Does endogenous formation of jurisdictions lead to wealth stratification?∗

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Abstract

This paper examines the validity of the “folk” intuition that endogenous formation of jurisdictions is likely to create stratification of households according to their wealth. More specifically, we examine a simple model of jurisdiction formation, close in spirit to that of Whestoff ([27]), in which a continuum of unequally wealthy households endowed with the same preferences for one public good and one private good make a location decision in a finite set. Households who choose the same location form a jurisdiction. Within each jurisdiction, the public good is financed by a proportional wealth tax whose rate is decided by a social choice mechanism. The only assumption imposed on the mechanism is to select the most preferred tax rate of one member of the jurisdiction. We define a jurisdiction structure to be stable if it gives to no household any incentive to move away from its jurisdiction. We raise the question of whether stable jurisdiction structures will be stratified in the precise sense that if two households belong to one jurisdiction, then so do all households with intermediate wealth. We provide a necessary and, if households preferences satisfy an additional regularity property, sufficient condition on the households preferences that guarantees that any stable jurisdictions structure involves stratification in this sense. The condition is that the household’s most preferred tax rate must be a strictly monotonic function of its wealth.

1 Introduction

A widely discussed feature of Tiebout’s ([26]) seminal work on endogenous jurisdictions formation is the asserted sorting property of the mechanism. If

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households choose their jurisdiction of residence and affect the bundle of public goods and taxes offered in every jurisdiction by “voting with their feet”, then, so the intuition goes, the resulting equilibrium jurisdictions structure should be characterized by homogenous jurisdictions, each inhabited by individuals with “similar” characteristics.

This sorting conjecture has many important policy implications (discussed by Benabou [3], Alesina and La Ferrara [1] among many others). Among other things, it may entail the stratification of individuals according to their wealth, implying in particular that “poor” households will live in different cities than “rich” ones. This possible stratification of the society that would result from an endogenous and decentralized process of cities formation may be considered undesirable from a normative standpoint. As a matter of fact, countries like France have designed laws that explicitly try to mitigate this stratification by requiring each municipality to have a quota of at least 20% of subsidized housing (usually rented to poor households) and by imposing fines on cities that fail to satisfy this requirement. On the other hand, there are authors like Alesina and La Ferrara [1] who argue that “homogeneous” communities are normatively better than “heterogeneous” ones because they facilitate the construction of “social capital”. As stratification is the source of such strong normative feelings, it appears important to investigate the general validity of the sorting property in plausible models of endogenous jurisdictions formation that are consistent with Tiebout’s [26] informal arguments.

As was noticed by Ellickson ([11], p. 39), “Tiebout’s vision has proved remarkably resistant to formalization”. This resistance certainly explains the energy spent on the problem of constructing precise models of endogenous jurisdictions formation and of proving the existence of more or less plausible notions of equilibrium for these. Another important question that has been considered in this literature is the investigation of Tiebout’s claim that a decentralized process of jurisdiction formation is likely to lead to a Pareto-efficient allocation of resources in the overall economy. Representative pieces of the literature that deal with either of these two questions (existence and Pareto-optimality) in various formal settings are Aumann and Drèze [2], Bewley [5], Drèze and Greenberg [8], Ellickson [10], Eppe, Filimon and Romer ([6], [7]), Greenberg ([12], [13]) Greenberg and Weber ([14]), Guesnerie and Oddou ([15], [16]), Konishi ([17]), Konishi, Le Breton and Weber ([19], [18]), Richter ([22], [23]), Rose Ackerman ([24]), Westhoff [27] and Wooders ([28],[29], [30]).

A large part of this literature has been rather vague with respect to the precise specification of the jurisdiction’s decision making process. This is the case of papers like Aumann and Drèze [2], Greenberg ([12], [13])

1 See e.g. the law SRU (Solidarité et Renouvellement Urbain) adopted by the french parliament in 2001.
Greenberg and Weber ([14]), Guesnerie and Oddou ([15], [16]) and Wooders ([28],[29],[30]) who have framed the problem in terms of cooperative game theory (by identifying a jurisdiction with a coalition that can secure to its members a set of utility vectors). This is also the case of papers such as Drèze and Greenberg [8] and Konishi, Le Breton and Weber ([19], [18]) who adopt a non-cooperative game theoretic framework to study the stability of jurisdictions structures but who model in an abstract way the payoff received by an individual when joining a particular coalition. While the rest of the literature has been more specific in describing the jurisdiction’s decision making process (usually by assuming a voting mechanism or a Lindahl scheme of taxation), it has focused attention on the delicate question of existence and/or Pareto-efficiency of equilibria rather than that of characterizing the positive properties possessed by these equilibria (for instance the fact of whether or not they lead to the stratification of individuals).

A noticeable exception to this is Westhoff ([27]) who proves, in a model with a continuum of agents and an arbitrary finite number of jurisdictions, the existence of a partition of the set of individuals into the jurisdictions (with each jurisdiction receiving a set of individuals with a strictly positive measure) that is stable in the sense that it gives to no individual any incentive to move to another jurisdiction. In Westhoff ([27]), each jurisdiction is assumed to choose a unique proportional tax rate on the individuals wealth by majority voting and to use the taxes collected to produce a public good. Individuals differ by their private wealth and by their preference for both the public good and their private consumption. In order to prove the existence of a stable jurisdiction structure Westhoff ([27]) imposes on households preferences the property that the individuals’ marginal rates of substitution between tax rate and public good can be completely ordered according to some a priori ranking of the households. While Westhoff uses this assumption to prove the existence of a stable jurisdiction structure, he also establishes that, under this assumption, any stable jurisdiction structure will put in the same jurisdiction a group of households who form an interval with respect to the a priori given ordering. Put differently, under this assumption, a stable jurisdiction structure will entail the stratification of individuals with respect to the a priori given ordering of individuals. However, in Westhoff ([27]), this a priori ordering of individuals need not be that induced by the comparison of their wealth.

An analogous property of the individual preferences was used by Greenberg and Weber [14] in a cooperative game theoretic setting. Greenberg and Weber are interested in proving the existence of a C-stable coalition structure (see Guesnerie and Oddou ([15]) defined as a partition of the set of individuals into coalitions that is immune to coalitional deviation. Greenberg and Weber assume a finite number of individuals. In the tradition of cooperative game theory, they also assume that a coalition can provide its members with any package of public good and taxes that satisfy the re-
quirement that taxes must be sufficient to finance the public good and that
the tax burden must be allocated to members either equally or according to
the member’s share of the aggregate wealth. Hence, they do not assume a
specific mechanism for choosing a particular package of tax and public good
within the jurisdiction. Within this framework, they prove that if individual’s preferences are additively separable with respect to the private good
and the public good, then the individuals can be ordered in such a way that
if two individuals $i$ and $j$ have the same ranking of two packages of taxes
and public good, then so do all individuals that are ordered between $i$ and $j$.
This property enables them to show the existence of a $C$-stable jurisdiction
structure. They also establish that any jurisdiction that is part of such a
$C$-stable jurisdiction structure is inhabited by households who form an in-
terval with respect to the ordering of the households. Greenberg and Weber
call *consecutive* coalition structures that satisfy this property.

The objective of this paper is to clarify further the role played by the
(similar looking) conditions on households preferences examined by Green-
berg and Weber [14] and Westhoff [27] to obtain equilibria which satisfy the
“sorting property” referred to above. Specifically, the model we examine has
one private good and one local public good produced in each jurisdiction.
There is a continuum of individuals that differ only by their endowment in
the private good. In particular, all households have the same preference for
the public good and the private good. We focus on wealth heterogeneity be-
cause it is usually the stratification of households with respect to wealth
which causes the most concern in common discussions about the sorting
properties of Tiebout model. In each jurisdiction, public good production
is financed by a proportional wealth tax, the rate of which is assumed to
be the most preferred tax rate of one member in the jurisdiction. A well-
known special case of such an intra-jurisdiction rule for deciding the package
of public good and taxes offered is majority voting where the tax rate that
beats any other by a majority of vote is the most preferred one of the median
individual. But our results do not ride heavily on the particular choice of the
median and carry over for any *positional dictatorship* rule à la Moulin [20].
In this setting, we focus on what we call “*stable jurisdictions structures*”
which we define, as in Westhoff [27], as partitions of the set of households
into jurisdictions that do not provide households with any individual incen-
tive to move.

