derive asymptotic distributions and standard errors of some test statistics, which allow to test for dominance.

B.1 Settings

We focus on a situation in which we have independence samples, possibly with different sample sizes, from the distributions discussed in the previous sections. Let \( X \) define a random variable with cumulative distribution function \( F \) and inverse \( F^{-1} \). Let \( \{X_i\}_{i=1}^n \) be a sequence of independent random variables with common distribution \( F \), and let \( \hat{F} \) be the corresponding empirical distribution function obtained from a sample of size \( n \).

In this section, we reduce notation by using \( \pi_{F_1} \) and \( \pi_{F_2} \) as the distribution functions corresponding to \( F(\cdot|c = 1, \pi) \) and \( F(\cdot|c = 2, \pi) \) respectively, where \( c = 1 \) and \( c = 2 \) are taken to be two different circumstances in the set \( C \). The empirical counterparts can be estimated through \( \hat{\pi}_{F_1} \) and \( \hat{\pi}_{F_2} \) respectively, with possibly \( \pi_{n1} \neq \pi_{n2} \). Using a similar notation, we define \( \pi_{\Lambda_1^k}(u) \) and \( \pi_{\Lambda_2^k}(u) \) as the \( k \)-th integral of \( \pi_{F_1^{-1}} \) and \( \pi_{F_2^{-1}} \), respectively, evaluated at percentile \( u \in [0, 1] \). The empirical counterparts of the two processes are given by \( \hat{\pi}_{\Lambda_1^k}(u) \) and \( \hat{\pi}_{\Lambda_2^k}(u) \) with possibly differing sample sizes.

We omit the lowerscript \( \pi \) when it is clear that comparisons are made under the same policy regime \( \pi \). Through this section we focus on comparison of either two outcomes distributions conditional on two (comparable) varieties under the same policy regime. Alternatively, we extend the same comparison to two different policy regimes. The style of the presentation of our arguments is closely related to Dardanoni & Forcina (1999).

B.2 Useful convergence results

Since the parametric form of \( F \) is not known, it is natural to use the empirical distribution function \( \hat{F} \) to estimate \( F \) and to use \( \hat{\Lambda}^k(u) \) to estimate \( \Lambda^k(u) \). Following Muliere & Scarsini (1989), one gets the following equivalent representation of the \( k \)-th integral of the GL curve:

\[
\hat{\Lambda}^k(u) = \frac{1}{(k-2)!} \int_0^u (u-t)^{k-2} \hat{F}^{-1}(t)dt, \quad u \in [0, 1], \quad k = 1, 2, \ldots \quad (10)
\]

Since \( \hat{F} \) is a consistent estimator of \( F \), \( \hat{\Lambda}^k(u) \) is a consistent estimator of \( \Lambda^k(u) \).

\[\text{The inverse empirical distributions can be estimated by referring to the quantile regression method studied by Havnes & Mogstad (2010). In this case, one obtains at any percentile the estimated outcome conditional on observable circumstances for the treatment group (\( \pi = 1 \)) and the control group (\( \pi = 0 \). We exploit the empirical distribution to perform the test.}\]
We base our study on the empirical process \( \hat{Q}(u) \), defined as:

\[
\hat{Q}(u) = \sqrt{n} \left( \hat{F}^{-1}(u) - F^{-1}(u) \right).
\]

In order to construct the limiting distribution and asymptotic variances of \( \hat{\Lambda}^k(u) \), we also consider the \( k \)-th integral of the process \( \hat{Q}(u) \):

\[
\hat{Q}^k(u) = \sqrt{n} \left( \hat{\Lambda}^k(u) - \Lambda^k(u) \right) = \frac{1}{(k-2)!} \int_0^u (u-t)^{k-2} \hat{Q}(t)dt.
\]

