Dynamic analysis of reducing public debts in an endogenous growth model with public capital.

Noritaka Maebayashi* Takeo Hori† Koichi Futagami‡

Abstract

We construct an endogenous growth model with productive public capital and government debt when government debt is adjusted to the target level. We examine how reducing public debts in the economy with large public debts affects transition of the economy and welfare. We find that the government faces the following trade-off when reducing its debt. In the short run, public investments are declined to reduce the debt and this causes negative effect on welfare. However, as the interest payment of debts becomes smaller, public investment begins to increase. Eventually, the government can extend the volume of public investment more than before. This has positive effect on welfare. We find that reducing the debt is welfare-improving. Furthermore, we find that the adjustment speed of reducing debts crucially affects welfare. The relationships between the welfare gains and the adjustment speed take U-shapes in many cases. However, they are monotonically decreasing when the tax rate is low and initial debt-GDP ratio is sufficiently large.

1 Introduction

Recently, many developed countries have been suffering large government debts. In Europe, Greece has experienced severe government financial failure. Public debt as the share of GDP in Greece amounts to 142.8 percent in 2010. The average level of the public debt in the European Union (EU) is close to 80 percent of GDP in 2010.¹ In the United States (US), the level is around 62 percent of GDP in 2010. In Japan, it amounts to 225.8 percent. If any measures are not taken and public debts keep increasing, these countries may become the same situation as in Greece.

In EU, there is a constraint on the size of government debts to keep away from government financial failure. The Maastricht Treaty states that EU Member States are required

¹The source of this data is Eurostat owned by European Commission. The debt-GDP ratios of some main countries of EU are following. Austria; 72.3%, Belgium; 96.8%, Denmark; 43.6%, Finland; 48.4%, France; 81.7%, Germany; 83.2%, Hungary; 80.2%, Ireland; 96.2%, Italy; 119%, Netherlands; 62.7%, Poland; 55%, Portugal; 93%, Spain; 60.1%, and United Kingdom; 80%.
to ensure that their government debts should be below 60 percent of GDP. However, many countries in EU violate this criterion. Then, European Commission has given the Member States projections for fiscal consolidations. In *Public finance in EMU – 2011* provided in 2011, European Commission states that the public deficits must shrink faster and also public debts should be forecasted to its inflection point in 2012, and to begin falling as a share of GDP from then on. In US, U.S. President Barack Obama sent a blueprint for how they can reduce public deficit and pay down public debt to the Congress in September 2011. He presented the plan to realize more than 3 trillion in net deficit reduction over the next 10 years. In his statement, the debt under this plan would be on a declining path as a share of the economy over the next decade, falling from a high of 77 percent of GDP in 2013 to 73 percent of GDP in 2021. In the trend of restructuring of public finance in many developed countries, it is important to investigate how reducing public debts affects economic growth and welfare.

In order to reduce the debt, there are mainly two ways. One is increasing the tax rate. The other is spending cut. Some countries insist that spending cuts should be chosen, and others insist they should use a mix of spending cuts and tax increases. In US, the president plans to choose both. On the other hand, in EU, European Commission proposes not to use tax increases but spending cut for their fiscal consolidations because evidence from the past indicates that expenditure based consolidations (spending cuts) tends to have greater success. In November 2011, Germany, Netherlands, Romania and the United Kingdom expressed spending cuts whereas France, Italy, Ireland, Greece, Portugal and Spain unveiled tax increases in addition to spending cuts.

This paper examine how reducing public debts affects economic growth and welfare when the government implement spending cuts which are commonly discussed in US and many European countries. When the governments reduce their debt levels by decreasing public spending, they must face the trade-off as follows. In the short run, a decline of public spending worsens welfare level because the public services or infrastructures which are necessary for economic activities are reduced. However, a decrease in public debt reduces its interest payment and hence it enables the government to spend more on public spending in the long run. This increases welfare level in the long run. Therefore, it is important to investigate whether reducing public debt by cutting public spending is really welfare-improving. If it is welfare-improving, should the government reduce its debt at slower pace or at more rapid pace? If the pace of the reduction is slow, the initial decline of public spending may be small. However, because the government takes a longer time to reduce its debts up to the target level, the declines of public spending may prolong for a longer time. On the other hands, the rapid reduction of the debts may result in large initial declines of public spending. However, the deep declines may be temporal and end up soon because the government takes a shorter time to reduce its debts. To clear out the relationships between welfare and the speed of reducing the debts is helpful for policy making of countries with large public debts.

Furthermore, even though there may be many kinds of government spending which the government can cut, we discuss the spending cut of public investment in this paper. This is because in some countries, public investments are decreased to reduce the debts. In the article published by European Union, Rubianes (2010) writes “In the event that the necessary adjustment of public finance is made at the expense of public investment, it could even fall below the current level and compromise competitiveness and future gains
of productivity over the medium and long run.” In addition, when the government holds too much debts, private investments in the economy are crowded out severely. In this situation, the public investment may not contribute to growth enough if there is the complementarity in goods production between the private and public capital\(^2\). Then, inefficient public investment financed by public debts may only increase interest payments for debts and lead to large outstanding debts. Therefore, spending cut in public investment to reduce the debts may be necessary to improve the efficiency of public investment.

We construct an endogenous growth model with public debt finance, where the growth engine is productive public capital as in Futagami et al. (1993) and Turnovsky (1997). In order to investigate welfare effects of reducing debts, we introduce the debt policy rule introduced by Futagami, Iwaisako and Ohdoi (2008). Under the rule, the government sets the target level of the debt-private capital ratio and reduces its debt gradually to the target level. The adjustment of reducing public debt is done by spending cut of public investment.

In this paper, we obtain the following three main results from numerical analyses. First, we obtain the transitional behavior of the economy when the governments reduce public debt. In the short run, reducing public debts decreases the growth rate of public capital and this enables more resources to be allocated to private investment. Therefore, the growth rate of private capital initially increases. As a result, the return from private capital (the interest rate) falls down and then the growth rate of consumption declines in the early stages of transition. However, as the public debt size becomes small, public investment begins to increase and this, in turn, crowds out private investment. Then, the growth rate of private capital begins to decline. This decline of private investment increases the interest rate and then the growth rate of consumption begins to recover. In the long run, the economy can attain higher growth rate levels of public capital, private capital and consumption than those in the state before the governments reduce debts. The reason of this is as follows. Because government debt is reduced, the interest payment becomes smaller than the initial state. Hence, public investment becomes larger than before the governments reduce debts, which makes it possible to attain higher economic growth in the long run.

Second, we find that reducing the debt at the expense of public investment improves welfare. In the short run, households change their consumption level right after the policy change. Whether this short-run effect becomes positive or negative depends on how the expected future income changes due to the policy change. Next, when the government reduces its debt, the growth rate of consumption decreases at the early stage of transition as we state above. This is the negative welfare effect during the transition. However, in the long run, the growth rate becomes higher than the initial level, which is the positive welfare effect in the long run. Moreover, countries whose initial debt-GDP ratio is larger can get the larger welfare gains. This is mainly because when the initial debt-GDP ratio is large, the growth rate of consumption in the new steady state is much higher than that in the initial steady state. In addition, although they have to reduce larger public debts and decline of public investment is deeper, more resources are released to private sector and the household can consume more.

Third, the adjustment speed of reducing debts crucially affects welfare. How the welfare

\(^2\)Futagami, Morita and Shibata (1993) and Turnovsky (1997) apply the complementarity between the private and public capital in goods production.
gains after decreases of government debts change as the governments increase the speed of reducing debts depends on the tax rate and how much the debt-GDP ratio initially exceed the target level. In many cases, (i) the relationships between the welfare gains and the adjustment speed take U-shapes. However, (ii) the welfare gains monotonically decrease in the adjustment speed when the tax rate is low and initial debt-GDP ratio is sufficiently large. If the governments increase the speeds of reducing public debts, they face the following trade-off. On the one hand, the initial decline of public investment becomes larger. On the other hand, the growth of consumption recovers sooner. When the tax rate is low and initial debt-GDP ratio is large when the government increases the adjustment speed. Then, the welfare gains monotonically decrease in the adjustment speed. In contrast, when the tax rate is high or the initial debt-GDP ratio is small, the initial decline of public investment becomes small. Then, when the adjustment speed is somewhat high, the positive effect of the sooner recovery of the growth of consumption dominates the negative effect of the initial decline of public investment. Thus, the relationships between the welfare gains and the adjustment speed take U-shapes.

Besides these main results, we state the possibility that the economy fails sustainable development. When the initial level of public capital is sufficiently small relative to private capital, if the public finance largely relies on issuing debts, the economy may fall into the development trap. In this trap, there exists no path the representative households with rational expectation and perfect foresight can trace. This happens because the tax revenue becomes too small due to scarce public capital to pay interest payments of debts. We show that the economy can avoid this situation by selecting tax finance rather than debt finance in its developing process.

In the theoretical literature, many studies examine fiscal policy of productive public spending with debt finance such as Greiner and Semmler (2000), Ghosh and Mourmouras (2004) and Yakita (2008).3 However, these studies do not deal with the debt size criterion but the deficit constraint in the Maastricht treaty.4 In contrast to these studies, Futagami et al. (2008) introduce the target rule of public debt size as we state above, where public debt are issued to finance productive public flow services like in Barro (1990). They show that there exist two steady states and the indeterminacy of transition paths exists.5 Because the reduction of public debts generates transitional dynamics, the occurrence of indeterminacy of transition paths causes difficulties in doing welfare analysis. Therefore, they do not investigate welfare effects of reducing public debts. We get over this problem by replacing the flow of public service with the stock of public capital. In this paper,

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3Greiner et al, (2000) studies how fiscal methods of debt issuance affect growth in the long run. They introduce the golden rule of public finance (GRPF). Under the rule, public debt is issued only for public investment and government consumption, the transfer and the interest payment must be financed by the tax revenue and the deficits is assumed constant rate of GDP. They show a less strict policy rule does not necessarily promote growth. Ghosh et al, (2004) show GRPF can avoid over-accumulation of public capital and be more efficient than the standard constraint of government. Yakita (2008) finds the threshold that determines whether government debt is sustainable or not in an overlapping generations model.