We then provide a *necessary* and, if households preferences for the private
good and the public good satisfy a regularity property, *sufficient* condition
for any stable jurisdiction structure to exhibit wealth stratification. We
consider a jurisdiction structure to be wealth-stratified when any jurisdiction

\footnote{For example, Rhode and Strumpf [21] who propose a way of testing the Tiebout
model using historical data base their whole discussion on income heterogeneity within
and between counties.}
containing two households also contain all those households whose wealth are strictly in between that of the two households. This notion of stratification is equivalent to the notion of consecutiveness considered in Greenberg and Weber ([14]) if the a priori given ordering of households is that induced by the comparison of their wealth. Greenberg and Weber actually argue that the ordering of households could be that of their private wealth if one was willing to interpret their general result in a setting very similar to that considered herein. If they were correct, then their result could be interpreted as implying that if individual preferences for the private good and the public good are additively separable, then a $C$-stable coalition structure entails the stratification of households according to their wealth. Yet, as we shall see in this paper (see remark 2), Greenberg and Weber’s argument is wrong when applied to the ranking of individuals according to their wealth.

The condition that we find to be necessary for any stable jurisdiction structure to be wealth-stratified requires household’s most preferred tax rate to be a strictly monotonic function of household’s wealth (taking the aggregate wealth of the jurisdiction as given). This condition is rather strong. Among other things, it amounts to requiring the household to consider the public good and the private good as being either always gross complement or gross substitutes. To illustrate the strength of this condition, we provide examples of economies with very standard additively separable preferences for a private good and a public good that fail nonetheless to satisfy this condition. In such economies therefore, stable jurisdictions structures that are not stratified can be constructed. In such structures, richer and poorer people cling together while middle class people will form a distinct jurisdiction.

The rest of this paper is organized as follows. The next section describes the formal framework. In Section 3, we provide an example of a quite standard and simple economy in which a stable jurisdiction structure is not wealth stratified. Section 4 contains the proof of our main results concerning the necessity and the sufficiency of our condition for the wealth stratification of any stable jurisdiction structure and section 5 concludes.

2 The formal framework

We consider economies with a continuum of households represented by the $[0,1]$ interval. An economy consists of four elements. First there is a Lebesgue measure $\lambda$ on $[0,1]$. Given any lebesgue measurable subset (coalition) $C$ of $[0,1]$, we interpret $\lambda(I)$ as “the number of individuals” in $I$. We assume that $\lambda$ is atomless in the sense that $\lambda(\{i\}) = 0$ for each $i \in [0,1]$. The second ingredient in the description of an economy is a wealth distribution modeled as a (Lebesgue measurable) bounded from above function $\omega : [0,1] \to \mathbb{R}_+$ which associates to each individual $i \in [0,1]$ its private wealth $\omega(i)$. The third ingredient in the description of an economy is a
specification of the households preferences. We assume that households have
the same preferences for a public good (the quantity of which is denoted by
$Z$) and a private good (the quantity of which being denoted by $x$) that are
represented by a twice differentiable, increasingly monotonic and strictly
concave\(^3\) direct utility function $U : \mathbb{R}_+^2 \rightarrow \mathbb{R}$. We denote by $\mathcal{U}$ the class of
all direct utility functions that satisfy these properties. Given any bundle
of public good and private good $(\mathbf{Z}, \mathbf{x}) \in \mathbb{R}_+^2$, we denote by $MRS^U(\mathbf{Z}, \mathbf{x})$
the marginal rate of substitution of public good to private good evaluated
at $(\mathbf{Z}, \mathbf{x})$ defined by

$$MRS^U(\mathbf{Z}, \mathbf{x}) = \frac{\partial U(\mathbf{Z}, \mathbf{x})}{\partial \mathbf{Z}} / \frac{\partial U(\mathbf{Z}, \mathbf{x})}{\partial \mathbf{x}}$$  \hspace{1cm} (1)

To provide link with standard consumer’s theory, we also denote by
$Z^M(p_Z, p_x)$ and $x^M(p_Z, p_x)$ the (Marshallian) normalized demands for public
good and private consumption (respectively) when the prices for these
two goods (expressed in fraction of wealth) are $p_z$ and $p_x$. These normalized
Marshallian demands are the solution of the program

$$\max_{x, Z} U(Z, x) \text{ subject to } p_Z Z + p_x x \leq 1$$

Given the assumption imposed on $U$, one can see easily that these normalized
Marshallian demands are differentiable functions of prices. For the proof of
proposition 2 below, we also require the consumer’s preference to satisfy the
following “regularity” condition.

**Condition 1**  \hspace{1cm} \forall (p_Z, p_x), (p'_Z, p'_x) \in \mathbb{R}_+^2, such that $p_Z Z^M(p_Z, p_x) \leq p'_Z Z^M(p'_Z, p'_x)$

\hspace{1cm} \forall \alpha \in [p_Z Z^M(p_Z, p_x), p'_Z Z^M(p'_Z, p'_x)], \forall p_Z \in [\min(p_Z, p'_Z), \max(p_Z, p'_Z)], \exists

\hspace{1cm} p_x \in \mathbb{R}_+ \text{ such that } \alpha = p_Z Z^M(p_Z, p_x)

This condition restricts somehow the household’s behavior in terms of its
optimal allocation of wealth between public good and private good spending
at various prices. Specifically, it requires the function that maps normalized
prices of public good and private good $(p_Z, p_x) \in \mathbb{R}_+^2$ into budget share
devoted to public good spending $p_Z Z^M(p_Z, p_x)$ to have an inverse in the set
of normalized private good prices between any two values taken by its image
and this, for every normalized price of public good lying between those who
give rise to the two values of its image. This condition will only be used in
the proof of proposition 2 below.

Finally, the fourth element of our description of an economy is a common
finite set $\mathbb{L}$ of localizations (whose typical members will be denoted by $l$, $l'$,
$l''$, etc.) available to households.

\(^3\)Our terminology for concavity and quasi-concavity of a function $f : A \rightarrow \mathbb{R}$ ($A \subseteq \mathbb{R}^k$)
is as follows: $f$ is strictly concave if, for every $\alpha \in [0, 1]$ and for every distinct $a, b \in A$,
$f(\alpha a + (1 - \alpha) b) > \alpha f(a) + (1 - \alpha) f(b)$ and $f$ is quasi-concave if, for every $a, b, x \in A$,
and every $\alpha \in [0, 1]$, $f(a) \geq f(x)$ and $f(b) \geq f(x)$ imply $f(\alpha a + (1 - \alpha) b) \geq f(x)$. 

6
A jurisdiction structure for the economy $(\lambda, \omega, U, \mathbb{L})$ is a (Lebesgue measurable) function $J : [0, 1] \to \mathbb{L}$. A jurisdiction structure $J$ induces a finite partition of $[0, 1]$ into (Lebesgue measurable) sets $C^J(l)$ defined by $C^J(l) = \{ i \in [0, 1] : J(i) = l \}$. $C^J(l)$ is thus the set - or the coalition - of individuals who live at localization $l$ given the jurisdiction structure $J$. In the same vein, we let, for every $l \in \mathbb{L}$, $n^J(l) = \lambda(C^J(l))$ denote the “number of individuals who live at $l$” in the jurisdiction structure $J$. The possibility that $n^J(l) = 0$ for some $l$ is, of course, not ruled out. Since, for every jurisdiction structure $J$, the sets $C^J(l)$ partition $[0, 1]$, we clearly have that $\sum_{l \in \mathbb{L}} n(l) = \lambda([0, 1])$. Given a jurisdiction structure $J$ and a localization $l$, we denote by $\omega^J(l)$ the aggregate wealth of the jurisdiction that is defined by $\omega^J(l) = \int_{C^J(l)} \omega(i) d\lambda$ where the integral is assumed to be the Lebesgue one (which is well-defined if $\omega$ is a bounded and measurable function).\footnote{See e.g. Royden ([25], ch.4.2).} Clearly $\omega^J(l) = \int_{C^J(l)} \omega(i) d\lambda = 0$ if $\lambda(C^J(l)) = 0$. Hence any coalition containing a set of measure 0 of households will have an aggregate wealth of 0 and will therefore not be able to provide positive amount of public good to its members. If such a coalition is to choose a tax rate by some collective choice mechanism to finance public good production, the only reasonable choice for such a tax rate is 0 (why taxing private wealth if no public good is to be provided?). Hence all coalitions with measure 0 will be characterized by a tax rate of 0, will provide 0 unit of public good and will leave their members enjoy all their private wealth as private consumption.

