POVERTY ORDERINGS

Buhong Zheng

University of Colorado at Denver

Abstract. This paper reviews the literature of partial poverty orderings. Partial poverty orderings require unanimous poverty rankings for a class of poverty measures or a set of poverty lines. The need to consider multiple poverty measures and multiple poverty lines arises inevitably from the arbitrariness inherent in poverty comparisons. In the paper, we first survey the ordering conditions of various individual poverty measures for a range of poverty lines; for some measures necessary and sufficient conditions are identified while for others only some easily verifiable sufficient conditions are established. These ordering conditions are shown to have a close link with the stochastic dominance relations which are based on the comparisons of cumulative distribution functions. We then survey the ordering conditions for various classes of poverty measures with a single or a set of poverty lines; in all cases necessary and sufficient conditions are established. These conditions again rely on the stochastic dominance relations or their transformations. We also extend the relationship between poverty orderings and stochastic dominance to higher orders and explore the possibility and the conditions of increasing the power of poverty orderings beyond the second degree dominance condition.

Keywords. Poverty ordering; Poverty measure; Poverty line; Stochastic dominance; Generalized Lorenz Curve; Distribution-sensitivity

It will be desirable, ..., not to rely upon the evidence of a single measure, but upon the corroboration of several. (Hugh Dalton, 1920)

1. Introduction

Since the publication of Sen's (1976) ground-breaking work on poverty measurement, much has been written on this subject and other related issues. The research has not only enhanced our understanding of the way poverty should be measured but also facilitated the empirical rankings of income distributions. Over the years, the literature of poverty measurement has evolved into two closely connected but distinct branches: the construction of summary poverty measures and partial poverty orderings. The first branch closely follows Sen's initial attempt in developing new aggregate poverty measures alternative to the official practice of head counting. As a consequence, the literature now contains more than a dozen new poverty measures, including the one proposed

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by Sen. In constructing and evaluating new poverty measures, most researchers have also adopted Sen's axiomatic approach and proposed various poverty axioms in addition to those of Sen's. Although there is no complete consensus on the exact properties a poverty measure ought to possess, researchers have identified the core set of axioms for poverty measurement. It is generally agreed that a poverty measure must be focused, continuous, monotonic and distribution-sensitive. By now, the literature of poverty-measure construction has been well reviewed and documented (e.g. Foster, 1984; Seidl, 1988; Chakravarty, 1990; Foster and Sen, 1997; and Zheng, 1997). These surveys have usefully clarified various issues surrounding poverty measurement and provided summaries of the literature.

The second branch of the literature, partial poverty orderings, concerns the unanimous rankings of income distributions with multiple poverty criteria. The word 'partial' here is as a counterpart of 'complete' and 'partial ordering' means that not all distributions of interest can be rank ordered. The shift in the focus of research from poverty-measure construction to partial poverty orderings is a natural development of the literature; if the research on poverty-measure construction was due to the lacking of an 'ideal' poverty measure then the research on partial poverty orderings was in part a consequence of the presence of too many 'ideal' measures. The axiomatic framework of poverty-measure construction that Sen (1976) pioneered and adopted by the following researchers does not yield a unique poverty measure. For a set of reasonable poverty axioms there always exist multiple satisfactory poverty measures; the characterization of an individual poverty measure must be based upon either a prior functional form or a questionable axiom. Thus 'any choice of a single measure ... is apt to be arbitrary' (Foster, 1984, p. 242) and so are the conclusions based on this measure. This arbitrariness, however, can be reduced by using *all* poverty measures that satisfy a set of reasonable axioms. That is, instead of choosing individual poverty measures, we may decide a set of appropriate poverty axioms which, in turn, determine a class of poverty measures. Obviously, it is impossible to check poverty orderings for all admissible poverty measures, some easily verifiable conditions must be developed. Also the use of a class of poverty measures, unlike a single measure, may make some distributions of interest not comparable; there may be no unanimous agreement among these measures on poverty comparisons. In general, however, the completeness of poverty orderings can be increased by introducing more restrictive axioms and limiting the admissible poverty measures. Thus, with respect to a class of poverty measures, the goal of the research on partial poverty orderings is twofold: (1) identify the circumstances under which two distributions can be unanimously ranked by various classes of poverty measures, and (2) explore the possibility of increasing the completeness or power of poverty orderings.