4By the Maastricht treaty, EU Member States are required to ensure the public deficits do not exceed 3 percent of GDP.

5Minea and Villieu (2012) find that if the governments set the the debt target defined as a ratio of bonds to GDP not to private capital as in Futagami et al. (2008), there exists a unique steady state and the indeterminacy is removed.
economically meaningful steady state is unique and then the transition path is uniquely determined. Therefore, we obtain more fruitful results as we state above: welfare effects and transitional responses of reducing the debt.

The remainder of this paper is organized as follows. Section 2 provides the basic model. Section 3 derives the equilibria and transition dynamics of the economy. Section 4 does comparative statics analyses of the equilibria and analyzes the policy effects on the economic growth and public investment. Section 5 examines transitional responses when the government lowers the target level of debts. Section 6 presents the policy effects on the welfare. Section 7 concludes the paper.

2 The model

Our model is based on the model of Futagami et al. (2008). The only difference between their model and ours is that productive government input for production of goods is public capital not public flow service. We consider an economy populated by an infinitely-lived representative household who has an infinite planning horizon and possesses perfect foresight. Time is denoted as $t \geq 0$. Without any loss of generality, we assume that there is no population growth and the size of population is normalized to be unity.

2.1 Production structure

A single final good, $Y_t$, is produced with private capital, $K_t$, public capital, $K_{g,t}$, and labor, $L_t$. The production function takes the Cobb-Douglas form as follows:

$$Y_t = AK_t^\alpha (K_{g,t}L_t)^{1-\alpha}, \quad \alpha \in (0, 1). \quad (1)$$

The first-order conditions for the profit maximization are given by:

$$r_t = \alpha Ak_{g,t}^{-\alpha}, \quad (2)$$

$$w_t = (1 - \alpha) Ak_{g,t}^{-\alpha} K, \quad (3)$$

where $r_t$ and $w_t$ denote the interest rate and the wage rate, and $k_{g,t}(\equiv K_{g,t}/K_t)$ is the public capital-private capital ratio.

2.2 Households

The utility function of the representative household is specified as:

$$U_0 = \int_0^\infty (\ln C_t)e^{-\rho t} dt, \quad (4)$$

where $C_t$ and $\rho(>0)$ stand for consumption and the subjective discount rate, respectively.

The household’s budget constraint is given by:

$$\dot{W}_t = (1 - \tau)(r_tW_t + w_t) - C_t, \quad (5)$$

where $W_t$ denotes assets and $\tau \in [0, 1)$ is the income tax rate, which is assumed to be time-invariant, as in Futagami et al. (2008). Taking the interest rate, the wage rate, and
the tax rate as given, the household chooses its consumption path so as to maximize (4) subject to (5). The intertemporal maximization yields the following condition:

$$\frac{\dot{C}_t}{C_t} = (1 - \tau)r_t - \rho. \quad (6)$$

In addition, the following transversality condition must hold:

$$\lim_{t \to \infty} C_t^{-1}W_t \exp(-\rho t) = 0. \quad (7)$$

2.3 Government

The government in this economy imposes income taxation, and issues bonds, $B_t$, to finance public capital investment, $G_t$. For simplicity, we assume that the depreciation rate of public capital is zero. The evolution of public capital is given by:

$$\dot{K}_{g,t} = G_t. \quad (8)$$

The budget constraint of the government is:

$$\dot{B}_t = r_t B_t + G_t - \tau(r_t W_t + w_t). \quad (9)$$

Following Futagami et al. (2008), we suppose that the government adjusts its bonds gradually to a target level. In this paper, we gauge the size of the economy by the level of private capital, $K_t$, as in Futagami et al. (2008). We assume the government adjusts $b_t \equiv B_t/K_t$ according to the following rule:

$$\dot{b}_t = -\phi(b_t - \bar{b}), \quad (10)$$

where $\bar{b}$ and $\phi(>0)$ stand for the target level of government bonds and the adjustment coefficient of the rule, respectively. We can set the ratio of debts to GDP as the target. By using $b_t$, we can calculate the ratio of debts to GDP because $B/Y = (B/K)(K/Y) = \bar{b}/(A(k_g)^{1-\alpha})$.

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3 Equilibrium

3.1 Dynamic system

In this section, we first derive the dynamic system of the economy, and then examine the existence and the stability of the steady state equilibria in Subsections 3.2 and 3.3. Then, the expenditure based reduction of public debt is formulated as following the discussion of fiscal consolidation in Public finances in EMU–2011 and Rubianes (2010).
We derive the equilibrium of this economy by using market equilibrium conditions. The labor market equilibrium condition is $L_t = 1$ because the population size is unity and each household supplies one unit of labor inelastically. The asset market clears as $W_t = K_t + B_t$. Substituting these into (9) and using (1), (2) and (3), we obtain:

$$\dot{B}_t = (1 - \tau)r_t B_t - (\tau Y_t - G_t).$$

(11)

We assume that the depreciation rate of private capital is zero for simplicity. Then, the good market equilibrium condition is given by:

$$\dot{K}_t = Y_t - C_t - G_t.$$  

(12)

Let us define $c_t \equiv C_t/K_t$ and $g_t \equiv G_t/K_t$. By substituting (2) into (6), and using (1) and (12), we obtain:

$$\dot{c}_t = [c_t + g_t - (1 - (1 - \tau)\alpha)Ak_{g,t}^{1-\alpha} - \rho]c_t.$$  

(13)

By using (1), (8) and (12) we have:

$$\dot{k}_{g,t} = (1 + k_{g,t})g_t - Ak_{g,t}^{2-\alpha} + c_t k_{g,t}.$$  

(14)

By using (11) with (1), (2) and the definition of $b_t$, we can rewrite (10) as:

$$\dot{b}_t = -\phi(b_t - \bar{b}) = \left\{ (1 - \tau)\alphaAk_{g,t}^{1-\alpha} - \frac{\dot{K}_t}{K_t} \right\}b_t - \tau Ak_{g,t}^{1-\alpha} + g_t.$$  

(15)

Substituting (12) into (15) and solving for $g_t$, we obtain the ratio of public investment to private capital, $g_t \equiv G_t/K_t$, as follows:

$$g_t = \frac{1}{1 + b_t} \left\{ \tau(1 + \alpha b_t)Ak_{g,t}^{1-\alpha} - \alpha b_t Ak_{g,t}^{1-\alpha} - \phi(b_t - \bar{b}) + (Ak_{g,t}^{1-\alpha} - c_t)b_t \right\}. $$  

(16)

The first term of right-hand side (RHS) represents the tax revenue of the government and shows that when other things equal, $g_t$ increases as $\tau$ rises. The second term represents the interest payment of the government for its debts. When other things equal, $g_t$ increases as $b_t$ decreases. The third and the last terms are brought by the debt policy rule (10). The third term shows that given $b_t$, if the government reduces the target level of its debt $\bar{b}$, the volume of public investment shrinks in the short run. When $\phi$ takes a larger value, decreases in public investment in the short run are also larger because the government must reduce its debt more rapidly and hence its budget becomes tight. However, as $b_t$ steadily decreases due to a decrease in $\bar{b}$ under the rule given by (10), the interest payment of government debt gradually decreases. This enables the governments to spend their revenues more on public investment. Finally, the last term implies that a larger level of the ratio of public to private capital, $k_{g}$, due to a high growth rate of $K_t$, has positive effects on public investment.\(^7\) When $K_t$ grows at a high rate, the government requires little effort to reduce $b_t \equiv B_t/K_t$ to $\bar{b}$. Then, a high growth of $K_t$ enables the government to increase public investment.

\(^7\)Eq.(12) shows that given $G_t$, when the difference between $Y_t/K_t = Ak_{g,t}^{1-\alpha}$ and $C_t/K_t = c_t$ becomes large, the growth rate of private capital rises.
Substituting (16) into (13) and (14) respectively, we obtain the following dynamic system with respect to $c_t$ and $k_{g,t}$.

$$\dot{c}_t = \frac{1}{1 + b_t}[c_t - \zeta(k_{g,t}, \tau, b_t) - \phi(b_t - \tilde{b})]c_t,$$  \hspace{1cm} (17)

$$\dot{k}_{g,t} = \frac{1}{1 + b_t}\left\{(k_{g,t} - b_t)c_t - \eta(k_{g,t}, \tau, b_t) - \phi(b_t - \tilde{b})(1 + k_{g,t})\right\},$$  \hspace{1cm} (18)

where

$$\zeta(k_{g,t}, \tau, b_t) \equiv (1 - \tau)(1 - \alpha)Ak_{g,t}^{1-\alpha} + \rho(1 + b_t),$$

$$\eta(k_{g,t}, \tau, b_t) \equiv [(1 - \tau)(1 + \alpha b_t)k_{g,t} - \{1 - (1 - \tau)\alpha\}b_t + \tau]\cdot Ak_{g,t}^{1-\alpha}.$$  

Equations (10), (17) and (18), together with the initial values $k_{g,0}$ and $b_0$, and the transversality condition (7), characterize the dynamics of the economy.