We note that this continuous measure-theoretic framework has a somewhat unpleasant interpretative feature for the purpose at hand. In particular, interpreting $\omega^J(l) = \int_{C^J(l)} \omega(i) d\lambda$ as coalition $C^J(l)$’s aggregate wealth may seem problematic since nothing in the framework prevents $\omega(i) > \omega^J(l)$ for some $i \in C^J(l)$. In this continuous setting therefore, a single household may have a wealth that is larger than the (Lebesgue) “sum” of the wealth possessed by all individuals living in the community.

We now turn more specifically to the mechanism used by jurisdictions to choose taxes and public good. We assume that the only collective choice made by a jurisdiction is the choice of a tax rate $t \in [0, 1]$ that is applied to the community’s aggregate wealth and that will finance public good production. Hence, if a jurisdiction structure $J$ assigns household $i$ to localization $l$ where the tax rate is $t$, household $i$ will have $(1 - t)\omega(i)$ units of wealth available for private consumption and will have access to $t\omega^J(l)$ units of public good. Hence the utility received by household $i$ assigned to localization $l$ in the jurisdiction structure $J$ if the tax rate is $t$ is $\Phi(\omega(i), t, \omega^J(l)) = U(t\omega^J(l), (1 - t)\omega(i))$.\footnote{See e.g. Royden ([25], ch.4.2).}
We record immediately for further references a few properties of the function $\Phi : \mathbb{R}_+ \times [0, 1] \times \mathbb{R}_+ \rightarrow \mathbb{R}$ so defined.

**Lemma 1** Under the assumptions made above on $U$, $\Phi$ is a twice differentiable function, is strictly concave with respect to $\omega(i)$ and $\omega^j(l)$ (taking $t \in [0, 1]$ as fixed), is strictly concave with respect to $t$ (taking $\omega(i)$ and $\omega^j(l) \in \mathbb{R}_+$ as fixed), is strictly single peaked with respect to $t^5$ (taking $\omega(i)$ and $\omega^j(l) \in \mathbb{R}_+$ as fixed) and is strictly increasing with respect to both $\omega(i)$ and $\omega^j(l)$ (taking $t$ as fixed).

**Proof.** We omit the obvious proof of the differentiability of $\Phi$ as well as that of strict increasingness with respect to $\omega(i)$ and $\omega^j(l)$ (taking $t$ fixed). The strict concavity with respect to the two variables $\omega(i)$ and $\omega^j(l)$ (taking $t \in [0, 1]$ as fixed) is also straightforwardly obtained from the assumed strict concavity of $U$. As for the strict concavity of $\Phi$ with respect to $t$ (taking $\omega(i)$ and $\omega^j(l) \in \mathbb{R}_+$ as fixed), consider any $\alpha \in [0, 1]$. We have, taking $\omega(i)$ and $\omega^j(l)$ in their domain as given, for every $t, t':$

$$\Phi(\omega(i), \alpha t + (1 - \alpha)t', \omega^j(l)) = U((\alpha t + (1 - \alpha)t')\omega^j(l), (1 - (\alpha t + (1 - \alpha)t'))\omega(i))$$

$$= U((\alpha t + (1 - \alpha)t')\omega^j(l), (\alpha(1 - t)\omega(i) + (1 - \alpha)(1 - t')\omega(i))$$

$$\geq \alpha U(t\omega^j(l), (1 - t)\omega(i)) + (1 - \alpha) U(t'\omega^j(l), (1 - t')\omega(i))$$

$$= \alpha \Phi(\omega(i), t, \omega^j(l)) + (1 - \alpha) \Phi(\omega(i), t', \omega^j(l))$$

(the third inequality being implied by the assumed strict concavity of $U$). For single peakedness, take any $\omega(i)$ and $\omega^j(l) \in \mathbb{R}_+$ as given and let $t^* \in \arg \max_{t \in [0, 1]} \Phi(\omega(i), t, \omega^j(l))$. Since $\Phi$ is a continuous function of $t$ defined on the compact set $[0, 1]$, $t^*$ exists by virtue of Weierstrass’ theorem. In order to prove strict single peakedness of $\Phi$ with respect to $t$, it is sufficient to prove that, for every pair $t$ and $t' \in [0, 1]$ such that $t < t' \leq t^*$ (case (i)) or such that (case (ii)) $t^* > t'$ one has $\Phi(\omega(i), t^*, \omega^j(l)) \geq \Phi(\omega(i), t', \omega^j(l)) > \Phi(\omega(i), t, \omega^j(l))$. By symmetry, only one case (say case (i)) need to be considered. Assume by contradiction that $\Phi(\omega(i), t^*, \omega^j(l)) \geq \Phi(\omega(i), t', \omega^j(l)) \geq \Phi(\omega(i), t', \omega^j(l))$ for $t < t' < t^*$. Clearly, there exists $\alpha \in [0, 1]$ such that $\alpha t + (1 - \alpha)t^* = t'$. By the just established strict concavity of $\Phi$ with respect to $t$, $\Phi(\omega(i), t', \omega^j(l)) > \alpha \Phi(\omega(i), t, \omega^j(l)) + (1 - \alpha) \Phi(\omega(i), t^*, \omega^j(l))$. Since, by the very definition of $t^*$, $\Phi(\omega(i), t^*, \omega^j(l)) \geq \Phi(\omega(i), t, \omega^j(l))$ and $\alpha \in [0, 1]$, we must have that

$$\Phi(\omega(i), t', \omega^j(l)) > \alpha \Phi(\omega(i), t, \omega^j(l)) + (1 - \alpha) \Phi(\omega(i), t^*, \omega^j(l))$$

$$\geq \Phi(\omega(i), t, \omega^j(l))$$

which establishes that $\Phi(\omega(i), t^*, \omega^j(l)) \geq \Phi(\omega(i), t', \omega^j(l)) > \Phi(\omega(i), t, \omega^j(l))$ as required. $\blacksquare$

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A function $f : A \rightarrow \mathbb{R}$ ($A \subset \mathbb{R}$) is strictly single peaked if, for all $a$, $b$ and $c \in A$ such that $a < b < c$, $f(c) < f(b) \Rightarrow f(b) > f(a)$ and $f(a) > f(b) \Rightarrow f(b) > f(c)$. 

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8
An important (and well-known) property of $\Phi$ is its strict single peakedness. It implies, among other things, that any individual $i$ with wealth $\omega(i)$ has a unique favorite tax rate $t^*(\omega(i), \omega^J(l))$ in any coalition $C$ of aggregate wealth $\omega_C$ to which it may belong. This unique favorite tax rate is the solution of the program:

$$\max_{t \in [0,1]} \Phi(\omega(i), t, \omega_C)$$

and as such, and given the strict concavity of $\Phi$ with respect to $t$, is a continuous function of $\omega(i)$ and $\omega_C$ thanks to Berge [4] maximum theorem.

We assume that the tax rate chosen in any coalition $C$ is the favorite tax rate of a specific individual in the coalition, the individual being chosen according to the position it occupies in the distribution of favorite tax rates. We define this notion of an $r$-positional dictator in a measurable coalition as follows.

**Definition 1** Given a jurisdiction structure $J$, a location $l$ such that $C^J(l)$ has a strictly positive measure, and a real number $r \in ]0, 1[$, we call a $r$-positional dictator in $C^J(l)$ any individual $i$ whose favorite tax rate is adopted by the coalition $C^J(l)$ and who is such that

$$\frac{\lambda\{h \in C^J(l) : t^*(\omega(h), \omega^J(l)) \leq t^*(\omega(i), \omega^J(l))\}}{\lambda(C^J(l))} = r$$

If $C^J(l)$ is a coalition of a strictly positive measure, such a $r$-positional dictator always exists as the set $\{h \in C^J(l) : t^*(\omega(h), \omega^J(l)) \leq t^*(\omega(i), \omega^J(l))\}$ is measurable and the function $t^*$ is continuous. The most well-known example of a $r$-positional dictator is the median individual which correspond to the case where $r = \frac{1}{2}$. Such a $\frac{1}{2}$-positional dictator would be given the power to decide the tax rate if this tax rate was chosen by majority voting. But one could think of many other kinds of “positional dictators” (for example, that of the household with the smallest most preferred tax rate which would correspond to $r = 0$). A nice feature of the results herein is that they hold for any rule that leave to some $a priori$ given $r$-positional dictator the task of choosing the coalition tax rate.