The partial nature of poverty rankings also arises inherently from the choice of a poverty line in poverty comparisons. In practice, the poverty line is also adjusted for the difference in family composition, which further increases the partiality of poverty comparisons. The determination of an appropriate poverty line has been an issue of debate ever since the concept was conceived by Rowntree (1901) and Booth (1902). Although various procedures for setting a poverty line have been developed based upon alternative concepts of poverty, 'a feature common to all proposed methods is a significant degree of arbitrariness in the value assigned to the poverty standard' (Foster and Shorrocks, 1988a). Even if a poverty line is regarded as the minimum nutritional need, the arbitrariness remains because 'there is no one level of food intake required for substance, but rather a range where physical efficiency declines with a falling intake of calories and protein' (Atkinson, 1983, p. 226). Since a poverty measure may produce contradictory conclusions at two different yet equally reasonable poverty lines, the use of any single poverty line, like the use of a single poverty measure, is also arbitrary and so are the conclusions based upon this single line. Naturally, this arbitrariness can be greatly reduced if the verdict of poverty comparisons holds for all reasonable poverty lines. Thus it is useful to describe situations where two distributions can be unanimously ranked for all possible poverty lines by a given poverty measure. This constitutes the second goal of the research on partial poverty orderings. For ease of presentation, in this review, we follow Zheng (1999) and refer to the ordering for a class of poverty measures with a fixed poverty line as *poverty-measure ordering* and refer to the ordering of a given poverty measure for a range of poverty lines as *poverty-line ordering*. It is helpful to point out from the outset that although these two dimensions of poverty orderings are conceptually distinct, the derived ordering conditions, as we will show below, are closely interconnected.

Foster raised concerns for both dimensions of partial poverty orderings in his 1984 survey article. In the subsequent years, a number of papers have appeared addressing these concerns and other related issues. In a series of papers, Foster and Shorrocks (1988a, 1988b and 1988c) characterized partial poverty orderings of the headcount ratio, poverty gap ratio and a member of the Foster *et al.* (1984) class when the poverty line varies over a wide range. They uncovered a remarkable link between the poverty-line orderings of these measures and the stochastic dominance conditions which have proven to be very useful ranking tools in both finance and economics. Recently, Foster and Jin (1998) further extended some of the results of Foster and Shorrocks to the Daltonian class of poverty measures. In an important contribution, Atkinson (1987) derived conditions of poverty-measure orderings for classes of additively separable poverty measures with a common poverty line. Interestingly, the ordering conditions again turn out to be based upon stochastic dominance. In some sense, Atkinson's paper also addresses poverty-line orderings since an interval rather than a single value is assumed for the poverty line. Zheng (1999) recently extended Atkinson's results to a more restrictive class of poverty measures and examined the implications and the limit of using higher orders of stochastic dominance in increasing the completeness of poverty orderings. Spencer and Fisher (1992), Jenkins and Lambert (1997, 1998a and 1998b), and Shorrocks (1998) characterized a new intuitive dominance device which can exhibit poverty incidence, poverty intensity and inequality within the poor simultaneously. Atkinson (1992) and Jenkins and Lambert (1993) investigated poverty-measure orderings when the poverty line is adjusted for the difference in family composition. In addition to these contributions, there are also several other interesting papers which further elaborate the two dimensions of poverty orderings.

The objective of this paper is to provide a comprehensive review of the literature on partial poverty orderings. Until now, although this branch of the literature has been briefly reviewed by several authors,[1] a through and updated analysis which integrates both dimensions of partial poverty orderings is lacking. In this paper, we assume away the difference in family composition and focus on partial poverty orderings for a class of poverty measures or a range of poverty lines or both. Although this assumption makes our review more concentrated, it is important to remember that the choice of an equivalence scale is critical in empirical applications (see Coulter *et al.*, 1992 for a detailed exposition). Also in the literature, partial poverty ordering is usually referred to as poverty orderings for simplicity. In the rest of this review we will also adopt this usage.

The remainder of the review is organized as follows. Section 2 provides preliminaries for the review. We briefly review poverty axioms, poverty measures and poverty lines. We also define two dominance criteria: stochastic dominance and generalized Lorenz dominance. Section 3 reviews poverty-line orderings. Section 4 surveys poverty-measure orderings. Section 5 concludes the survey with a summary and some remarks.