### 3.2 Steady States

Now we derive the steady state of the economy where $c_t$, $k_{g,t}$, and $b_t$ become constant over time. Setting $\dot{c}_t = 0$ and $\dot{k}_{g,t} = 0$ and $b_t = \bar{b}$ in (17) and (18), respectively, results in:

$$c = \zeta(k_g, \tau, \bar{b}), \hspace{1cm} \text{and} \hspace{1cm} (k_g - \bar{b})c = \eta(k_g, \tau, \bar{b}).$$  \hspace{1cm} (19)

We omit time index $t$ in the above because $c$ and $k_g$ become constant over time in the steady state. By eliminating $c$ from the two equations of (19), we have:

$$[(1 - \tau)\alpha k_g - \tau]Ak_g^{1-\alpha} = \rho(k_g - \bar{b}).$$  \hspace{1cm} (20)

This equation determines the steady state value $k_g^*$. Substituting $k_g^*$ into the first equation of (19), we obtain the steady state value $c^*$. Let us denote the left-hand side (LHS) of (20) as $\Lambda(k_g)$. As shown in Figure 1, $\Lambda(k_g)$ is a convex function of $k_g$ that becomes equal to zero when $k_g = 0$ and $k_g = \frac{\tau}{(1 - \tau)\alpha}$ (see Appendix A). We denote the RHS of (20) as $\Pi(k_g)$. Apparently, $\Pi(k_g)$ is a straight line whose slope is $\rho$, and becomes equal to zero when $k_g = \bar{b}$ holds. Let us define $\tilde{k}_g$ by $\Lambda'(\tilde{k}_g) = \rho$. As shown in Appendix A, there exists a unique $\tilde{k}_g > 0$. Note that at $k_g = \tilde{k}_g$, both sides of (20) have the same slopes.

Figure 1 shows that if $\Lambda(0) > \Pi(0)$ and $\Lambda(\tilde{k}_g) < \Pi(\tilde{k}_g)$ hold, there exist two steady states. Because of $\Lambda(0) \equiv 0$, $\Pi(0) \equiv -\rho \bar{b}$ and $\bar{b} > 0$, the first inequality is satisfied. The second inequality is equivalent to:

$$\bar{b} < \tilde{k}_g - \frac{[(1 - \tau)\alpha \tilde{k}_g - \tau]Ak_g^{1-\alpha}}{\rho}. \hspace{1cm} (21)$$

If $\bar{b}$ is larger than the RHS of the above inequality, there exist no paths that the rational representative households can choose.\(^8\) Therefore, we henceforth assume (21). We denote

\(^8\) Transversality condition (7) excludes all paths if there exists no steady states. We will see it in the next subsection.
the steady state value of \(k_g\) in each steady state as \(k_{g,H}^*\) and \(k_{g,L}^*\) \((k_{g,H}^* > \hat{k}_g > k_{g,L}^* > 0)\). From (2) and (6), the long-run growth rate, \(\gamma = \dot{C}_t/C_t = \dot{K}_t/K_t = \dot{K}_g/K_g\), is given by:

\[
\gamma = (1 - \tau)\alpha A(k_g^*)^{1-\alpha} - \rho,
\]

where \(k_g^*\) is equal to \(k_{g,H}^*\) or \(k_{g,L}^*\). We can immediately confirm that the long-run growth rate increases with \(k_g^*\). This is because the return of private capital increases with \(k_g^*\) and the households save more. From now on, we call the steady state with \(k_{g,H}^*\) the high-growth steady state, whereas we call the steady state with \(k_{g,L}^*\) the low-growth steady state. From the discussion so far, we obtain the following proposition.

**Proposition 1**

*There exist two steady states, the high-growth and the low-growth steady states, if (21) holds.*

As in Futagami et al. (2008), there exists two steady states in our model. In their model, both of the two steady states are economically meaningful. In our model, only the high-growth steady state is economically meaningful as we will show in the next subsection.

### 3.3 Stability

We next examine the stability of the steady states. The dynamic adjustment of \(b_t\) is autonomously determined by (10) and it is always stable, whereas \((c_t, k_{g,t})\) evolves over time according to (17) and (18). When \(\dot{b}_t = 0 \ (b_t = \bar{b})\) holds, from (17) and (18), dynamics of \(c_t\) and \(k_{g,t}\) are obtained as:

\[
\begin{align*}
\dot{c}_t &= \frac{1}{1 + \bar{b}} [c_t - \zeta(k_{g,t}, \tau, \bar{b})]c_t, \\
\dot{k}_{g,t} &= \frac{1}{1 + \bar{b}} \{ (k_{g,t} - \bar{b})c_t - \eta(k_{g,t}, \tau, \bar{b}) \}.
\end{align*}
\]

Using these, we can depict the movement of \((c_t, k_{g,t})\) in \((c, k_g)\) space as shown in Figure 2. Depending on the value of \(\bar{b}\), the \(k_g = 0\) locus takes different shapes. As a result, there are five cases of phase diagrams. When \(\bar{b}\) is relatively small: \(\bar{b} < \frac{(1 - \tau)\alpha}{(1 - \tau)\alpha}\), we obtain the phase diagram as in Figure 2-(b). When \(\bar{b}\) equals to \(\frac{(1 - \tau)\alpha}{(1 - \tau)\alpha}\), we have two kinds of diagrams represented in Figure 2-(a). Figure 2-(a)-(i) is the case when \(k_g = \bar{b}\) is the high-growth steady state, while \(x \equiv \left\{ \frac{\rho(1+\tau(\alpha-1))}{A(1-(1-\tau)(1-\alpha))} \right\} \frac{1}{\alpha}\) is the low-growth steady state. Figure 2-(a)-(ii) is, in contrast, \(k_g = \bar{b}\) is the low growth steady state, while \(x\) is the high-growth steady state. Finally, when \(\bar{b}\) is relatively large: \(\bar{b} > \frac{(1 - \tau)\alpha}{(1 - \tau)\alpha}\), two kinds of diagrams emerge. Figure 2-(c)-(i) is the case where both of two steady state values \(k_g\) become smaller than \(\bar{b}\), whereas Figure 2-(c)-(ii) is the case where both of them become larger than \(\bar{b}\). In any cases, we have following common features as for the dynamic behaviors. The low-growth steady states is unstable, whereas there exists a saddle path converging to the high-growth steady state. The stable arm is always upward sloping. If (21) is violated, there are no steady

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9 Please see Appendix B about the shapes and positions of \(\dot{c} = 0\) and \(\dot{k}_{g,t} = 0\) loci in the phase diagrams.
states. In this case, there are no paths which satisfy transversality condition (7). The paths which go to \( k_g = 0 \) or \( c = 0 \) do not satisfy (7) obviously. The paths in which both \( k_g \) and \( c \) continue to increase are excluded because private capitals, \( K_t \), keep decreasing and become 0 eventually in those paths.\(^{10}\)

Next, we check the local stability of the system including the case of \( b_t \neq \tilde{b} \). We obtain the following proposition on the stability of the dynamic system (10), (17) and (18).

**Proposition 2**

The high-growth steady state is locally saddle-point stable, whereas the low-growth steady state is unstable.

**Proof.** See Appendix C.

Proposition 2 and the phase diagrams state the following three facts. First, although there are two steady states, only the high-growth steady state is economically meaningful. Second, it is determinate. Third, given \( b_0 \approx \tilde{b} \), the economy falls into the development trap if the initial level of public-private capital ratio \( k_{g,0} \) is smaller than \( k_{g,L}^{*} \). In the trap, there exists no path that the representative households with perfect foresight and rational expectation trace, and hence the economy cannot develop.\(^{11}\)

The intuition for the development trap is as follows: When \( k_g \) is small, production relative to private capital is small.\(^{12}\) The tax revenue of the government becomes small relative to private capital, which tightens the budget constraint of the government. Then, the public investment becomes small relative to private capital. This can be verified as follows: By using \( b_t = \tilde{b} \), (15) and (22), we can derive:

\[
g = \tau A k_g^{1-\alpha} - \rho \tilde{b}.
\]

This equation shows that in the steady state, \( g \) is small when \( k_g \) is small. A small \( g \) implies that the government can not accumulate enough public capital. Then, the economy falls into the development trap when \( k_{g,0} \) is sufficiently small.

Our results are totally different from Futagami et al. (2008), where a productive input of the government is a flow variable. The results in Futagami et al. (2008) are as follows: First, the two steady states are both stable and economically meaningful. Second, the high-growth steady state can become locally indeterminate. Finally, because the low-growth steady state is stable and economically meaningful, the economy can be stuck in

\(^{10}\)The reason of this is as follows. From (12), if private capital keeps decreasing, \( A k_{g,t}^{1-\alpha} - c_t - g_t < 0 \) holds. By using (16), This condition can be rewritten as \( c_t > (1-\tau)(1+\tilde{b})A k_{g,t}^{1-\alpha} \). Next, from (24), \( \dot{k}_{g,t} = 0 \) locus is rewritten as \( c = \{ (1-\tau)(1+\tilde{b}) + \frac{(1-\tau)A k_{g,t}^{1-\alpha}}{k_{g,b}^{1-\alpha}} (1 + \tilde{b}) \} A k_{g,t}^{1-\alpha} \). Because the paths in which both \( k_g \) and \( c \) continue to increase go over \( \dot{k}_{g,t} = 0 \) locus, they satisfy \( c_t > \{ (1-\tau)(1+\tilde{b}) + \frac{(1-\tau)A k_{g,t}^{1-\alpha}}{k_{g,t}^{1-\alpha}} (1 + \tilde{b}) \} A k_{g,t}^{1-\alpha} \). Because \( \tilde{b} > \frac{\tau}{(1-\tau)p} \) when (21) is violated, these paths satisfy \( c_t > (1-\tau)(1+\tilde{b})A k_{g,t}^{1-\alpha} \) and then \( K_t \) keeps decreasing on the paths.

\(^{11}\)The transversality condition (7) excludes any paths that go to the states that \( k_g \) becomes zero.

\(^{12}\)The production relative to private capital is given by \( Y_t/K_t = A k_{g,t}^{1-\alpha} \).
the low-growth trap. The low-growth trap in Futagami et al. (2008) occurs when the household has the pessimistic expectations that the low-growth steady state will realize. The development trap in this paper occurs when $k_{g,0}$ is smaller than $k^*_{g,L}$. Therefore, the low-growth trap in Futagami et al. (2008) has a feature different from the development trap in this paper.

In Futagami et al. (2008) where public services enter the production function, public productive expenditure immediately comes into operation in the final good production. If the government increases public productive expenditure, the final good production and also the tax revenue of the government immediately increase, which allows the government to spend more on public productive expenditure. There is a complementarity between public productive expenditure and the final good production. As a result, the development trap does not occur and equilibrium indeterminacy can arise in Futagami et al. (2008).