We assume therefore that any (Lebesgue measurable) coalition $C \subseteq [0, 1]$ will leave to some $r$-positional dictator the task of choosing the tax rate. As there may be many such $r$-positional dictators, we assume specifically that the coalition will actually select the *infinum*, taken over all $r$-positional dictators in the coalition, of these tax rates.\(^7\) Given a Lebesgue measurable

\(^6\)This concept is a generalization to the continuous case of the notion of positional dictator introduced in Moulin ([20]).

\(^7\)This assumption is not important. We only need a way to make a selection among the (possibly) many $r$-positional dictators in a coalition. We could as well take the supremum or any other fixed method to make the selection.
coalition $C \subseteq [0, 1]$, we denote by $t^C$ its tax rate chosen by the $r$-positional dictator in the way just described.

We are now equipped to define what we mean by a stable jurisdiction structure in an economy.

**Definition 2** A jurisdiction structure $J : [0, 1] \to \mathbb{L}$ is stable in the economy $(\lambda, \omega, U)$ if, for all $l \in \mathbb{L}$ and for all $i \in C^J(l)$, \( \Phi(\omega(i), t^C(l), \omega^J(l)) \geq \Phi(\omega(i), t^C(l'), \omega^J(l')) \) for every $l' \in \mathbb{L}$.

In words, a jurisdiction structure is stable if it does not give households any strict incentive to move from their location. This notion of stability obviously corresponds to the usual game theoretic concept of a Nash equilibrium. It should be contrasted with the somewhat strongest notion that is commonly encountered in the literature on endogenous jurisdictions formation (see for instance Greenberg and Weber ([14])) and which concerns stability with respect to group deviations.

Notice that, since any individual has measure 0, the move of some individual to some location does not affect the aggregate wealth nor the chosen tax rate in the jurisdiction. This “infinitesimality” of households that it allows, which greatly simplify the analysis, is actually the main motivation for using a continuous framework rather than a discrete one. This advantage comes however at the cost of the interpretative difficulties alluded to above.

It is also worth noticing that, when considering the possibility of moving away from a jurisdiction, a household may contemplate going to some localization inhabited by a set of measure 0 of households where the tax rate and the quantity of public good are both 0. Hence, if households preferences satisfy the (Inada) condition that \( U(Z, x) > U(0, x') \) for every $Z, x, x' \in \mathbb{R}_{++}$, the jurisdiction structure $J^G$ defined, for some given $l \in \mathbb{L}$ by $J^G(i) = l$ for every $i \in [0, 1]$ will be stable if $\int_{[0,1]} \omega(i) d\lambda > 0$. This obvious remark guarantees that, at least if the Inada condition holds and if the aggregate wealth in the economy is not 0, there always exist at least one stable jurisdiction structure in the economy: The trivial one where all households are pooled together in the same jurisdiction. If the Inada condition is not imposed, then even this degenerate jurisdiction structure may not be stable.

As this paper is not concerned with existence, we do not need to impose the Inada condition. Rather, our purpose is to provide a necessary and sufficient condition on the households preferences which ensures that any stable jurisdiction structure, if there exists one, is wealth-stratified. In order to do this, we need first to define what we mean by a wealth-stratified jurisdiction structure.

**Definition 3** A jurisdiction structure $J$ in the economy $(\lambda, \omega, U)$ is wealth-stratified if, for every $l, l' \in \mathbb{L}$, $i, j, k \in [0, 1]$, $(i, k \in C^J(l)$, $\omega(i) < \omega(j) < \omega(k)$.
\[\omega(k), j \in C^J(l') \Rightarrow [(t^{C^J(l')}, \omega^J(l')) = (t^{C^J(l)}, \omega^J(l))]\]

In words, a jurisdiction structure is wealth-stratified if whenever a jurisdiction contains two households \(i\) and \(k\) with different levels of wealth, it must also contain all households whose wealth are strictly in between that of \(i\) and \(k\). Hence, in a wealth-stratified jurisdiction structure, households with adjacent wealth levels are clustered together in a strong manner. This property of stratification of a jurisdiction structure is called “consecutiveness” by Greenberg and Weber [14]. These authors however use the word “consecutive” more generally to designate a coalition who forms an interval with respect to some \textit{a priori} given ordering of the players. Yet, this ordering of players need not be that induced by the comparison of their wealth. As a matter of fact, when Greenberg and Weber [14] interpret their general notion of consecutiveness in a setting analogous to the one considered here where coalitions provide a public good to their members (who have identical additively separable preferences for the public good and the private good) and finance this public good by proportional wealth taxation, they claim that the \textit{a priori} ordering of individuals is that induced by the wealth. However, as we shall see in the remark below, this claim is false.

We note in passing that our definition of stratification allows households with identical wealth to belong to distinct jurisdictions. This may, perhaps, sound at odd with one’s intuitive conception of the essence of stratification. Yet, this feature is unavoidable in the continuous setting adopted herein where individual households have no impact on the jurisdictions’s tax rates and public good packages and where two households with identical wealth may be indifferent between two distinct jurisdictions. If such an indifference take place, it is easy to imagine a stable jurisdiction structure in which the population of households with identical wealth split up into the two jurisdictions between which each household is indifferent.

3 Main results

Although the intuition that stable jurisdiction structures involve stratifications of households into homogeneous group seems quite plausible, its validity requires that the households preference satisfy a rather strong condition. To illustrate this, we first provide an example of a standard and simple economy in which a stable jurisdiction structure is not wealth stratified.

\textbf{Example 1} The economy considered has households that are split-up into three groups, all defined with respect to some strictly positive real number \(\varepsilon\). A first group consists of a continuum of mass 3 households with wealth uniformly distributed on the interval \([2 + \sqrt{2} - \varepsilon, 2 + \sqrt{2} + \varepsilon]\). In the second
group, there is a continuum of mass 3 households with wealth levels uniformly distributed on the interval $[2 - \sqrt{2} - \varepsilon, 2 + \sqrt{2} - \varepsilon]$ while in the third group, there is a continuum of mass 8 households with wealth uniformly distributed on the interval $[\frac{3}{2} - \varepsilon, \frac{3}{2} + \varepsilon]$. There is therefore a total mass of 14 consumers. We assume that households preferences are represented by the utility function $U(Z, x) = \ln Z + 4x - x^2$. This utility function satisfy all of the above properties (it is in particular strictly increasing with respect to the private good if $x \leq 2$, which will be the case in this economy). The favorite tax rate of a household with wealth $\omega(i)$ living in a jurisdiction where the aggregate wealth is $\omega_C$ solves the program

$$\max_t \ln t \omega_C + 4(1 - t)\omega(i) - (1 - t)^2 \omega(i)^2$$

whose first order conditions are

$$\frac{1}{t^*} - 4\omega(i) + 2(1 - t^*) \omega(i)^2 = 0$$

Solving for $t$ (and using the positive root) yields

$$t^* = \frac{\omega(i) - 2 + \sqrt{(\omega(i) - 2)^2 + 2}}{2\omega(i)}$$

Hence, $t^* = \frac{1}{2}$ if $i$ is a household with wealth $2 - \sqrt{2}$ or $2 + \sqrt{2}$ and $t^* = \frac{1}{3}$ if the household has a wealth $\frac{3}{2}$. Consider now the jurisdiction structure where, for some localizations $l$ and $l' \in \mathbb{L}$ ($l \neq l'$), $J(i) = l \iff \omega(i) \in [2 - \sqrt{2} - \varepsilon, 2 - \sqrt{2} + \varepsilon] \cup [2 + \sqrt{2} - \varepsilon, 2 + \sqrt{2} + \varepsilon]$ and $J(i) = l' \iff \omega(i) \in [\frac{3}{2} - \varepsilon, \frac{3}{2} + \varepsilon]$. This jurisdiction structure is clearly not wealth-stratified since $2 - \sqrt{2} + \varepsilon < \varepsilon - \frac{3}{2} < \frac{3}{2} + \varepsilon < 2 + \sqrt{2} - \varepsilon$ for a suitable choice of $\varepsilon$. Yet it is easy to see that it is stable if, for instance, the 1/2-positional dictatorship rule is adopted. Under this rule, one will have $t_{C^J(l')} = \frac{1}{3}$, $\omega^J(l') = 12 = \omega^J(l)$, $t_{C^J(l)} = \frac{1}{2}$. A household belonging to $C^J(l')$ and endowed with a wealth of $\frac{3}{2}$ enjoys a utility level of