2. Preliminaries and notation

In this review, we consider continuous income distributions; all discussions on discrete distributions, whenever needed, are relegated to footnotes. Let x be a random variable of income and assume the income interval is $[0, \infty)$ and the set of income distributions is $\Psi := \{F: [0, \infty) \to [0, 1] \mid F$ is nondecreasing and continuous; $F(0) = 0$ and $F(\infty) = 1\}$. For a population proportion $r \in [0, 1)$, the corresponding income quantile of distribution F is defined as $\xi_r(F) := \inf\{x \mid F(x) \geqslant r\}$. A poverty line separates the population into the poor and nonpoor subgroups and the proportion of the people below a given poverty line z is denoted as $r_z(F)$, i.e. $r_z(F) := \sup\{F(x) \mid x < z\}$.[2] A censored distribution of x at z is defined as the distribution of x^* which sets all values above z to z. For a distribution F censored at the poverty line, its income quantile is denoted as $\tilde{\xi}_r(F)$. Thus $\tilde{\xi}_r(F) = \xi_r(F)$ for $r < r_z$ and $\tilde{\xi}_r(F) = z$ for $r \geqslant r_z$. Also for convenience, we omit the symbol F from various expressions as long as no confusion may arise.

A poverty measure is a function P whose value indicates the degree of poverty associated with a given distribution and a poverty line. There are two broad categories of poverty measures: additively separable measures of poverty and rank-based measures of poverty.[3] In what follows, we first briefly describe these two classes.

2.1. *Poverty measures*

For a given poverty line z and an individual poverty deprivation function $p(x, z)$, an additively separable poverty measure can be written as

$$P(F;\ z) = \int_0^z p(x, z)\ \mathrm{d}F(x)$$

where $p(x, z)$ is assumed to be differentiable in both z and x for $x \in [0, z)$.[4] If $p(x, z) = \ln z - \ln x$ then the poverty measure is what Watts (1968) proposed; If $p(x, z) = \beta^{-1}[1 - (x/z)^{\beta}]$ with $\beta < 1$ then the measure is a monotonic transformation of the second measure proposed by Clark, Hemming and Ulph (1981); the Chakravarty (1983) measure is the Clark *et al.* second measure with $\beta > 0$. For $\beta = 0$ and 1, the Clark *et al.* second measure becomes the Watts measure and the poverty gap ratio, respectively. The class of poverty measures proposed by Foster, Greer and Thorbeke (1984) corresponds to $p(x, z) = (1 - x/z)^{\alpha}$ with $\alpha \geqslant 2$. If α were allowed to take values 0 and 1 then the Foster *et al.* class also contains the official measures of the headcount ratio and the poverty gap ratio. It is of course possible to introduce more additively separable poverty measures along this line. An example of such new measures is the CDS (constant distribution-sensitivity) measure which corresponds to $p(x, z) = e^{\gamma(z-x)} - 1$ with $\gamma > 0$. This new measure, as we will show later, has constant poverty aversion and can be used to characterize the infinite order (mixed) dominance condition for all $\gamma > 0$.

The class of rank-based poverty measures contains the poverty measure proposed by Sen (1976) and its variations. A feature in common is that all use the rank of each poor person within the poor subpopulation (or the whole population) as an indicator of the relative deprivation. Besides the Sen measure, this class includes the measures proposed by Takayama (1979), Thon (1979, 1983) and Kakwani (1980). In general, a rank-based measure can be written as

$$P(F;\ z) = \int_0^z q(F, x;\ z)\ \mathrm{d}F(x).$$

If $q(F, x;\ z) = 2[1 - F(x)/r_z](1 - x/z)$ then $P(F;\ z)$ is the Sen measure; if $q(F, x;\ z) = 2[1 - F(x)](1 - x/\mu_F)$, where $\mu_F = r_z\mu_p + (1 - r_z)z$ is the mean income of the censored distribution and μ_p is the mean income of the poor, then $P(F;\ z)$ is the Takayama measure;[5] If $q(F, x;\ z) = 2[1 - F(x)](1 - x/z)$ then $P(F;\ z)$ is the measure that Thon proposed in 1979; If $q(F, x;\ z) = (1/(c - 1))[c - 2F(x)](1 - x/z)$ with $c \geqslant 2$ then $P(F;\ z)$ is the more general class of measures that Thon proposed in 1983. Finally if $q(F, x;\ z) = (k + 1)[1 - F(x)/r_z]^k(1 - x/z)$ with $k > 0$ then $P(F;\ z)$ is the Kakwani measure. A monotonic transformation of the first measure proposed by Clark *et al.* (1981) corresponds to $q(F, x;\ z) = r_z^{\alpha-1}(1 - x/z)^{\alpha}$ with $\alpha \geqslant 1$ although it does not involve the rank of the poor $F(x)$.