4 Characters of Steady States

This section analyzes effects of changes in income tax rate and in the target level of government bonds in the steady states. Because the economically meaningful steady state is unique (the high-growth steady state), we should examine the policy effect in the high-growth steady state. Because effects of policy changes on $k^*_{g,L}$ are crucial for the development trap, we also pay attention to the policy effects on $k^*_{g,L}$.

We first examine the effects of changes in $\bar{b}$. As shown in Figure 3-(a), when $\bar{b}$ rises, only the straight line $\Pi(k_g)$ that represents RHS of (20) shifts downward. As a result, $k^*_{g,L}$ rises whereas $k^*_{g,H}$ reduces. We next examine effects of changes in $\tau$. As shown in Figure 3-(b), when $\tau$ rises, only the U-shaped curve $\Lambda(k_g)$ that represents LHS of (20) shifts downward. Then, $k^*_{g,L}$ falls whereas $k^*_{g,H}$ rises. We obtain the following lemma.

Lemma

The effects of changes in policy variables $\bar{b}$ and $\tau$ are the followings;

\begin{align*}
(\text{i}) & \quad \frac{\partial k^*_{g,L}}{\partial \bar{b}} > 0, \quad \frac{\partial k^*_{g,H}}{\partial \bar{b}} < 0. \\
(\text{ii}) & \quad \frac{\partial k^*_{g,L}}{\partial \tau} < 0, \quad \frac{\partial k^*_{g,H}}{\partial \tau} > 0.
\end{align*}

This lemma tells us the followings. In the stable high-growth steady state, increases in $\bar{b}$ ($\tau$) have negative (positive) effects on the ratio of public to private capital, $k^*_{g,L}$. In the unstable low-growth steady state, the ratio of public to private capital, $k^*_{g,L}$, increases (decreases) when $\bar{b}$ ($\tau$) increases. Hence, when the public capital is scarce relative to the private capital, the economy is more (less) likely to fall into the development trap.

When $\bar{b}$ increases, the interest payment of the government debt increases in the steady state as (25) shows. As a result, the public investment becomes small. The accumulation of public capital is depressed. Consequently, when the public capital is scarce relative to the private capital, the economy is more likely to fall into the development trap. Besides,

\footnote{In Futagami et al. (2008), the welfare level in the low-growth steady state is lower than that in the high-growth steady state.}
the depressed accumulation of public capital decreases the ratio of public to private capital, \(k_{g,H}^*\), in the high-growth steady state. When \(\tau\) increases, the tax revenue increases. The public investment relative to private capital, \(g\), increases, as (25) shows. Then, an increase in \(\tau\) has effects opposite to an increase in \(\bar{b}\).

In Futagami et al. (2008) where public services enter the production function, the economy falls into the low-growth trap when the household has the pessimistic expectations. Because it is not easy for the government to control households’ expectations, the government has difficulties in inducing the economy to escape from the low-growth trap. In contrast, in our case where stock of public capital enters the production function, the government can relatively easily induce the economy to escape from the development trap by decreasing \(\bar{b}\), or increasing \(\tau\).

We next investigate the relationship between the long-run growth rate and the main policy variable, \(\bar{b}\).\(^{14}\) From now on, we only focus on the high-growth steady state that is stable. Since the growth rate in the high-growth steady state is given by \(\gamma_H = (1 - \tau)\alpha A(k_{g,H}^*)^{1-\alpha} - \rho\), we can prove the next proposition by using Lemma-(i).

**Proposition 3**

A decrease in the target level, \(\bar{b}\), enhances the growth rate in the high-growth steady state.

In the rest of this section, we investigate the effect of \(\bar{b}\) on the ratio of public investment to private capital in the high growth steady state, \(g_H^*\). From (25) and Lemma-(i), we obtain immediately:

\[
\frac{\partial g_H^*}{\partial \bar{b}} = \tau(1 - \alpha)A(k_{g,H}^*)^{-\alpha} \frac{\partial k_{g,H}^*}{\partial \bar{b}} - \rho < 0.
\] (26)

This states the following proposition.

**Proposition 4**

A decrease in the target level, \(\bar{b}\), boosts the ratio of public investment to private capital in the high-growth steady state.

Proposition 4 states that if the government reduces \(\bar{b}\), \(g_H^*\) always increases in the long run. The reason is easy. As \(\bar{b}\) decreases, the outstanding public debt in the long run is reduced. Therefore, the interest payment is reduced and then the budget constraint of the government becomes loose. Please remember that in the short run, a decrease in \(\bar{b}\) shrinks public investment. However, this steadily decreases outstanding debt and the interest payment and can gradually increase public investment. In the long run, Proposition 4 says that the volume of public investment can become larger than that in the original state.

\(^{14}\)How the tax rate influence the growth rate in the long run may be an important issue. We obtain a tax rate which maximizes the growth rate in the high-growth steady state in this model. And we find this tax is increasing in \(\bar{b}\).
5 Transitional Dynamics

This section examines the transitional dynamics numerically. Our scenario is as follows: The economy is initially on a high-growth steady state with $\bar{b} = \bar{b}_{init}$, where $\bar{b}_{init}$ denotes the initial level of $\bar{b}$. At time 0, the government reduces $\bar{b}$ from $\bar{b}_{init}$ to $\bar{b}_{new}$ unexpectedly, where $\bar{b}_{new}$ is the level of $\bar{b}$ after the policy change. Then, the economy begins to move toward the new high-growth steady state along the saddle path and the transitional dynamics is generated.

In Section 4, we have examined the long-run effects of changes in $\bar{b}$: decreases in $\bar{b}$ have positive effects both on the growth rate and on public investment in the long-run. This section shows the policy effects of $\bar{b}$ in the short run and on the transition path. We also show that the transitional dynamics and the strength of policy effects in the short run and on the transition path heavily depend on the value of $\phi$ that has no effects in the long-run. Remember that $\phi$ determines the adjustment speed of $b_t$ (see (10)).

We analyse the transition paths numerically by using the relaxation algorithm. As a benchmark, we choose the parameter values as follows: The subjective time discount rate, $\rho$, is set to 0.05, which is an often-used value in the growth literature. The elasticity of production with respect to public capital, $1 - \alpha$, is set to 0.25. This value is commonly accepted in studies such as Barro (1990), Greiner (2007) and Gómez (2004) and so on. We use $\tau = 0.1$ following Greiner (2007) who studies how public investment financed by public debts affects the transitional dynamics of the economy. The value of $A$ is chosen so that the long-run growth rate of the new steady state is equal to 0.02, which results in $A = 0.1313$. The values of $\bar{b}_{init}$ and $\bar{b}_{new}$ are chosen so that the debt-GDP ratio in the initial and new steady states are equal to 80 percent and 50 percent, respectively. This yields $\bar{b}_{init} = 0.0802$ and $\bar{b}_{new} = 0.0519$. The reason of setting the initial debt-GDP ratio to 80 percent is that an average debt-GDP ratio in EU27 is almost 80 percent in 2010 as we mention in Section 1. In the recent crises of public finance in EU, selecting the case of EU may be appropriate. Because the member countries of the EU must maintain the debt-GDP ratio of less than 60 percent under the Maastricht criterion, we assume that the debt-GDP ratio in the new steady state is less than 60 percent. It is also noted that the debt-GDP ratio in the high-growth steady state is given by $B/Y = \bar{b}/(A(k_{g,Y}^{*})^{1-\alpha})$ and hence it increases with $\bar{b}$ in the high-growth steady state (see Lemma-(i)). Under these parameter values, $k_{g,H}^{*} (\equiv K_{g,Y}/K)$ is around 0.34 and 0.39 in the initial and new steady states, respectively.

The value of $\phi$ has no effects on the steady state. However, as for transitional dynamics, $\phi$ has important roles since $\phi$ governs the dynamics of $b_t$, and hence time paths of other endogenous variables are strongly affected by the value of $\phi$. We use five values: $\phi = 0.05, 0.1, 0.2, 0.3$ and 0.4. In case of $\phi = 0.05$, it takes about 25.5 years until the debt-GDP ratio reduces to around 60 percent. As $\phi$ increases, it takes shorter years. When $\phi = 0.1, 0.2, 0.3$ and 0.4, the debt-GDP ratio reduces to around 60 percent in around 13, 6.5, 4.5 and 3.5 years.

---


16 Greiner (2008) also set $\rho = 0.05$.

17 When $\phi = 0.05, 0.1, 0.2, 0.3$ and 0.4, $b_t - \bar{b}$ reduces to 50 percent of its initial size in around 14, 7,
Under these parameter values, we numerically examine the transitional dynamics generated by the reduction of $b$ from $b_{\text{init}}$ to $b_{\text{new}}$, by using the relaxation algorithm. Figure 4 presents the results. Panels (a) and (b) show that $b_t$ and the debt-GDP ratio monotonically decrease and converge to their new steady state values. As $\phi$ increases, the speeds of convergence of $b_t$ and the debt-GDP ratio increase.

Panel (c) shows that just after the policy change, $g_t(\equiv G/K)$ jumps and decreases below the initial level, and then begins to increase up to the new steady state level that is higher than the initial level. To reduce $b_t$ to $b_{\text{new}}$, the government must initially reduce its expenditure, which results in the short-run decreases in $g_t$, as shown by the third term in parentheses of (16). However, as $b_t$ steadily reduces, the interest payment of the debt of the government gradually decreases and its budget constraint gradually becomes loose. Then, $g_t$ gradually increases and eventually exceeds the initial level (see (25) and Proposition 4). When the values of $\phi$ are larger, the initial declines in $g_t$ also become larger. A large $\phi$ means that the government must reduce its debt at a higher speed and hence its budget becomes much tight just after the policy change. As a result, the government must reduce public investment in large amount. When $\phi$ is as large as 0.3 or 0.4, $g_t$ becomes even negative just after the policy change, which implies that the government has to sell its capital in order to meet its budget. Though the initial declines in $g_t$ increase with $\phi$, it takes shorter periods until $g_t$ recovers its initial level when the value of $\phi$ is large: When $\phi$ is equal to 0.05, 0.1, 0.2, 0.3 and 0.4, it takes about 14.5, 11.5, 8.5, 6.5 and 6 years until $g_t$ returns back to its initial level. Because public investment is the source of the accumulation of public capital $K_{g,t}$, the growth rates of public capital follow the dynamical paths similar to those of $g_t$ as shown in Panel (d).