$$3 + \ln(4) \simeq 4.3863$$

while a move to coalition $C^J(l)$ (where the tax is $\frac{1}{2}$) would provide to the household with a utility of :

$$3 - 9/16 + \ln(6) \simeq 4.2293$$

As the inequality $3 + \ln(4) > 3 - 9/16 + \ln(6)$ is strict, it will be robust with respect to the replacement of the household of wealth $\frac{3}{2}$ by any household in
the interval \([\frac{3}{2} - \varepsilon, \frac{3}{2} + \varepsilon]\) is \(\varepsilon\) is small enough. Hence no household located at \(l'\) has incentive to move to \(l\). Analogously, a household with wealth \(2 + \sqrt{2}\) located at \(l\) receives utility of:

\[
4 \left(1 + \frac{\sqrt{2}}{2}\right) - \left(1 + \frac{\sqrt{2}}{2}\right)^2 + \ln(6) \simeq 5.706
\]

while a move to \(l'\) would provide this household with a utility level of

\[
4 \left(\frac{4}{3} + \frac{2\sqrt{2}}{3}\right) - \left(\frac{4}{3} + \frac{2\sqrt{2}}{3}\right)^2 + \ln(4) = 5.31
\]

Here again, the comparison of these two numbers is robust to a change of household in the interval \([2 + \sqrt{2} - \varepsilon, 2 + \sqrt{2} + \varepsilon]\) for a sufficiently small \(\varepsilon\).

Finally, the household at \(l\) whose wealth is \(2 - \sqrt{2}\) obtains at home a utility of:

\[
4 \left(1 - \frac{\sqrt{2}}{2}\right) - \left(1 - \frac{\sqrt{2}}{2}\right)^2 + \ln(6) \simeq 2.8775
\]

while a move to \(B\) would only give this household a utility of

\[
4 \left(\frac{4}{3} - \frac{2\sqrt{2}}{3}\right) - \left(\frac{4}{3} - \frac{2\sqrt{2}}{3}\right)^2 + \ln(4) \simeq 2.7959
\]

(the comparison of these two numbers being robust to the choice of any household in the interval \([2 - \sqrt{2} - \varepsilon, 2 - \sqrt{2} + \varepsilon]\) for \(\varepsilon\) sufficiently small.

The preferences used in the example are quite standard. They are additively separable, strictly concave, differentiable and so on. What is therefore the property of these preferences that enables the construction of this example? This property is exhibited on the picture below which shows the relationship, for the example above, between a household’s most preferred tax rate in a jurisdiction and its private wealth (denoted \(y\) on the picture), for a given aggregate wealth.\(^8\) A specificity of this relationship as depicted on the picture is its non-monotonicity. If households have preferences of this kind, their most preferred tax rates are first decreasing with respect to their wealth and, for wealth level above \(3/2\), increasing with wealth. If households most preferred tax rates are not monotonic with respect to their wealth in a jurisdiction, the ordering of households according to their most preferred tax rates is insufficient to provide us with a ranking of individuals according to their wealth. As it turns out, it is precisely this property that must necessarily hold for wealth stratification to be a characteristic of any stable jurisdiction structure.

\(^8\)As it turns out, the function \(t^*\) does not depend upon the jurisdiction’s aggregate wealth in the specific case of the preferences considered in the example.
We define as follows this property of strict monotonicity of \( t^* \) with respect to \( \omega(i) \).

**Definition 4** The function \( t^* : \mathbb{R}^2_{++} \Rightarrow \mathbb{R} \) that solves (2) is strictly monotonic with respect to \( \omega \) if, for every aggregate wealth level \( \overline{\omega} \), either \( \hat{\omega}(i) > \overline{\omega} \Rightarrow t^*(\hat{\omega}(i), \overline{\omega}) > t^*(\overline{\omega}(i), \overline{\omega}) \) or \( \overline{\omega}(i) > \hat{\omega}(i) \Rightarrow t^*(\hat{\omega}(i), \overline{\omega}) < t^*(\overline{\omega}(i), \overline{\omega}) \).

As the example above makes clear, the non-monotonicity of the household’s most preferred tax rates with respect to wealth is perfectly compatible with additively separable preferences. This gives us the opportunity to mention in passing, in the following remark, that part b) of lemma 1 in Greenberg and Weber [14] is false.

**Remark 1** Greenberg and Weber [14] claim (see part b) in the proof of their lemma 1) that if households preferences can be represented by an additively separable utility function, then, for all quantities \( \overline{Z} \) and \( \hat{Z} \) of public good, all pairs of aggregate wealth \( \overline{\omega}_C \) and \( \hat{\omega}_C \) and all triplets of individuals \( i, j \) and \( k \) such that \( \omega(i) < \omega(j) < \omega(k) < \min [\overline{\omega}_C, \hat{\omega}_C] \), if

\[
\begin{align*}
    f(\overline{Z}) + g(\omega(i)(1 - \frac{\overline{Z}}{\overline{\omega}_C + \omega(i)})) &< f(\hat{Z}) + g(\omega(i)(1 - \frac{\hat{Z}}{\hat{\omega}_C})) \quad (3) \\
    f(\overline{Z}) + g(\omega(k)(1 - \frac{\overline{Z}}{\overline{\omega}_C + \omega(k)})) &< f(\hat{Z}) + g(\omega(k)(1 - \frac{\hat{Z}}{\hat{\omega}_C})) \quad (4)
\end{align*}
\]
then
\[ f(Z) + g(\omega(j)(1 - \frac{Z}{\omega_C})) < f(\tilde{Z}) + g(\omega(j)(1 - \frac{\tilde{Z}}{\omega_C + \omega(j)})) \]  \hspace{1cm} (5)

To show the falsity of such a claim, take individual \( i, j \) and \( k \) as in the example above with \( \omega(i) = 2 - \frac{\sqrt{2}}{2} < \omega(j) = \frac{3}{2} < \omega(k) = 2 + \frac{\sqrt{2}}{2} \) with \( \omega_C = 120000 \) and let \( \tilde{Z} = \frac{1}{2} \times 120000 = 60000 \) and \( \bar{Z} = \frac{1}{3} \times 120000 = 40000 \). Assume also that the utility functions are just as in the example with \( f(Z) = \ln Z \) and \( g(x) = 4x - x^2 \). Clearly both (3) and (4) are satisfied since

\[
\ln(40000) + 4(2 - \frac{\sqrt{2}}{2})(1 - \frac{40000}{120000 + 2 - \frac{\sqrt{2}}{2}}) - (2 - \frac{\sqrt{2}}{2})^2(1 - \frac{40000}{120000 + 2 - \frac{\sqrt{2}}{2}})^2 \approx 12.006 < \ln(60000) + 4(2 - \frac{\sqrt{2}}{2})(1 - \frac{60000}{120000}) - (2 - \frac{\sqrt{2}}{2})^2(1 - \frac{60000}{120000})^2 \\
\approx 12.088
\]

and

\[
\ln(40000) + 4(2 + \frac{\sqrt{2}}{2})(1 - \frac{40000}{120000 + 2 + \frac{\sqrt{2}}{2}}) - (2 + \frac{\sqrt{2}}{2})^2(1 - \frac{40000}{120000 + 2 + \frac{\sqrt{2}}{2}})^2 =: 14.52 < \ln(60000) + 4(2 + \frac{\sqrt{2}}{2})(1 - \frac{60000}{120000}) - (2 + \frac{\sqrt{2}}{2})^2(1 - \frac{60000}{120000})^2 \\
= 14.916
\]

Yet

\[
\ln(40000) + 4(\frac{3}{2})(1 - \frac{40000}{120000}) - (\frac{3}{2})^2(1 - \frac{40000}{120000})^2 \approx 13.597 > \ln(60000) + 4(\frac{3}{2})(1 - \frac{60000}{120000 + \frac{3}{2}}) - (\frac{3}{2})^2(1 - \frac{60000}{120000 + \frac{3}{2}})^2 \\
\approx 13.44
\]

in violation of (5). Hence contrary to what Greenberg and Weber claim, a jurisdiction structure that is immune to coalitional deviation does not lead to jurisdictions that are “consecutive” with respect to the ordering of households according to their income.

Before stating formally the result that the strict monotonicity of the individual’s optimal tax rate with respect to wealth is necessary to guarantees the stratification of any stable jurisdiction structure, we recall a few facts about the behavior of this optimal tax rate \( t^*(\omega(i), \omega_C) \) as a solution of the program (2).