2.2. *Poverty axioms*

In poverty measurement, the desirability of a poverty measure is evaluated by the properties (axioms) it satisfies. Sen (1976, 1981) identified several such axioms which are still viewed as the key requirements for a poverty measure. Over the years, researchers have also introduced several other axioms. As shown in Zheng (1997), some of these axioms are ad hoc and some can be implied by others combined and hence are not independent. For a poverty measure P, the reasonable axioms include:[6]

1. *focus*: P is not affected by changes in nonpoor incomes.
2. *symmetry*: P is not affected if two persons switch their incomes.
3. *continuity*: P is a continuous function of income.
4. *replication invariance*: P is not affected by the pooling of several identical populations.
5. *monotonicity*: P increases if a poor person's income decreases.
6. *strong transfer*: P increases if income is transferred from a poor person to someone richer (may or may not be poor).
7. *weak transfer sensitivity*: the increase in P due to a regressive transfer (from a poor person to someone richer) within the poor is inversely related to the income levels of the transfer; no one crosses the poverty line as a result of the transfer.
8. *increasing poverty line*: P is an increasing function of the poverty line.

In the literature, two other weak version axioms, *restricted continuity* and *weak transfer*, are also often mentioned. *Restricted continuity* requires *continuity* over $[0, z)$ instead of $[0, \infty)$ and *weak transfer* limits *strong transfer* to the situation where no one crosses the poverty line as a result of the transfer.

For both types of poverty measures (additively separable and rank-based), *focus*, *symmetry* and *replication invariance* are automatically satisfied because they are defined over the poverty domain $[0, z)$ and the income distribution is continuous. For an additively separable poverty measure, *restricted continuity* is also automatically satisfied because $p(x, z)$ is assumed to be differentiable in x over $[0, z)$. In addition, *continuity* requires $p(x, z)$ to be a continuous function of x at the poverty line z; *monotonicity* requires $\partial p(x, z)/\partial x < 0$ over $[0, z)$; *strong transfer* requires *continuity* and $\partial^2 p(x, z)/\partial x^2 > 0$ over $[0, z)$; *weak transfer sensitivity* requires $\partial^3 p(x, z)/\partial x^3 < 0$ over $[0, z)$, and *increasing poverty line* requires $\partial p(x, z)/\partial z > 0$. Clearly all additively separable poverty measures given above satisfy these axioms (for the Foster *et al.* measure, $\alpha \geqslant 3$ is required to satisfy *weak transfer sensitivity*).

For those rank-based measures, all satisfy *increasing poverty line*; only the transformed Clark *et al.* first measure satisfies *weak transfer sensitivity*. The Sen measure, the Kakwani measure and the transformed Clark *et al.* first measure violate both *continuity* and *strong transfer* but satisfy *restricted continuity* and *weak transfer*. The Takayama measure violates *monotonicity*, *strong transfer* and *increasing poverty line*. The Thon measures satisfy all axioms except *weak transfer sensitivity*.

To facilitate our presentation, we define the following four inclusive classes of poverty measures according to the axioms that each measure satisfies. Denote Φ as the class of poverty measures satisfying *focus*, *symmetry*, *replication invariance*, *restricted continuity* and *increasing poverty line*.

$$\Phi_M := \{P \in \Phi \mid P \text{ satisfies } \textit{monotonicity}\},$$
$$\Phi_{WT} := \{P \in \Phi_M \mid P \text{ satisfies } \textit{weak transfer}\},$$
$$\Phi_{ST} := \{P \in \Phi_{WT} \mid P \text{ satisfies } \textit{continuity}\},$$

and

$$\Phi_{WTS} := \{P \in \Phi_{ST} \mid P \text{ satisfies } \textit{weak transfer sensitivity}\}.$$

The corresponding subclasses of additively separable poverty measures are denoted as Ξ, Ξ_M, Ξ_{WT}, Ξ_{ST} and Ξ_{WTS}, respectively.[7]

2.3. *Poverty orderings, stochastic dominance and generalized Lorenz dominance*

Poverty orderings require unanimity in poverty rankings for a class of poverty measures or a set of poverty lines. Before we proceed with our discussion, it is important to clarify the meaning of ‘unanimity’. In the literature, researchers have interpreted this concept differently and there exist three levels of orderings: weak, semi-strict and strict. Consider, for example, poverty-measure orderings for two distributions $F \in \Psi$ and $G \in \Psi$ and F poverty dominates G in the sense that F has less (or no more) poverty than G. The weak ordering requires $P(F; z) \leqslant P(G; z)$ for all poverty measures, the semi-strict ordering requires $P(F; z) \leqslant P(G; z)$ for all poverty measures with strict inequality holding for some poverty measures, and the strict ordering requires $P(F; z) < P(G; z)$ for all poverty measures. Thus strict ordering implies semi-strict ordering which, in turn, implies weak ordering. It also follows that the ordering conditions derived for weak ordering may not hold for semi-strict or strict orderings. The three levels of poverty-line orderings can be similarly distinguished.