In contrast to $g_t$, the growth rate of private capital $K_t$ jumps and increases beyond the initial level just after the policy change, and then decreases (see Panel (e)). After the policy change, public investment decreases as we have just discussed, which has crowding-in effects on private investment. Then, the growth rate of $K_t$ initially increases. However, as $g_t$ gradually increases, the crowding-out effects on private investment begin to prevail, which leads to the gradual declines in the growth rate of $K_t$. When $\phi$ is as large as 0.3 or 0.4, the crowding-out effects of public investment become so strong that the growth rate of $K_t$ follows the nonmonotonical transitional path.

The dynamics of the growth rates of $K_{g,t}$ and $K_t$ give rise to the non-monotonic transitional paths of $k_{g,t} \equiv K_{g,t}/K_t$, as shown in Panel (f). At the early stages of transition, $k_{g,t}$ gradually decreases because of decreased public investment and increased private investment. The larger $\phi$ is, the larger the declines in $k_g$ are. However, because of gradual increases in public investment and gradual decreases in private investment, $k_{g,t}$ starts to increase to its new steady state level in several years after the policy change. When $\phi$ becomes larger, it takes shorter time until $k_{g,t}$ begins to increase: When $\phi$ is equal to 0.05, 0.1, 0.2, 0.3 and 0.4, $k_{g,t}$ begins to increase in about 13, 10, 7, 6 and 5 years. From (2) and (6), the growth rate of private consumption becomes a function of $k_g$. Then, as shown in Panel (g), the growth rates of private consumption follow the dynamical paths similar to those of $k_{g,t}$.

3.5, 2.5 and 1.5 years, respectively.
Panel (h) provides the transitional paths of $c_t$. From the panel, we know the effects of the policy change on the initial consumption levels, $C_0$, since $C_0 \equiv c_0 k_0$ holds. The effects on $C_0$ apparently depend on the value of $\phi$. When $\phi$ is small ($\phi = 0.05$), $C_0$ decreases in reaction to the policy change. As $\phi$ becomes larger, $C_0$ jumps to the larger level. For sufficiently large values of $\phi(=0.4)$, $C_0$ jumps to the larger level than the initial level as a result of the policy change. As the value of $\phi$ becomes larger, the initial declines in public investment also becomes large, and hence more resources are released to private sector (see (12)). Consequently, the household can consume more as $\phi$ increases. To see why, let us integrate (5) and use (6).

$$C_0 = \rho(W_0 + \tilde{w}_0).$$

(27)

$C_0$ increases with the initial asset, $W_0$, and the present value of wage income, $\tilde{w}_0 = \int_0^\infty w_t \exp \left\{ - \int_0^t r_{\omega} d\omega \right\} dt$. $w_t$ depends on $k_{g,t}$ and $K_t$ (see (3)). After the policy change, the growth rate of private capital $K_t$ increases beyond the initial level (see Panel (e)), which enhances $w_t$ and exerts positive effects on $\tilde{w}_0$. In contrast, $k_g$ decreases at the early stages of transition as shown in Panel (f), which lowers $w_t$ and has negative effects on $\tilde{w}_0$. As $\phi$ increases, the positive effects tend to dominate the negative effects. Therefore, $C_0$ tends to be larger (smaller) than the initial level when $\phi$ is large (small) and $C_0$ increases with $\phi$.

6 Welfare

As in Section 5, we consider the following scenario: The economy is initially on a high-growth steady state with $\bar{b} = \bar{b}_{init}$. At time 0, the government reduces $\bar{b}$ from $\bar{b}_{init}$ to $\bar{b}_{new}$ unexpectedly. Then, the transitional dynamics is generated. The purpose of this section is to investigate the welfare effects of this policy change.

Our welfare measure is (4). Because the growth rate of consumption is given by $\gamma_{C,t} \equiv (1 - \tau)A k_{g,t}^{1-\alpha} - \rho$ and we have $C_t = C_0 e^{\int_0^t \gamma_{C,u} du}$, we can rewrite (4) as:

$$U_0 = \ln \frac{C_0}{\rho} + \int_0^\infty \left( \int_0^t \gamma_{C,u} du \right) e^{-\rho t} dt.$$ 

The above equation shows that decreases in $\bar{b}$ affect welfare level through the effects on $C_0$ and the paths of $\gamma_{C,t}$. As we have observed in Section 5, the effects of the policy change on $C_0$ and the transitional paths of $\gamma_{C,t}$ depend heavily on the value of $\phi$. Consequently, we will observe that welfare effects of the policy change are influenced by the value of $\phi$.

We examine the welfare effects numerically by considering the case where the government reduces its debt-GDP ratio from $B_0/Y_0$ to 0.5, where $B_0/Y_0 = 0.6, 0.7, 0.8, 0.9$ and 1. As in Section 5, we assume that the debt-GDP ratio in the new steady state is less than 60 percent because the member countries of the EU must maintain the debt-GDP ratio of less than 60 percent under the Maastricht criterion. In the benchmark, the values of $A$, $\rho$ and $\alpha$ are the same as those employed in Section 5. The values of $\phi$ we consider are $\phi = 0.01, 0.025, 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.75$ and 1. When $\phi$ is smaller (larger) than 0.05 (0.4), it takes extremely longer (shorter) years until the debt-GDP ratio reduces to 60 percent. However, to clarify the welfare effects of the policy change, we include extremely small and large values of $\phi$. We consider three values of income tax rates: $\tau = 0.1, 0.2$.
and 0.3. The values of $\bar{b}_{int}$ and $\bar{b}_{SS}$ are chosen for each $\tau$ and $B_0/Y_0$ so that the initial debt-GDP ratio becomes equal to $B_0/Y_0$ and the debt-GDP ratio in the new steady state becomes equal to 0.5.

Let us denote the welfare level without the policy change by $U_{0,N}^*$. We have $U_{0,N}^* = \ln(c_{H,N}^* K_0)/\rho + \gamma_{C,N}^*/\rho^2$ where $c_{H,N}^*$ and $\gamma_{C,N}^*$ are the initial steady state values of $c$ and $\gamma_C$. To calculate $U_{0,N}^*$, we need the initial value of private capital, $K_0$. Here, we set $K_0 = 1$.

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The welfare level just after the policy change is denoted by $U_{0,N}^{**}$. We calculate $U_{0,N}^{**}$ by setting $K_0 = 1$ and using the relaxation algorithm (see Appendix D in more detail).

Table 1]

In Table 1, we present the difference between $U_{0,N}^{**}$ and $U_{0,N}^*$. The table shows that when $\tau = 0.1$, $B_0/Y_0 = 1$ and $\phi = 0.05$ hold, social welfare can be improved by 0.04048 ($= U_{0,N}^{**} - U_{0,N}^*$) if the government reduces its debt-GDP ratio. In addition, the table shows that it takes about 37 years until the debt-GDP ratio reduces to about 60 percent.¹⁸

The first notable result in table 1 is that in all cases we consider, reductions in $\bar{b}$ have positive welfare effects. As shown in Section 5, when the government reduces $\bar{b}$, whether $C_0$ may decrease or increase in the short-run depending on the values of $\phi$. Furthermore, the growth rate of consumption, $\gamma_C$, decreases at the early stage of transition, which has negative effects on welfare. However, in the long-run, the growth rate becomes higher than the initial level, which has positive welfare effects. Our results show that in all the cases, the positive effects dominate the negative effects, and hence social welfare improves if the government reduces $\bar{b}$ and decreases its debt-GDP ratio.

Table 1 next tells that given $\tau$ and $\phi$, as the initial debt-GDP ratio is larger, the welfare gains are also larger. For example, when $\tau = 0.1$ and $\phi = 0.05$, the welfare gains decrease from 0.03379 to 0.02627 as $B_0/Y_0$ decreases from 0.9 to 0.8. A large $B_0/Y_0$ means that there are large gaps between the initial and target debt-GDP ratios. To fill this large gap, the government must initially reduce $g_t$ in large amount, as shown in Table 2.¹⁹

As discussed in Section 5, the initial declines in $g_t$, leading to the decreases in $k_{g,t}$ at the early stages of transition, are the source of the negative welfare effect (the declines in $\gamma_{C,t}$). Then, the negative welfare effect is large when $B_0/Y_0$ is large. However, the large initial declines in $g_t$ suggests that more resources are released to private sector and hence the household can consume more compared to the case where $B_0/Y_0$ is small. Then, the short-run effect (changes in $C_0$) tends to be weakly negative or even positive when $B_0/Y_0$ is large. In addition, a large $B_0/Y_0$ indicates a large difference between the values of $k_g$ in the initial and new steady states. Then, the growth rate of consumption in the new steady state is much higher than that in the initial steady state, which suggest that the positive long-run effect is large when $B_0/Y_0$ is large. Because the dominance of the positive effects over the negative effect increases as $B_0/Y_0$ increases, the welfare gains from the policy change are large when $B_0/Y_0$ is large.

Table 2]

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¹⁸ When $B_0/Y_0$ is equal to 0.6, we do not provide years that it takes until the debt-GDP ratio reduces to about 60 percent.