In a recent paper, Aaberge, Havnes and Mogstad (2011) have shown that if \( F \) has a continuous nonzero derivative \( f \) on a finite support, then \( \hat{Q}^k(u) \) converges in distribution to the process \( Q^k(u) = \frac{1}{(k-2)!} \int_0^u (u-t)^{k-2} W_0(t) f(F^{-1}(t)) dt \) where \( W_0(t) \) is the Brownian bridge on \([0,1]\), a Gaussian process whose covariance function is \( s(1-t) \), for \( 0 \leq s \leq t \leq 1 \). Equivalently, the process \( Q^k(u) \) has the same probability distribution as the Gaussian process

\[
\sum_{j=1}^{\infty} h_j(u)Z_j, \quad \forall u \in [0,1]
\]

where \( h_j(u) \) is given by

\[
h_j(u) = \frac{1}{(k-2)!} \int_0^u (u-t)^{k-2} \sqrt{2 \frac{\sin(j\pi t)}{j\pi}} f(F^{-1}(t)) dt.
\]

and \( \{Z_j\}_{j=1}^{\infty} \) is a collection of independent \( \mathcal{N}(0,1) \) variables. This gaussian process has covariance function \( \text{Cov}(Q^k(u), Q^k(v)) = \sigma[u, v] \) for \( u, v \in [0,1] \). By applying Fubini’s theorem and a change in variable operations, it is clear that the covariance function admits the following representation:

\[
\sigma[u, v] = \text{Cov} \left[ Q^k(u), Q^k(v) \right] = \sum_{j=1}^{\infty} h_j(u)h_j(v)
\]

\[
= \frac{1}{((k-2)!)^2} \int_0^{F^{-1}(u)} \int_0^{F^{-1}(v)} [(u - F(x))(v - F(y))]^{k-2} F(x)(1 - F(y))dx \, dy.
\]

Consider the case in which the processes \( Q^k(.) \) and \( \hat{Q}^k \) are defined only on a finite number \( m \) of percentiles levels \( 0 < u_1 \leq \ldots \leq u_m \leq 1 \). This occurs if, for instance, the distribution between percentiles is assumed to be uniformly distributed. Empirical distribution are often estimated for a finite number of percentiles. It is therefore possible to reconstruct all the objects that we have shown by substituting the integrals with

\[\text{The formula makes use of an alternative representation of the covariance function for the process } W_0(t), \text{ that is:}\]

\[
s(1-t) = 2 \sum_{j=1}^{\infty} \frac{\sin(j\pi s)\sin(j\pi t)}{(j\pi)^2},
\]

This formula turns out to be useful in the determination of the asymptotic variance of the test statistics we describe in this section of the paper.
summations over ordered percentiles.

We make use of a more compact vectorial notation to represent the stochastic process $Q^k(u)$ at any percentile $u \in \{u_1, \ldots, u_m\}$. Hence, $Q^k := (Q^k(u_1), \ldots, Q^k(u_m))^t \in \mathbb{R}^m$, where upperscript $t$ is for transpose. The variance-covariance matrix for $Q^k$, denoted by $\Sigma^k$ with size $m \times m$, is constructed as follows:

$$\Sigma^k := \text{Cov}[Q^k, Q^k] = E\left[Q^k \cdot (Q^k)^t\right].$$

A typical element of $\Sigma^k$ corresponding to percentiles $u_i, u_j$ is denoted by the covariance function $\sigma[u_i, u_j]$, which is equivalently represented by expression (11).

The empirical processes can also be represented in compact vector notation for all $u \in \{u_1, \ldots, u_m\}$:

$$\hat{Q}^k = (\hat{Q}^k(u_1), \ldots, \hat{Q}^k(u_m))^t \in \mathbb{R}^m.$$  

We also write $\Lambda^k$ for the $m \times 1$ vector of the $k$-th order integral $GL$ ordinates for the population under analysis, with $\hat{\Lambda}^k$ being the corresponding vector of sample estimates. By exploiting the previous results, since $\hat{F}$ is a consistent estimator of $F$, then

$$\hat{\Lambda}^k \quad \text{is asymptotically distributed as} \quad \mathcal{N}\left(\Lambda^k, \frac{\Sigma^k}{n}\right).$$  

A consistent estimator of the asymptotic variance of $\hat{\Lambda}^k$ is given by $\hat{\Sigma}^k$, whose element $\hat{\sigma}[u_i, u_j]$ is obtained by replacing $F$ with the empirical counterpart $\hat{F}$ in (11). Since $\hat{F}$ converges to $F$, $\hat{\sigma}[u_i, u_j]$ converges to $\sigma[u_i, u_j]$ as $n$ grows.