**Lemma 2** Let \( (\omega(i), \omega_C) \in \mathbb{R}^2_+ \). Then, \( \frac{Z^M}{\omega_C - \omega(i)} \) is the unique solution of (2)
Proof. That the solution of (2) is unique is guaranteed by the strict concavity of the objective function with respect to $t$. We note also that, since $\frac{Z^M(\frac{1}{\omega_C}, \frac{1}{\omega_i})}{\omega_C} \geq 0$ and $\frac{x^M(\frac{1}{\omega_C}, \frac{1}{\omega_i})}{\omega_i} \geq 0$, the fact that $Z^M(\frac{1}{\omega_C}, \frac{1}{\omega_i})$ satisfies the budget constraint $\frac{Z^M(\frac{1}{\omega_C}, \frac{1}{\omega_i})}{\omega_C} + \frac{x^M(\frac{1}{\omega_C}, \frac{1}{\omega_i})}{\omega_i} = 1$ implies that $\frac{Z^M(\frac{1}{\omega_C}, \frac{1}{\omega_i})}{\omega_C} \in (0, 1]$. Suppose by contradiction that $\frac{Z^M(\frac{1}{\omega_C}, \frac{1}{\omega_i})}{\omega_C}$ does not solve (2). That is, suppose that there exists $\bar{t} \in [0, 1]$ such that $U(\bar{t} \omega_C, (1 - \bar{t}) \omega(i)) > U(Z^M(\frac{1}{\omega_C}, \frac{1}{\omega_i}), x^M(\frac{1}{\omega_C}, \frac{1}{\omega_i}))$. But since the bundle $(\bar{t} \omega_C, (1 - \bar{t}) \omega(i))$ satisfies the budget constraint $\frac{\omega_C}{\omega_C} + \frac{(1 - \bar{t}) \omega(i)}{\omega(i)} = 1$, this inequality is incompatible with the very definition of $Z^M(\frac{1}{\omega_C}, \frac{1}{\omega_i})$ and $x^M(\frac{1}{\omega_C}, \frac{1}{\omega_i})$. 

Hence lemma 2 just states the fact that the household’s most preferred tax rate can be viewed as the expenditure (using the fraction of the household’s wealth as the numéraire) that the household would like to devote to the public good if the price of this public good was $\frac{1}{\omega_C}$ and the price of private good was $\frac{1}{\omega_i}$. This easy link with classical consumer’s theory is convenient because it enables one to interpret somewhat differently this monotonicity of the individual’s most preferred tax rate. For instance, one can easily see the following.

Lemma 3 $t^*(\omega(i), \omega_C)$ is strictly monotonic with respect to $\omega(i)$ if and only if public good is either always a gross complement to the private good or always a gross substitute to the private good.

Proof. Recalling that the public good is a gross complement (resp. a gross substitute) to the private good at normalized prices $(\pi_z, \pi_x) \in \mathbb{R}_{++}^2$ iff $\frac{\partial Z^M(\pi_z, \pi_x)}{\partial \pi_z} \leq 0$ (resp. $\geq 0$), the result is obtained as a straightforward combination of definition 4 and lemma 2.

We now establish that the monotonicity of $t^*$ with respect to $\omega(i)$ (equivalently the fact that the relation of gross substitutability between the private good and the public good is independent from the price of the private good) is a necessary condition for a stable jurisdiction structure to be wealth stratified.

Proposition 1 The monotonicity of $t^*(\omega(i), \omega_C)$ with respect to $\omega(i)$ is a necessary condition for a stable jurisdiction structure to be stratified.

Proof. Assume that $t^*(\omega(i), \omega_C)$ is not monotonic with respect to $\omega(i)$. This implies that, for some aggregate wealth level $\omega_C$, there exists individual wealth levels $a, b, c \in \mathbb{R}_{++}$ such that $a < b < c$ and such that either (i) $t^*(a, \omega_C) = t^*(c, \omega_C) > t^*(b, \omega_C)$ or (ii) $t^*(a, \omega_C) = t^*(c, \omega_C) < t^*(b, \omega_C)$. Assume first case (i). Consider then an economy where $\lambda\{i \in [0, 1] : \omega(i) = a\} = A$, $\lambda\{i \in [0, 1] : \omega(i) = c\} = C$ and $\lambda\{i \in [0, 1] : \omega(i) = b\} = B$ and where $\lambda([0, 1]) = A + C + B$ where $A, B$ and $C$ are such that $a A + c C + b B =$.
\(\omega_C\). Consider then the jurisdiction structure \(J\) defined by \(J(i) = 1\) if \(\omega(i) = a\) or \(\omega(i) = c\) and \(J(i) = l'\) if \(\omega(i) = c\). Both jurisdictions have aggregate wealth \(\omega_C\). Jurisdiction \(C^J(l)\) obviously chooses \(t^*(a, \omega_C) = t^*(c, \omega_C)\) as its tax rate while the choice of jurisdiction \(C^J(l')\) is \(t^*(b, \omega_C)\). By the very definition of the function \(t^*\) and the uniqueness of the solution of the program (2), this jurisdiction structure is stable. The proof for the case (ii) is similar.

Proposition 1 thus shows that unless \(t^*\) is monotonic with respect to \(\omega(i)\), for every aggregate wealth level \(\omega_C\), it is possible to find stable jurisdiction structures that are not wealth-stratified. As the example above illustrates the stringency of the requirement of monotonicity of \(t^*\) with respect to \(\omega(i)\), we conclude from proposition 1 that stratification of households according to their wealth is not the necessary outcome of a decentralized process of jurisdictions formation. It is worth emphasizing that this result does not depend crucially upon the fact that there is a continuum of households and would hold as well if the number of households was assumed to be finite. A counter example analogous to the one presented in the proof of proposition 1 could be constructed in a finite economy if a “large” number of individuals of wealth \(a, b\) and \(c\) was assumed in such a way as to render negligible the impact of individual’s entry on the jurisdiction’s tax rate and aggregate wealth. Hence even in a finite economy, monotonicity of \(t^*\) would be a necessary condition for any stable jurisdiction structure to be wealth stratified.

We now turn to the question of the sufficiency of the monotonicity of \(t^*\) with respect to \(\omega(i)\) for the wealth stratification of any stable jurisdiction structure. The examination of this question requires some knowledge of the structure of the households’ indifference curves in the \((t, \omega_C)\) plane obtained from the function \(\Phi\) defined in the preceding section. As usual, the indifference curve of a household of wealth \(\omega(i)\) passing thorough point \((\overline{t}, \overline{\omega}_C)\) such that \(\Phi(\omega(i), \overline{t}, \overline{\omega}_C) = \overline{\Phi}\) is the graph of the implicit function \(\omega_{\overline{\Phi}}(.; \omega(i)) : [0, 1] \rightarrow \mathbb{R}_+\) defined by \(\Phi(\omega(i), t, \omega_{\overline{\Phi}}(t; \omega(i))) \equiv \Phi\). The assumption imposed on \(U\) guarantees that the function \(\omega_{\overline{\Phi}}\) is derivable everywhere and that its derivative \(\frac{\partial \omega_{\overline{\Phi}}(t; \omega(i))}{\partial t}\) evaluated at \(\overline{t}\) is given by:

\[
\frac{\partial \omega_{\overline{\Phi}}(t; \omega(i))}{\partial t} = \frac{1, U_x(\overline{\omega}_C, (1 - \overline{t})\omega(i))}{\overline{\Phi}, U_Z(\overline{\omega}_C, (1 - \overline{t})\omega(i))} \omega(i) - \overline{\omega}_C
\]  

Using (6), one can represent the indifference maps of individuals with different levels of wealth as in the figure 2 below. Indifference curves of a household with private wealth \(\omega(i)\) are \(U\)-shaped and reach a minimum at this individual’s most preferred tax rate for the corresponding aggregate

\[\text{It is straightforward to establish that } \omega_{\overline{\Phi}}(.; \omega(i)) \text{ is indeed a function.}\]
wealth level) (at the minimum of an indifference curve, the term within the bracket is zero thanks to the first order condition of the program (2)). Despite what the picture may suggest, indifference curves need not be convex. The only property that these indifference curves possess is that of being “single caved” (monotonically decreasing at the left of the minimum and monotonically increasing at the right).