¹⁹ The initial decline of $g_t$ is calculated by $(g_0 - g_{int})/g_{int}$ where $g_{int}$ is the initial steady state values and $g_0$ is the value just after the policy change.
As expected, the values of $\phi$ affect the welfare effects. Table 1 shows that as for the relationship between the welfare gains and $\phi$, following two patterns are observed. In many cases, (i) the welfare gains decrease in $\phi$ when $\phi$ is small but turn to increase in $\phi$ as $\phi$ becomes relatively large. Thus, the relationships between the welfare gains and $\phi$ take U-shapes in many cases. However, (ii) the welfare gains monotonically decrease in $\phi$ when $\tau$ is low and $B_0/Y_0$ is large: $\tau = 0.1$ and $B_0/Y_0 \geq 0.9$.

To understand the intuition for these results, the following points should be noted. Table 2 illustrates that the size of the initial declines of $g_t$ under $\phi = 0.05$ are crucially depend on $\tau$ and $B_0/Y_0$. When $\tau$ is lower (higher), and $B_0/Y_0$ is larger (smaller), the initial declines in $g_t$ become larger (smaller). This is because when the tax rate is lower (higher), the government can rely less (more) on tax revenues to reduce its debt-GDP ratio. Furthermore, when $B_0/Y_0$ is larger (smaller), the initial declines of $g_t$ become also larger (smaller) to fill larger gaps between the initial and target debt-GDP ratios. Next, the increase in $\phi$ has following two opposite effects on welfare. It enlarges the decline of $g_t$ as we see in Figure 4-(c). However, it leads to recover the growth rate of consumption sooner as we see in Figure 4-(g). Which effect is dominant depends on the size of the initial declines in $g_t$. When the tax rate is low and initial debt-GDP ratio is large as $\tau = 0.1$ and $B_0/Y_0 \geq 0.9$, the initial decline of $g_t$ is too large when the government increases $\phi$. Then, the welfare gains monotonically decrease in $\phi$. In contrast, when the tax rate is high or $B_0/Y_0$ is small, the initial decline of $g_t$ becomes small. Then, when $\phi$ is somewhat large, the positive effect of the sooner recovery of the growth of consumption dominates the negative effect of the initial decline of $g_t$. Thus, the relationships between the welfare gains and $\phi$ take U-shapes.

It is possible that which of welfare effects in the short run (the effects on $C_0$), on the transition path and in the long run dominate may depend on the value of the subjective discount rate. When $\rho$ is small (large), it is likely that welfare effects in the long-run (in the short run and on the transition path) dominate those in the short run and on the transition path (in the long run). Hence, the total welfare effects may depend on $\rho$. We conduct the same exercise under the different values of discount rates (see Table 3). The values of $A$, $\alpha$ and $\phi$ are the same as those employed in Table 1. The values of $\hat{b}_{\text{init}}$ and $\hat{b}_{\text{new}}$ are chosen for each $\rho$, $\tau$ and $B_0/Y_0$ so that the initial debt-GDP ratio becomes equal to $B_0/Y_0$ and the debt-GDP ratio in the new steady state becomes equal to 0.5. Table 2 shows again that (i) in all cases we consider, reductions in $\hat{b}$ have positive welfare effects, and (ii) as the initial debt-GDP ratio is larger, the welfare gains are also larger. With respect to the influences of $\phi$, the findings when $\rho$ is 0.03 are completely same as those in Table 1. When $\tau$ is low and $B_0/Y_0$ is large: $\tau = 0.1$ and $B_0/Y_0 \geq 0.9$, they are monotonically decreasing. The relationships between the welfare gains and $\phi$ take U-shapes in many cases. In the case of $\rho = 0.07$, results are also same when $B_0/Y_0$ is small. Therefore, results are almost the robust.

However, in the case of $\rho = 0.07$, some results different from Table 1 are obtained when $B_0/Y_0$ is large. When $\tau \geq 0.2$ holds, the welfare gains monotonically increase with $\phi$ if $B_0/Y_0$ is large enough. As discussed in the above, when $B_0/Y_0$ is large, the short-run effect (changes in $C_0$) tends to be weakly negative or even positive. In addition, $C_0$ increases with $\phi$ as shown in Panel (h) of Figure 4. Because a large $\rho$ indicates that the household cares more about the short-run effect, the welfare gains monotonically increase with $\phi$ when $B_0/Y_0$ is large and $\tau \geq 0.2$ holds. In case of $\tau = 0.1$ and $B_0/Y_0 = 0.9$, the welfare
gains increases with $\phi$ for large values of $\phi$, in contrast to the result obtained in Table 1. These increases in the welfare gains also reflect the short-run effect. Also, when $\tau = 0.1$ and $B_0/Y_0 = 1$ hold, the short-run effect (changes in $C_0$) tends to be weakly negative or even positive and $C_0$ increases with $\phi$. However, the negative effect coming from the decline of $g_t$ is so strong that the welfare gains decrease with $\phi$ even when the discount rate is large ($\rho = 0.07$).

[Table 3]

In the benchmark cases, when $\tau$ is equal to 0.2 or 0.3, the long-run growth rates become much larger than 0.02. When $\tau$ is equal to 0.2 and 0.3, the long-run growth rate is equal to about 0.026 and 0.027 in benchmark case, respectively. A small difference from 0.02 in the long-run growth rate may lower the reliability of the result. Therefore, in the case that $\tau$ is 0.2 and 0.3, we conduct the same numerical exercise when we change the value of TFP, $A$, so that the long-run growth rate becomes equal to 0.02. However, Table 4 shows that the results are not different from those are obtained in Table 1 qualitatively. When $\tau = 0.2$, welfare gains decrease at first and then begin to increase as the value of $\phi$ increases. On the other hand, welfare gains increase with $\phi$ when $\tau$ is as large as 0.3.

[Table 4]

7 Conclusion

We investigate the policy effects when the government reduces the debt-GDP ratio to below the 60 percent criterion. In the manner of expenditure based reduction of public debt, the government decreases public investment initially to reduce its debt. This policy change causes the following response during the transition and in the steady state, and brings opposite effects on welfare between on the transition path and in the long run. In the short run, the investment in private capital rises initially due to the fall of public investment and then the return from private capital decreases. Then, the growth rate of consumption begins to decline after the policy change. This exerts negative effects on welfare on the transition path. However, as the size of public debt becomes smaller, the interest payment of the debts shrinks and the government can spends its tax revenue more on public investment. The recovery of public investment, in turn, hinges on private investment and then the return from private capital decreases. As a result, in the long run, the growth rate of consumption, private capital and public capital exceed those in the initial state. This exerts the positive effects on welfare in the long run. We find positive effects dominate the negative effects and then the policy change improves welfare.

The adjustment speed of reducing debts crucially affects welfare. How the welfare gains are influenced as the governments change the speed of reducing debts depends on the tax rate and how much the debt-GDP ratio initially exceed the target level. The relationships between the welfare gains and the adjustment speed take U-shapes in many cases. However, they are monotonically decreasing when the tax rate is low and initial debt-GDP ratio is sufficiently large. When the tax rate is low, the governments have to reduce the gap of the debt-GDP ratio between the initial debt-GDP ratio and the target without enough tax revenues. Then the initial decline of public investment becomes large.
When the initial debt-GDP ratio is large, the government must fill the large gap between the initial and the target level of the debt. Then the initial decline of public investment also becomes large. The increase in the speeds of reducing public debts leads to the following trade-off. On the one hand, the initial decline of public investment becomes larger. On the other hand, the growth of consumption recovers sooner. When the tax rate is low and initial debt-GDP ratio is large, the initial decline of public investment is too large when the government increases the adjustment speed. Then, the welfare gains monotonically decrease in the adjustment speed. In contrast, when the tax rate is high or the initial debt-GDP ratio is small, the initial decline of public investment becomes small. Then, when the adjustment speed is somewhat high, the positive effect of the sooner recovery of the growth of consumption dominates the negative effect of the initial decline of public investment. Then, the relationship between the welfare gains and the adjustment speed takes U-shape.

Besides these main results, this paper shows the possibility that the economy fails sustainable development when the initial level of public capital is sufficiently small relative to private capital. In the development process, the government should rely more on tax finance rather than debt to avoid falling into the development trap.

Finally, we provide the directions for the future research. For countries facing government’s financial crises, there may be two ways to improve the financial condition. First one is reducing government spending. Second one is increasing the tax rate. We investigate the first one in this paper. However, countries such as France, Italy, Ireland, Greece, Portugal, Spain and US try to implement both spending cuts and tax increases. Investigating debt reduction by including tax increases is very important. Further possibility is that under some conditions, reduction in tax rates rather than increases in tax rates may be better policy for reducing debt. Bruce and Turnovsky (1999) show that both expenditure cuts and cut in the tax rate generate higher economic growth which leads to higher tax revenues in the future. However, they do not investigate the policy effects on the transition dynamics but those in the steady state. Investigating which is better, increasing or decreasing tax rates for welfare including transition path is also important. Furthermore, we assume public infrastructure only influences production of goods as Futagami et al, (1993) and Turnovsky (1997) and so on. However, it may also affect households’ utility function if public capital includes public health capital as in Agénor (2008). How this extension leads to change the policy effects on the economy may be worth being analized. These are left to be done for future research.

Acknowledgments

We would like to thank Akira Momota, Akira Yakita and Kazuki Hiraga for helpful comments and suggestions. We are responsible for all remaining errors.
Appendix

A Properties of $\Lambda(k_g)$

We can show that $\Lambda(k_g)$ has the following properties:

\[
\Lambda'(k_g) = Ak_g^{-\alpha}\{(1 - \tau)\alpha(2 - \alpha)k_g - \tau(1 - \alpha)\} < (>)0 \quad \text{if and only if} \quad k_g < (>)\bar{k}_g,
\]

\[
\Lambda''(k_g) = (1 - \tau)\alpha(1 - \alpha)(2 - \alpha)Ak_g^{-\alpha} + \tau\alpha(1 - \alpha)Ak_g^{-\alpha - 1} > 0,
\]

\[
\lim_{k_g \rightarrow 0} \Lambda'(k_g) = -\infty, \quad \lim_{k_g \rightarrow \infty} \Lambda'(k_g) = \infty, \quad \text{and} \quad \Lambda(0) = \Lambda\left(\frac{\tau}{(1 - \tau)\alpha}\right) = 0,
\]

where $\bar{k}_g \equiv \frac{(1 - \alpha)\tau}{(1 - \tau)(2 - \alpha)a}$. Apparently, $\Lambda(k_g)$ is a convex function of $k_g$ as shown in Figure 1. The above properties ensure the existence and the uniqueness of $\hat{k}_g(>0)$ that satisfies $\Lambda'(\hat{k}_g) = \rho$.