### B.3 Testing for inverse stochastic dominance

The objective of the section is to propose an empirical test and its limiting distribution for inverse stochastic dominance at order $k$, constructed for any pair of distributions $F(.|c = 1, \pi)$ and $F(.|c = 2, \pi)$ under the same policy regime. The test allows to implement step $k(c, c', e, \pi)$ in the implementation algorithm for ezOP.

The general test for inverse stochastic dominance would require a null and alternative hypothesis formulated as in Barrett & Donald (2003). We take therefore inverse stochastic dominance at order $k$ ($F_1 \succ_{SDK} F_2$) as the null hypothesis, and intersection of the $k$-th order integrals of the quantile function as the alternative. Hence:

$$H_0^k : \Lambda^k_{1}(u) \geq \Lambda^k_{2}(u) \quad \text{for all} \ u \in [0, 1];$$

$$H_1^k : \Lambda^k_{1}(u) < \Lambda^k_{2}(u) \quad \text{for some} \ u \in [0, 1].$$

One can easily test for equality by reversing the role of $c = 1$ and $c = 2$ and testing if dominance is accepted also in this case.

In practice, dominance can be tested only for a finite number of percentiles $\{u_1, \ldots, u_m\}$. We take a similar stance as in Dardanoni & Forcina (1999) and Lefranc et al. (2009), among others, by testing dominance for a finite number $m$ of linear constraint. We write $\Lambda^k_c(u_i)$ for the ordinate of the $k$-th integral of the quantile function, corresponding to the $i$-th fraction of the population with circumstances $c$. We also write $\Lambda^k_c$ for the $m \times 1$ vector of the integral quantile function ordinates for population with circumstances $c$, with $\hat{\Lambda}^k_c$ being the corresponding vector of sample estimates.

Let $\Lambda^k$ be the $2m \times 1$ vector obtained by staking the vectors $\Lambda^k_1$ and $\Lambda^k_2$, corresponding
to population with any two circumstances \( c = 1 \) and \( c = 2 \). The sample estimates are collected in \( \hat{\Lambda}^k \), and we use \( n = n_1 + n_2 \) to indicate the overall population, while \( r_c = n_c/n \) depict the relative size of the sample whose circumstances are \( c \).

The hypothesis of dominance can be reformulated as a sequence of \( m \) linear constraints placed on the vector \( \Lambda^k \). Let \( R = (I_m, -I_m) \) be the \( m \times 2m \) difference matrix, with \( I_m \) indicating the \( m \times m \) identity matrix. Define the parametric vector \( \delta_k \in \mathbb{R}^m \) as:

\[
\delta_k = R \Lambda^k
\]

We maintain the (nontestable) assumption that \( F_1 \) and \( F_2 \) can be described by independent processes. The various hypothesis of dominance or equality can be written in terms of linear inequalities involving \( \delta_k \). By exploiting the result in (12), the hypothesis of interest specify restrictions of an asymptotic multivariate normal variable:

\[
\sqrt{n} \hat{\delta}_k = \sqrt{n} R \hat{\Lambda}^k \quad \text{is asymptotically distributed as} \quad \mathcal{N} \left( \sqrt{n} R \Lambda^k, \Omega \right) \quad (13)
\]

where \( \hat{\delta}_k \) denotes the sample estimate of \( \delta_k \), and

\[
\Omega = R \text{diag} \left( \frac{\Sigma^k}{r_1}, \frac{\Sigma^k}{r_2} \right) R^t
\]

The empirical estimator of the asymptotic variance, \( \hat{\Omega} \), is obtained by plugging \( \hat{\Sigma}_c^k \) in the previous formula. We exploit the convergence result to test the discretized versions of the dominance or equality hypothesis, which is defined on the set of proportions \{\( u_1, \ldots, u_m \}\).

**B.3.1 Testing equality**

In the case of equality testing, the null and alternative hypothesis concerning the \( k \)-th integration order can be stated as follows:

\[
H^k_0: \delta_k = 0 \quad H^k_1: \delta_k \neq 0
\]

Under the null hypothesis, it is possible to resort to a Wald test static \( T^k_1 \):

\[
T^k_1 := n \delta_k^t \hat{\Omega}^{-1} \hat{\delta}_k.
\]

Given the convergence results in (13), the asymptotic distribution of the test \( T^k_1 \) is \( \chi^2_m \). The p-value tabulation follows the usual rules.