We first establish an important property satisfied by the slopes of these indifference curves when the household’s most preferred tax rate is a monotonic function of private wealth for every aggregate level $\omega_C$: that of being ordered in the same way as households’ wealth at every minimum of an indifference curve. This result is the content of the following lemma.

**Lemma 4.** Assume that households preferences are represented by a utility function in $U$. Then, if, for any given $\omega_C \in \mathbb{R}_+$, the most preferred tax rate function $t^*$ is monotonically decreasing (resp. increasing) with respect to $\omega(i)$, we will have, at any $(T, \omega) \in \mathbb{R}_+ \times [0,1]$ such that $t = t^*(\omega(i), \omega)$ for some $\omega(i) \in \mathbb{R}_+$, $\frac{\partial^2 \Phi}{\partial t^2} (T, \omega(k)) > (\text{resp. } <) \frac{\partial^2 \Phi}{\partial t^2} (T, \omega(j))$ for every $\omega(j)$,
\(\omega(k) \in \mathbb{R}_+\) such that \(\omega(j) < \omega(k)\) where for every \(i\), \(\Phi_i = \Phi(\omega(i), \overline{\omega}, \overline{x}C)\).

**Proof.** Let \((\overline{t}, \overline{\omega}) \in [0,1] \times \mathbb{R}_+\) be a tax-rate and aggregate wealth package such that, for some \(\omega(i) \in \mathbb{R}_+,\) \(\overline{t} \in \arg\max_{t \in [0,1]} \Phi(x(i), t, \overline{\omega}).\) Since \(\Phi\) is derivable and strictly concave, the solution of this program is unique and characterized by the first order condition

\[
MRS^U(t^\ast(\omega(i), \overline{\omega}), (1 - t^\ast(\omega(i), \overline{\omega})))\omega(i) = \frac{\omega(i)}{\overline{\omega}}
\]

Differentiating this identity with respect to \(\omega(i)\) and using the derivability of \(t^\ast\) (obtained from the derivability of the Marshallian demand function thanks to lemma 2) yields (after some manipulation)

\[
\frac{\partial t^\ast(.)}{\partial \omega(i)} = \frac{U_x(.) + (1 - \overline{t})U_{xx}(.\omega(i) - U_{Zx}(.)\overline{\omega})}{\omega(i)^2U_{xx}(.) - 2\omega(i)\overline{\omega}U_{Zx}(.) + \overline{\omega}^2U_{ZZ}(.)}
\]

(7)

As the denominator of this expression is negative due to the strict concavity of \(U\), the function \(t^\ast\) is monotonic with respect to \(\omega(i)\) if and only if the sign of \(U_x(\overline{t}, \overline{\omega}) + (1 - \overline{t})[\omega(i)U_{xx}(\overline{t}, \overline{\omega}) - \overline{\omega}U_{Zx}(\overline{t}, \overline{\omega})]\) is constant. Now, consider any indifference curve represented by (6) and differentiate it with respect to \(\omega(i)\). This yields

\[
\frac{\partial^2 \omega C^\ast(t, \omega(k))}{\partial t \partial \omega(i)} = \frac{U_{Zx}(.)U_x(1 - \overline{t})U_{xx}(.), \omega(i) - U_{x}(.), \omega(i)U_{Zx}(.)}{\overline{\omega}^2U_{Zx}(.)^2}
\]

Evaluating this expression at \(\overline{t} = t^\ast(\omega(k), \omega C^\ast(\overline{t}, \omega(k)))\) and using the first order conditions of the program (2), one obtains

\[
\frac{\partial^2 \omega C^\ast(t, \omega(k))}{\partial t \partial \omega(i)} = \frac{U_x(.) + (1 - \overline{t})[\omega(i)U_{xx}(.) - \overline{\omega}U_{Zx}(.)]}{\overline{\omega}U_{Zx}(.)}
\]

which can be signed using (7) above. Hence, at the minimum of any indifference curve, increasing household’s wealth increases (resp. decreases) the slope of the indifference curve if \(U_x(.) + (1 - \overline{t})[\omega(i)U_{xx}(.) - \overline{\omega}U_{Zx}(.)] \geq 0\) (resp. \(\leq 0\)) everywhere, a case which arises if and only \(t^\ast\) is monotonically decreasing (resp. increasing) with respect to private wealth for the aggregate wealth level at which the minimum is examined. 

In the next lemma, we establish that if, for every aggregate wealth level \(\overline{\omega}\), \(t^\ast\) is strictly monotonic with respect to \(\omega(i)\), then the ordering of the individuals according to their most preferred tax rates does not depend upon the aggregate wealth. Put differently, the following lemma establishes that if \(t^\ast\) is increasing (resp. decreasing) with respect to \(\omega(i)\) at some aggregate wealth level \(\overline{\omega}\), then it will be increasing (resp. decreasing) with respect to \(\omega(i)\) at every aggregate wealth level.
Lemma 5  Assume that the function $t^*$ is strictly monotonic with respect
to $\omega(i)$ as per definition 4 and that the household’s utility function belongs
to $\mathcal{U}$. Then, for every $\omega(i)$, $\omega(j)$, $\omega$, $\omega$ $\in \mathbb{R}_{++}$, $(t^*(\omega(i), \omega) > t^*(\omega(j), \omega))$ \iff $(t^*(\omega(i), \omega) > t^*(\omega(j), \omega))$

Proof. Assume that $t^*$ is strictly monotonic with respect to $\omega(i)$ as per definition 4. We recall that, by virtue of lemma 2 and the assumptions
implied by the utility function, $t^*$ is a continuous function of its
range. Then, for every $\omega(i)$, $\omega(j)$, $\omega$, $\omega$ $\in \mathbb{R}_{++}$, \lim_{\omega(i)\rightarrow0} t^*(\omega(i), \omega)$ and $\lim_{\omega(i)\rightarrow\infty} t^*(\omega(i), \omega)$. As $t^*$ has a compact range, these limits are well-defined. Since $t^*$ is a continuous function that
is strictly monotonic with respect to its first argument, we know that the
mapping \( t^{\omega-1} : \mathbb{R}_{++} \rightarrow \mathbb{R}_{++} \)
\begin{align*}
\min\left[ t(0, \omega), t(\infty, \omega) \right], \max\left[ t(0, \omega), t(\infty, \omega) \right] \rightarrow \mathbb{R}_+, \\
\end{align*}
defined by \( t^{\omega-1}(t) = \omega(i) \iff t = t^*(\omega(i), \omega) \) is a continuous and one-to-one strictly monotonic function for any given $\omega$. Consider then any $\omega \in \mathbb{R}_{++}$ and Let $\omega(i)$ and $\omega(j)$ be two distinct wealth levels such that $t^*(\omega(j), \omega) > t^*(\omega(i), \omega)$. Suppose by contradiction that, for some $\omega \in \mathbb{R}_{++}$ (with $\omega \neq \omega$), one has $t^*(\omega(j), \omega) < t^*(\omega(i), \omega)$ (the weak inequality being ruled out by the
requirement of strict monotonicity). Such an assumption obviously implies that
\begin{align*}
\min(t(\infty, \omega) - t(0, \omega), t(\infty, \omega) - t(0, \omega)) < 0 < \max(t(\infty, \omega) - t(0, \omega), t(\infty, \omega) - t(0, \omega))
\end{align*}
Define the function $\Upsilon : [\min[\omega, \omega], \max[\omega, \omega]] \rightarrow \mathbb{R}_+$ by $\Upsilon(\omega) = t(\infty, \omega) - t(0, \omega)$. The remark made above on the properties of $t^{\omega-1}$ guarantees the
continuity of $\Upsilon$. Hence, by virtue of the mean value theorem, inequality (8) implies the existence of $\hat{\omega}$ such that $t(\infty, \hat{\omega}) - t(0, \hat{\omega}) = 0$. But this implies that $t^*$ is not strictly monotonic with respect to $\omega(i)$ at aggregate wealth level $\hat{\omega}$.