B Phase diagram of $(k_g, c)$

From (23), the $\dot{c} = 0$ locus is given by $c = \zeta(k_g, \tau, \bar{b})$. It is easy to show $\zeta(0, \tau, \bar{b}) > 0$, $\partial_\zeta(k_g, \tau, \bar{b})/\partial k_g > 0$ and $\partial_2^2 \zeta(k_g, \tau, \bar{b})/\partial k_g^2 < 0$. Eq (23) shows that $\dot{c}_t \geq (\leq)0$ holds when $c_t \geq (\leq)\zeta(k_g, \tau, \bar{b})$. Summarizing these results, Figures 2 (a)-(c) show the shape of $\dot{c} = 0$ locus and the motion of $c_t$.

Next, we move on to the $\dot{k}_g = 0$ locus. Depending on the value of $\bar{b}$, the $\dot{k}_g = 0$ locus takes different shapes. We consider the following three cases; (i) $\bar{b} = \tau/(1 - \tau)\alpha$, (ii) $\bar{b} < \tau/(1 - \tau)\alpha$ and (iii) $\bar{b} > \tau/(1 - \tau)\alpha$.

(i) We consider the case of $\bar{b} = \tau/(1 - \tau)\alpha$. Because $\eta(k_{g,t}, \tau, \bar{b}) = (k_{g,t} - \bar{b})Ak_g^{-\alpha}$ holds, we know from (24) that the $k_g = 0$ locus is given by:

\[
k_g = \bar{b} = \frac{\tau}{(1 - \tau)\alpha} \quad \text{and} \quad c = Ak_g^{1 - \alpha}.
\]

(B.1)

The first equation of (B.1) tells that one of the steady state values of $k_g$ is given by $k_g^* = \bar{b}$. Solving the $\dot{c} = 0$ locus and the second equation of (B.1) for $k_g$, we know that the other steady state value of $k_g$ is given by:

\[
x \equiv \left\{ \frac{\rho(1 + \frac{\tau}{(1 - \tau)\alpha})}{A[1 - (1 - \tau)(1 - \alpha)]} \right\}^{\frac{1}{1 - \alpha}}.
\]

The upper panel of Figure 2-(a) presents the $\dot{k}_g = 0$ locus in the case when $\rho$ is small enough to satisfy $\bar{b} > x$. When $\rho$ is large enough to satisfy $\bar{b} < x$, the $\dot{k}_g = 0$ locus is shown in the lower panel of Figure 2-(a).

We know from (24) that:

\[
\dot{k}_{g,t} \geq 0, \quad \text{(a) if } c_t \leq Ak_g^{1 - \alpha} \quad \text{and} \quad k_g \leq \bar{b}, \quad \text{or} \quad \text{(b) if } c_t \geq Ak_g^{1 - \alpha} \quad \text{and} \quad k_g \geq \bar{b},
\]

\[
\dot{k}_{g,t} < 0, \quad \text{(c) if } c_t < Ak_g^{1 - \alpha} \quad \text{and} \quad k_g > \bar{b}, \quad \text{or} \quad \text{(d) if } c_t > Ak_g^{1 - \alpha} \quad \text{and} \quad k_g < \bar{b}.
\]

Based on the discussion so far, the phase diagrams are represented in Figure 2-(a) when $\bar{b} = \frac{\tau}{(1 - \tau)\alpha}$ holds.
We next move on to the cases (ii) and (iii) where \( \bar{b} \neq \frac{\tau}{(1-\rho_{\alpha})} \). From (24) and the definition of \( \eta(k_{g,t}, \tau, \bar{b}) \), the \( \hat{k}_g = 0 \) locus is given by:

\[
c = \Gamma(k_g, \tau, \bar{b}) = \frac{(1-\tau)(1+a\bar{b})k_g - \{1-(1-\tau)\alpha\bar{b} + \tau\}}{k_g - \bar{b}} A k_g^{1-\alpha}. \tag{B.2}
\]

Apparently, \( \Gamma(k_g, \tau, \bar{b}) \) becomes equal to zero when \( k_g = 0 \) and \( k_g = \hat{k}_g \equiv \frac{\{1-(1-\tau)\alpha\bar{b} + \tau\}}{(1-\tau)(1+\alpha \bar{b})} \). Then, the \( \hat{k}_g = 0 \) locus passes through \((0,0)\) and \((\hat{k}_g,0)\) as shown in Figures 2-(b) and (c). \( \Gamma(k_g, \tau, \bar{b}) \) is continuous with respect to \( k_g \) except at \( k_g = \bar{b} \).

(ii) We consider the case where \( \bar{b} < \frac{\tau}{(1-\rho_{\alpha})} \) holds. In this case, \( \bar{b} < \hat{k}_g \) holds because we have \( \hat{k}_g - \bar{b} = \frac{(1+\bar{b})(\tau - (1-\alpha)\alpha \bar{b})}{(1-\tau)(1+\alpha \bar{b})} \). We rewrite (B.2) as:

\[
\Gamma(k_g, \tau, \bar{b}) = \left\{ (1-\tau)(1+a\bar{b}) + \frac{(1-\tau)\alpha \bar{b} - \tau}{k_g - \bar{b}} (1+\bar{b}) \right\} A k_g^{1-\alpha}. \tag{B.3}
\]

Because \( \bar{b} < \frac{\tau}{(1-\rho_{\alpha})} \) holds, \( \partial \Gamma(k_g, \tau, \bar{b})/\partial k_g \) has a positive sign if \( k_g \neq \bar{b} \). We also have \( \lim_{k_g \rightarrow 0} \Gamma(k_g, \tau, \bar{b}) = +\infty \). Because we have \( \Gamma(0, \tau, \bar{b}) = 0 < \zeta(0, \tau, \bar{b}) \), the continuity of \( \Gamma(k_g, \tau, \bar{b}) \) and \( \zeta(k_g, \tau, \bar{b}) \) indicates that the \( \hat{k}_g = 0 \) locus has at least one intersection with the \( \hat{c} = 0 \) locus in the region of \( k_g < \hat{b} \) as shown in Figure 2-(b). We can show \( \lim_{k_g \rightarrow -\infty} \Gamma(k_g, \tau, \bar{b})/\zeta(k_g, \tau, \bar{b}) = (1+a\bar{b})/(1-\alpha) > 1 \), which means that for large \( k_g \), the \( \hat{k}_g = 0 \) locus is located above the \( \hat{c}_1 = 0 \) locus. Because we have \( \Gamma(k_g, \tau, \bar{b}) = 0 < \zeta(k_g, \tau, \bar{b}) \), the continuity indicates that the \( \hat{k}_g = 0 \) locus intersects with the \( \hat{c} = 0 \) locus at least once in the region of \( k_g > \hat{b} \). Because there exist two steady states, we now know that one of the two intersection of the \( \hat{c} = 0 \) and \( \hat{k}_g = 0 \) loci is in the region of \( k_g < \hat{b} \) and the other intersection is in the region of \( k_g > \hat{b} \). From the discussion so far, we depict the \( \hat{k}_g = 0 \) locus as shown in Figure 2-(b). From (24), we obtain:

\[
\begin{align*}
\hat{k}_{g,t} &\geq 0, \quad (a) \text{ if } c_t \leq \Gamma(k_{g,t}, \tau, \bar{b}) \text{ and } k_g \leq \bar{b}, \quad \text{or (b) if } c_t \geq \Gamma(k_{g,t}, \tau, \bar{b}) \text{ and } k_g \geq \bar{b}, \tag{B.4} \\
\hat{k}_{g,t} &< 0, \quad (c) \text{ if } c_t < \Gamma(k_{g,t}, \tau, \bar{b}) \text{ and } k_g > \bar{b}, \quad \text{or (d) if } c_t > \Gamma(k_{g,t}, \tau, \bar{b}) \text{ and } k_g < \bar{b}. \tag{B.5}
\end{align*}
\]

Thus, we can draw the phase diagram as shown in Figure 2-(b).

(iii) We finally turn to the case of \( \bar{b} > \frac{\tau}{(1-\rho_{\alpha})} \). The second term in parentheses of RHS of (B.3) is negative (positive) when \( k_g < (>) \bar{b} \) holds. In the region where \( k_g < (>) \bar{b} \) holds, the graph of \( c = \Gamma(k_g, \tau, \bar{b}) \) is located below (above) the graph of \( c = (1-\tau)(1+a\bar{b})Ak_g^{1-\alpha} \) as shown in Figure 2-(c). Eq (B.3) also reveals \( \lim_{k_g \rightarrow \bar{b}+0} \Gamma(k_g, \tau, \bar{b}) = +\infty \). In addition, the graph of \( c = \Gamma(k_g, \tau, \bar{b}) \) asymptotically becomes close to the graph of \( c = (1-\tau)(1+a\bar{b})Ak_g^{1-\alpha} \) as \( k_g \) increases to +\( \infty \). Therefore, in the region of \( k_g > \bar{b} \), the \( \hat{k}_g = 0 \) locus takes a shape that is shown in Figure 2-(c). By using (B.2), we next show that \( \Gamma(k_g, \tau, \bar{b}) > 0 \) holds in the region of \( k_g < \hat{k}_g \). From (B.2), we know that the denominator of \( \Gamma(k_g, \tau, \bar{b}) \) becomes negative when \( k_g < \bar{b} \) and the numerator becomes negative when \( k_g < \hat{k}_g \). Because we have \( \hat{k}_g < \bar{b} \) when \( \bar{b} > \frac{\tau}{(1-\rho_{\alpha})} \) holds, both the denominator and numerator of \( \Gamma(k_g, \tau, \bar{b}) \) are negative in the region where \( k_g < \hat{k}_g \) holds. Therefore, \( \Gamma(k_g, \tau, \bar{b}) > 0 \) holds in the region of \( k_g < \hat{k}_g \). From the discussion so far, together with (B.4) and (B.5), we can draw the phase diagram as shown in Figure 2-(c). The upper (lower) panel shows the case where both \( k_{g,L}^* \) and \( k_{g,H}^* \) are smaller (larger) than \( \bar{b} \).
Appendix E, we derive these by solving this equation:

\[
\begin{pmatrix}
\dot{b} \\
\dot{c} \\
\dot{k}_g
\end{pmatrix} = 
\begin{pmatrix}
-\phi & 0 & 0 \\
J_{cb} & J_{cc} & J_{ckg} \\
J_{k_g b} & J_{k_g c} & J_{k_g k_g}
\end{pmatrix}
\begin{pmatrix}
b_t - \bar{b} \\
c_t - c^* \\
k_g - k_g^*
\end{pmatrix}.
\]

\[J = (J_{ij})\] denotes the coefficient matrix of the former system:

\[
J_{cb} = -\frac{\rho + \phi}{1 + b} c^*, \quad J_{cc} = \frac{c^*}{1 + b}, \quad J_{ckg} = -\frac{(1 - \alpha)2(1 - \tau)c^*}{1 + b} A(k_g^*)^{-\alpha},
\]

\[
J_{k_g b} = -\frac{1 + k_g^*}{1 + b} \eta(k_g^*, \tau, \bar{b}), \quad J_{k_g c} = \frac{k_g^* - \bar{b}}{1 + b},
\]

\[
J_{k_g k_g} = -\frac{(1 - \alpha)\eta(k_g^*, \tau, \bar{b})}{(1 + b)k_g^*} - [(1 - \tau)\alpha A(k_g^*)^{1-\alpha} - \rho].
\]

where \(k_g^* = k_g^{*L}\) or \(k_g^{*H}\), and \(c^* = \zeta(k_g^*, \tau, \bar{b})\).

Let us denote eigenvalues of the coefficient matrix \(J\) as \(\nu_i\) \((i = 1, 2, \text{and } 3)\). The structure of the first column of \(J\) entails that \(-\phi\) is an eigenvalue \(\nu_1\). The remaining eigenvalues, \(\nu_2\) and \(\nu_3\), of \(J\) are those of the matrix, \(\bar{J}\), derived by deleting the first row and column from \(J\). The eigenvalues, \(\nu_2\) and \(\nu_3\), are the solution of the characteristic equation, \(\nu^2 - (J_{cc} + J_{k_g k_g})\nu + J_{cc}J_{k_g k_g} - J_{k_g c}J_{ckg} = 0\). Because we will use \(\nu_2\) and \(\nu_3\) in Appendix E, we derive these by solving this equation:

\[
\nu_2 = \frac{J_{cc} + J_{k_g k_g} + \sqrt{(J_{cc} + J_{k_g k_g})^2 - 4(J_{cc}J_{k_g k_g} - J_{k_g c}J_{ckg})}}{2},
\]

\[
\nu_3 = \frac{J_{cc} + J_{k_g k_g} - \sqrt{(J_{cc} + J_{k_g k_g})^2 - 4(J_{cc}J_{k_g k_g} - J_{k_g c}J_{ckg})}}{2}.
\]

To check the stability, we examine the sign of \(\det \bar{J} = J_{cc}J_{k_g k_g} - J_{k_g c}J_{ckg}\). Using (C.2), we obtain:

\[
\det \bar{J} = -\frac{c^*}{(1 + b)^2} \{ (2 - \alpha)(1 - \tau)\alpha A(k_g^*)^{1-\alpha} - (1 - \alpha)\tau A(k_g^*)^{-\alpha} - \rho \}.
\]

To simplify the above equation, we subtract \(\Pi'(k_g^*)\) from \(\Lambda'(k_g^*)\):

\[
\Lambda'(k_g^*) - \Pi'(k_g^*) = (2 - \alpha)(1 - \tau)\alpha A(k_g^*)^{1-\alpha} - (1 - \alpha)\tau A(k_g^*)^{-\alpha} - \rho.
\]

Thus, (C.5) can be rewritten as:

\[
\det \bar{J} = -\frac{c^*}{(1 + b)^2} \{ \Lambda'(k_g^*) - \Pi'(k_g^*) \}.
\]

Figure 1 shows that in the high-growth steady state, \(\Lambda'(k_g^{*H}) > \Pi'(k_g^{*H})\) holds, and hence \(\det \bar{J} < 0\). One of the eigenvalues of \(\bar{J}\) has a positive real part and the other has a negative real part. Moreover, because the inequality \(\det \bar{J} < 0\) implies \((J_{cc} + J_{k_g k_g})^2 - 4(J_{cc}J_{k_g k_g} - J_{k_g c}J_{ckg}) > 0\), both \(\nu_2\) and \(\nu_3\) are real numbers. We then have \(\nu_2 > 0\) and \(\nu_3 < 0\). The high-growth steady state is saddle-point stable.
Figure 1 shows that in the low-growth steady state, $\Lambda'(k_{g,L}) < \Pi'(k_{g,L})$ holds and then $\det J > 0$ holds. This implies that the real parts of $\nu_2$ and $\nu_3$ have the same signs. To determine the sign of the real parts, we check the sign of $\text{Tr} \cdot \bar{J} = J_{cc} + J_{k_g k_g}$. As we have just shown, we have $\det \bar{J} = J_{cc} J_{k_g k_g} > 0$ in the low-growth steady state. This inequality can be written as:

$$J_{cc} J_{k_g k_g} > J_{k_g}^2 > (1 - \alpha)^2 (1 - \tau) (k_g^* - \bar{b}) A(k_g^*)^{-\alpha}.$$ 

We divide the both sides of the above inequality by $J_{cc} = \zeta(k_g^*, \tau, b)$, we obtain:

$$J_{k_g} > \frac{(1 - \alpha)^2 (1 - \tau) (k_g^* - \bar{b}) A(k_g^*)^{-\alpha}}{1 + \bar{b}}.$$

Adding $J_{cc}$ to the both sides of the above inequality and using $c^* = \zeta(k_g^*, \tau, b)$, we obtain:

$$J_{cc} + J_{k_g} > \frac{1}{1 + \bar{b}} \left\{ c^* - (1 - \alpha)^2 (1 - \tau) (k_g^* - \bar{b}) A(k_g^*)^{-\alpha} \right\},$$

$$= \frac{1}{1 + \bar{b}} \left\{ (1 - \tau)(1 - \alpha)[\alpha + (1 - \alpha)\bar{b}] A(k_g^*)^{1-\alpha} + \rho(1 + \bar{b}) \right\},$$

$$> 0.$$ 

In the low-growth steady state, $\text{Tr} \cdot \bar{J} = J_{cc} + J_{k_g k_g} > 0$ holds. The real parts of $\nu_2$ and $\nu_3$ are positive. The low-growth steady state is unstable.

D Welfare Effects of $\bar{b}$

To calculate the value of $U_0^{**}$, we calculate the dynamic path of and the initial value of $U_t \equiv \int_t^{\infty} (\ln C_v) e^{-\rho(v-t)} dv$ by using the relaxation algorithm. However, we can not calculate the dynamic path of and the initial value of $U_t$ directly because $U_t$ does not remain constant at the high-growth steady state. Let us define $X_t \equiv U_t - \ln K_t/\rho$. Because of $C_t \equiv c_t K_t$, we have:

$$X_t = \rho X_t - \ln c_t - \frac{1}{\rho} (Ak_g, 1-\alpha - c_t - \rho).$$

$X_t$ becomes constant over time at the high-growth steady state. Then, we calculate the dynamic path of and the initial value of $X_t$ by using the relaxation algorithm. Since $K_0$ is normalized to one, we have $U_0^{**} = X_0$.

References


\[ \Lambda(k_g) : [(1 - \tau)\alpha k_g - \tau] A k_g^{1-\alpha} \]

\[ \Pi(k_g) : \rho (k_g - \bar{b}) \]

Figure 1
Figure 2-(a) $\bar{b} > x$  

(i) $\bar{b} > x$ 

(ii) $\bar{b} < x$ 

$\dot{k}_g = 0 (k_g = \bar{b})$ 

$\dot{k}_g = 0 (c = Ak_g^{1-\alpha})$ 

$\dot{c} = 0$ 

$\rho(1 + \bar{b})$ 

Figure 2-(a) $\bar{b} = \frac{\tau}{(1-\tau)\alpha}$
Figure 2-(b) \( \bar{b} < \frac{\tau}{(1-\tau)\alpha} \)
Figure 2-(c) $\bar{b} > \frac{\tau}{(1-\tau)\alpha}$
(i) Policy effect with respect to \( \bar{b}(\bar{b} \uparrow) \)

\[
\Lambda(k_g) : [(1 - \tau)\alpha k_g - \tau]A k_g^{1-\alpha}
\]

\[\Pi(k_g) : \rho(k_g - \bar{b})\]

Figure 3

(ii) Policy effect with respect to \( \tau(\tau \uparrow) \)

\[
\Lambda(k_g) : [(1 - \tau)\alpha k_g - \tau]A k_g^{1-\alpha}
\]

\[\Pi(k_g) : \rho(k_g - \bar{b})\]

Figure 3
Figure 4. Transitional Dynamics
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Table 1. Welfare Effects: when debt-GDP ratio reduces to 50%
The number in the parenthesis expresses time to take through the debt-GDP ratio becomes approximately 60%.
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Table 2. The Initial Declines of $g_t$ under $\phi = 0.05$. 
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Table 3. Welfare Effects under $\rho = 0.03$ and 0.07.
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Table 4. Welfare Effects When the Long-run Growth rate is equal to 0.02.