**B.3.2 Testing dominance**

In the case of strong dominance testing, such that \( F_1 \succ_{ISDk} F_2 \), the null and alternative hypothesis concerning the \( k \)-th integration order can be stated as follows:

\[
H^k_0: \delta_k \in \mathbb{R}_+^m \quad H^k_1: \delta_k \not\in \mathbb{R}_+^m
\]

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The Wald test statistics with inequality constraints has been developed by Kodde & Palm (1986). For this set of hypothesis, the test statistics $T^k_2$ is defined as:

$$T^k_2 = \min_{\delta_k \in \mathbb{R}^m_+} \left\{ u \left( \delta_k - \delta_k \right)' \hat{\Omega}^{-1} \left( \delta_k - \delta_k \right) \right\}.$$  

Kodde & Palm (1986) have shown that the statistic $T^k_2$ is asymptotically distributed as a mixture of $\chi^2$ distributions, since $\delta_k$ is consistently estimated by $\hat{\delta}_k$:

$$T^k_2 \sim \chi^2 = \sum_{j=0}^m w \left( m, m-j, \hat{\Omega} \right) \Pr \left( \chi^2_j \geq c \right),$$

with $w \left( m, m-j, \hat{\Omega} \right)$ the probability that $m-j$ elements of $\delta_k$ are strictly positive.

To test the reverse dominance order, that is $F_2 \succ_{ISDk} F_1$, it is sufficient to replace $-\hat{\delta}_k$ and $-\delta_k$ for their positive counterparts.

### B.4 Testing dominance in the Gap curve

We extend the previous analysis on stochastic dominance by constructing a test for dominance at order $k$ in the difference in gap curves, defined by $(\Lambda_k^1(u) - \Lambda_k^2(u))$ for all $u \in [0,1]$. It is assumed in this section that there exists a degree of inverse stochastic dominance, $\kappa$, for which $\pi F_1 \succ_{ISDk} \pi F_2$ for all $\pi$. Therefore, it makes sense to perform the test only for $k \leq \kappa$. The null hypothesis takes the form:

$$H^k_0 : 0 \Lambda_k^1(u) - 0 \Lambda_k^2(u) \geq 1 \Lambda_k^1(u) - 1 \Lambda_k^2(u) \quad \text{for all } u \in [0,1];$$

$$H^k_1 : 0 \Lambda_k^1(u) - 0 \Lambda_k^2(u) < 1 \Lambda_k^1(u) - 1 \Lambda_k^2(u) \quad \text{for some } u \in [0,1].$$

One can easily test for equality by reversing the role of $c = 1$ and $c = 2$ and testing if inverse dominance at order $k$ is accepted also in this case.

In practice, dominance can be tested only for a finite number of percentiles $\{u_1, \ldots, u_m\}$. We follow the previous section in defining $\Lambda_k^k$ as the $4m \times 1$ vector obtained by staking the vectors $0 \Lambda^k$ and $1 \Lambda^k$ in this precise order. The element $\pi \Lambda_k^k(u_i)$ depict the $k$-th integral of the quantile function corresponding to the $i$-th fraction of the population with circumstances $c$ under policy $\pi$. The sample estimates are collected in the $4m \times 1$ vector $\hat{\Lambda}_k^k$, and we use $n = n_1 + n_2 + \frac{1}{2} n_1 + \frac{1}{2} n_2$ to depict the overall sample size with circumstances $c = 1$ and $c = 2$ under $\pi = 0$ and $\pi = 1$, while $x_{R_t} = x_{n_c}/n$ is the relative size of each subsample.

The hypothesis of dominance in the differences of the gap curve can be reformulated as a sequence of $m$ linear constraints placed on the vector $\Lambda_k^k$. Let $R_G = (R_c, -R_c)$ be the $m \times 4m$ difference matrix, where $R$ is defined as above. Define the parametric vector $\gamma_k = R_G \Lambda_k^k$.