This lemma enables one to establish that, if the household’s preference
satisfy the regularity condition 1 stated in the preceding section, the strict
monotonicity of the household’s most preferred tax rate with respect to
wealth is sufficient to guarantee that any stable jurisdiction structure involves
wealth stratification. Actually, in the continuous setting considered
herein, we can not exclude the possibility of observing two distinct jurisdic-
tions offering exactly the same package of tax rate and aggregate wealth
inhabited by households with identical characteristics. As these distinct jurisdictions offering identical package of tax rates and public goods
to its citizens would trivially violate stratification, we propose the following
qualified statement of the fact that, if the regularity condition holds, the
strict monotonicity of the household’s most preferred tax rate is sufficient
to ensure the stratification of any stable jurisdiction structure.
Proposition 2. Let the household's preferences be represented by a utility function in $U$ satisfying condition 1. Then, for any economy $(\lambda, \omega, U, L)$, if $t^*$ is a strictly monotonic function of the private wealth, and $J$ is a stable jurisdiction structure such that, for some locations $l$ and $l' \in L$ (with $l \neq l'$), and households $i$, $j$ and $k \in [0, 1]$ such that $\omega(i) < \omega(j) < \omega(k)$, $l = J(i) = J(k)$ and $J(j) = l'$, we must have $(t_{C_j^l}^*, \omega^j(l')) = (t_{C_l^j}^*, \omega^j(l'))$.

Proof. Let $J : [0, 1] \to L$ be a jurisdiction structure in some economy $(\lambda, \omega, U, L)$ such that, for some locations $l$ and $l' \in L$ (with $l \neq l'$), and households $i$, $j$ and $k \in [0, 1]$ such that $\omega(i) < \omega(j) < \omega(k)$, $l = J(i) = J(k)$, $J(j) = l'$ $(t_{C_j^l}^*, \omega^j(l')) \neq (t_{C_j^l}^*, \omega^j(l'))$. We wish to show that such a jurisdiction structure is not stable if the tax rate chosen in any jurisdiction is the one favored by the $r$-positional dictatorship rule for some real number $r \in [0, 1]$. Without loss of generality, one can assume that $\omega(i) = \inf_{h \in C^l} \omega(h)$ and $\omega(k) = \sup_{h \in C^l} \omega(h)$. Denote by $j^*(l)$ and $j^*(l')$ respectively the individuals who are the $r$–positional dictators in the coalitions $C_j^l$ and $C_j^l$. Stability of $J$ requires of course that

$$\Phi(\omega(i), t_{C_j^l}^*, \omega^j(l)) \geq \Phi(\omega(i), t_{C_j^l}^*, \omega^j(l'))$$

$$\Phi(\omega(j), t_{C_j^l}^*, \omega^j(l)) \leq \Phi(\omega(j), t_{C_j^l}^*, \omega^j(l'))$$

$$\Phi(\omega(k), t_{C_j^l}^*, \omega^j(l)) \geq \Phi(\omega(k), t_{C_j^l}^*, \omega^j(l'))$$

We provide the proof for the case where $t^*$ is strictly increasing with respect to private wealth, leaving to the reader the proof of the converse symmetric case. By lemma 5, this must be so at every aggregate wealth level $\omega$. As $t^*$ is strictly monotonic with respect to private wealth, we have that $t^*(\omega(i), \omega^j(l)) < t^*(\omega(j), \omega^j(l)) < t^*(\omega(k), \omega^j(l))$. Assume that $t_{C_j^l}^* \in [t^*(\omega(i), \omega^j(l)), t^*(\omega(j), \omega^j(l))]$ (we leave the symmetric argument for the case where $t_{C_j^l}^* \in [t^*(\omega(j), \omega^j(l)), t^*(\omega(k), \omega^j(l))]$ to the reader). We illustrate the argument on figure 3. Assume first that $t_{C_j^l}^* \geq t_{C_j^l}^*$. By lemma 4, the slopes of households indifference curves are ordered in such a way that

$$\frac{\partial \Phi_i^j(t_{C_j^l}^*, \omega^j(j))}{\partial t} < \frac{\partial \Phi_i^j(t_{C_j^l}^*, \omega^j(j))}{\partial t} < \frac{\partial \Phi_i^j(t_{C_j^l}^*, \omega^j(i))}{\partial t} = 0 < \frac{\partial \Phi_i^j(t_{C_j^l}^*, \omega^j(i))}{\partial t}$$

and are as illustrated on figure 3. One notices that, for any package of tax rate and aggregate wealth $(\bar{T}, \bar{\omega})$ such that $\bar{T} \geq t_{C_j^l}^*$ in a neighborhood of $(t_{C_j^l}^*, \omega^j(l))$ one has

$$\Phi(\omega(j), t_{C_j^l}^*, \omega^j(l)) < \Phi(\omega(j), \bar{T}, \bar{\omega})$$

$$\Phi(\omega(k), t_{C_j^l}^*, \omega^j(l)) \geq \Phi(\omega(k), \bar{T}, \bar{\omega})$$
Hence, for household $k$ to weakly prefer staying at $l$ rather than moving to $l'$, there must be a package of tax rate and aggregate wealth $(t_0, \omega_0)$ (with $t_0 \in [t_{CJ}^i, t_{CJ}^l]$) such that $\Phi(\omega(k), t_{CJ}^i, \omega^J(l)) = \Phi(\omega(k), t_0, \omega_0)$ and $\Phi(\omega(j), t_{CJ}^i, \omega^J(l)) = \Phi(\omega(j), t_0, \omega_0)$. Without loss of generality, let $(t_0, \omega_0)$ be such that $t_0 \leq \tilde{t}$ for every tax rate $\tilde{t} \in [t_{CJ}^i, t_{CJ}^l]$ such that $\Phi(\omega(j), \tilde{t}, \tilde{\omega}) = \Phi(\omega(j), t_{CJ}^i, \omega^J(l))$ and $\Phi(\omega(k), \tilde{t}, \tilde{\omega}) = \Phi(\omega(k), t_{CJ}^i, \omega^J(l))$ for some $\tilde{\omega} \in \mathbb{R}_{++}$. Since $\frac{\partial \omega(\omega_0)}{\partial t} < \frac{\partial \omega(\omega(j))}{\partial t}$, we must have, by definition of $t_0$, that $\frac{\partial \omega(t_0 \omega(k))}{\partial t} \geq \frac{\partial \omega(t_0 \omega(j))}{\partial t}$. Yet this later inequality is incompatible with lemmas 4 and 5 and the fact, guaranteed by the regularity condition 1 and lemma 2, that there exists some individual wealth level $\omega(h)$ such that we have that

$$t_0 \in \arg\max_{t \in [0,1]} U(t \omega_0, (1 - t) \omega(h))$$

The role played by the regularity condition in the proof of this proposition deserves, perhaps, a few comments. A difficulty in establishing the
sufficiency of the strict monotonicity of $t^*$ with respect to the private wealth as a condition for stratification arises when one tries to extend the information provided by the “local” ordering of the slopes of indifference curves at every minimum (which is unique and unambiguously related to the ordering of wealth thanks to lemmas 4 and 5) at some other combination of taxes and aggregate wealth that does not necessarily correspond to the minimum of an indifference curve. The regularity condition ensures that this extension can be made, at least over a certain range of tax rates and aggregate wealth combinations. Put differently, the regularity condition ensures that the indifference curves of individuals with different wealth in the $(t, \omega)$ plane cross only once between any two points that correspond to the minimum of some indifference curves.

4 Conclusion

The main object of this paper was to investigate the general plausibility of the intuitive claim that endogenous and decentralized process of jurisdiction formation lead to a sorting, or a stratification, of households according to their wealth. Using a simple model of jurisdiction formation in which the only trade off made by households is that between the tax they pay and the public good they get, we find that the plausibility of this claim was somewhat weak. For we showed that in order for a stable jurisdiction structure to always lead to a stratification of households according to their wealth, it is necessary and sufficient that the household preference for the private good and the public good be such that they either consider the public good to either always be a gross complement, or always be a gross substitute, to the private good.

Yet the model we use provides a somewhat too stylized representation of the process by which jurisdictions actually form in the “real world”. A missing ingredient of our model is clearly the housing market. In the model examined herein, borrowed roughly from Westhoff [27], households can locate everywhere without paying any cost. In the real world, this is not the case. If I want to live in a particular jurisdiction that offers a package of taxes and public good that I find particularly attractive, I must be able to afford the cost of renting (or buying) a house in this jurisdiction. Endogenous models of jurisdiction formation with housing markets and intra-jurisdiction collective decision making with respect to taxes and public good have been constructed and studied, notably by Rose-Ackerman [24], Epple, Filimon and Romer ([6],[7]) Dunz [9]and Konishi [17]. However these authors have focus their investigation on the question of existence of a (general) equilibrium for these models and have not specifically examined the positive properties of this equilibrium with respect to stratification. Such an examination is an absolute priority for future research.
References