We maintain the (nontestable) assumption that $\pi F_1$ and $\pi F_2$ can be described by independent processes for all $\pi$. Moreover, we introduce the assumption of independence between $\pi F_c$ and $\pi F_c$ for all $c$. This latter assumption is verified when the sampling scheme is based upon randomized assignment to treatment and control groups. The various hypothesis of dominance or equality can be written in terms of linear inequalities involving $\gamma_k$. By exploiting the result in [12], the hypothesis of interest specify restrictions of an
asymptotic multivariate normal variable:
\[
\sqrt{n} \hat{\gamma}_k = \sqrt{n} R_G \hat{A}_G^k \text{ is asymptotically distributed as } \mathcal{N} \left( \sqrt{n} R_G A_G^k, \Phi \right) \tag{14}
\]
where \( \hat{\gamma}_k \) denotes the sample estimate of \( \gamma_k \), and
\[
\Phi = R_G \text{ diag } \left( \frac{\Sigma_1^k}{0 r_1}, \frac{\Sigma_2^k}{0 r_2} \right) R_G^t.
\]
The empirical estimator of the asymptotic variance, \( \hat{\Phi} \), is obtained by plugging \( \hat{\Sigma}_c^k \) in the previous formula. We exploit the convergence result to test the discretized versions of the dominance or equality hypothesis, which is defined on the set of proportions \( \{u_1, \ldots, u_m\} \).

**B.4.1 Testing equality in gap curve difference**

In the case of equality testing, the null and alternative hypothesis concerning the \( k \)-th integration order can be stated as follows:
\[
H_0^k : \gamma_k = 0 \quad H_1^k : \gamma_k \neq 0
\]
Under the null hypothesis, it is possible to resort to a Wald test static \( G_{T_1}^k \):
\[
G_{T_1}^k := n \hat{\gamma}_k^t \hat{\Phi}^{-1} \hat{\gamma}_k.
\]
Given the convergence results in (14), the asymptotic distribution of the test \( G_{T_1}^k \) is \( \chi^2_m \). The p-value tabulation follows the usual rules.

**B.4.2 Testing dominance in gap curves difference**

In the case of strong dominance testing, such that \( G(0 \Lambda_1^k, 0 \Lambda_2^k, u) \geq G(1 \Lambda_1^k, 1 \Lambda_2^k, u) \) for all \( u \in [0, 1] \), the null and alternative hypothesis concerning the \( k \)-th integration order can be stated as follows:
\[
H_0^k : \gamma_k \in \mathbb{R}_+^m \quad H_1^k : \gamma_k \not\in \mathbb{R}_+^m
\]
The Wald test statistics with inequality constraints has been developed by Kodde & Palm (1986). For this set of hypothesis, the test statistics \( G_{T_2}^k \) is defined as:
\[
G_{T_2}^k = \min_{\gamma_k \in \mathbb{R}_+^m} \left\{ n (\hat{\gamma}_k - \gamma_k)^t \Phi^{-1} (\hat{\gamma}_k - \gamma_k) \right\}.
\]
Kodde & Palm (1986) have shown that the statistic \( T_2^k \) is asymptotically distributed as a mixture of \( \chi^2 \) distributions, since \( \gamma_k \) is consistently estimated by \( \hat{\gamma}_k \):
\[
T_2^k \sim \chi^2 = \sum_{j=0}^m w \left( m, m - j, \hat{\Phi} \right) \Pr \left( \chi_j^2 \geq c \right),
\]
with \( w \left( m, m - j, \hat{\Phi} \right) \) the probability that \( m - j \) elements of \( \gamma_k \) are strictly positive.

To test the reverse dominance order, that is \( G(1 \Lambda_1^k, 1 \Lambda_2^k, u) \geq G(0 \Lambda_1^k, 0 \Lambda_2^k, u) \) for all \( u \in [0, 1] \), it is sufficient to replace \( -\hat{\gamma}_k \) and \( -\gamma_k \) for their positive counterparts.
C Figures and Tables

Figure 1: Proof of proposition 1. The curves $\Delta F^{-1}(p)$ (solid black) and $\tilde{\Delta F}^{-1}(p)$ (dashed red).

Figure 2: Proof of proposition 6. $[W(F_0) - W(F'_0)] - [W(F_1) - W(F'_1)] \geq 0 \iff \Delta W(F_0, F'_0) - \Delta W(F_1, F'_1) \geq 0$.