Ideally, one would like to review conditions for all three levels of poverty orderings since various conditions have been derived by researchers using different definitions of orderings.[8] However, to put these different conditions into a single paper may well complicate the presentation. For a review paper, it is important to keep discussions as simple as possible and as consistent as possible. With this objective in mind, we adopt the weak definition of poverty orderings for both poverty measures and poverty lines, although the strict definition is probably the most useful one in empirical applications. It is necessary to remind ourselves that not all results for weak orderings can be carried directly over to the other two definitions of poverty orderings. In most cases, however, the ordering conditions for the other two definitions can be similarly derived.

Since the poverty ordering conditions that we will review are closely related to stochastic dominance and generalized Lorenz dominance, it is helpful to define them before we state the results. To conform with the definition of poverty orderings, we also define both dominances in their weak version.

For two distributions $F \in \Psi$ and $G \in \Psi$ and an integer $k = 1, 2, ...$, distribution F kth degree stochastic dominates G, denoted as $F\mathbf{D}_k G$, over an income interval $[a, b] \subset [0, \infty)$ if $F_k(x) \leqslant G_k(x)$ for all $x \in [a, b]$ where $F_1 := F$ and $F_k(x) := \int_0^x F_{k-1}(t)\,\mathrm{d}t$ for $k \geqslant 2$. It is important to point out that our definition here is different from what is usually defined in the stochastic dominance literature. The difference lies in that we define the dominance relation over a subinterval $[a, b]$ instead of the whole income space and that we do not define the dominance relation differently for $k \leqslant 2$ and $k > 2$.

Following Saposnik (1981), we may define that distribution F rank dominates G if $\xi_r(F) \geqslant \xi_r(G)$ for all $r \in [0, 1]$. Following Gastwirth (1971) and Shorrocks (1983), we may also define that distribution F generalized Lorenz dominates G if $\int_0^r \xi_t(F)\,\mathrm{d}t \geqslant \int_0^r \xi_t(F)\,\mathrm{d}t$ for all $r \in [0, 1]$.

3. Poverty-line orderings

In this section, we will survey poverty-line orderings, i.e. poverty orderings by a poverty measure with a range of poverty lines. We begin with the Foster *et al.* class.

3.1. *The Foster* et al. *poverty measures*

In a series of papers, Foster and Shorrocks (1988a, 1988b and 1988c) raised the issue of poverty orderings for multiple poverty lines and described the ordering conditions for the Foster *et al.* (1984) measures. Foster and Shorrocks (1988a) dealt with continuous distributions while their 1988b paper dealt with discrete distributions. In both papers, the interval $(0, \infty)$ is specified as the range for the poverty line which is the same for all distributions. In other words, the poverty line can be arbitrarily set between zero and infinity incomes. For this range of poverty lines, they proved a nice link between poverty orderings and stochastic dominance. Denote the Foster *et al.* measure as FGT_α, i.e. $FGT_\alpha(F; z) = \int_0^z (1 - x/z)^\alpha\,\mathrm{d}F(x)$, Foster and Shorrocks' main result can be stated as follows:

PROPOSITION 3.1 (Foster and Shorrocks, 1988a, 1988b and 1988c). For two distributions $F \in \Psi$ and $G \in \Psi$ and a nonnegative integer α, $FGT_\alpha(F; z) \leqslant FGT_\alpha(G; z)$ for all $z \in (0, \infty)$ if and only if $FD_{\alpha+1}G$ over $[0, \infty)$.

In proving this proposition, Foster and Shorrocks (1988a) utilized a relation between lower partial moments and stochastic dominance, $\int_0^z (z - x)^\alpha\,\mathrm{d}F(x) = \alpha! F_{\alpha+1}(z)$, which is well known in financial economics (e.g. Fishburn, 1976; O'Brien, 1984; and Bawa, 1975). Thus $FGT_\alpha(F; z) = \alpha! z^{-\alpha} F_{\alpha+1}(z)$ and to require $FGT_\alpha(F; z) \leqslant FGT_\alpha(G; z)$ for all $z \in (0, \infty)$ amounts to requiring that the $(\alpha + 1)$th degree stochastic dominance curve of F lies nowhere above that of G. Given this unique relationship, it seems unlikely that stochastic dominance can be used to characterize poverty-line orderings for measures other than the Foster *et al.* class.

The significance of this result can be summarized by the following few remarks.

First, it provides us with an easy way to check unanimous ordering of income distributions. Although for a given pair of distributions one may test poverty dominance directly by plotting poverty values for all poverty lines, it is often easier to implement some well-established dominance criteria which may also yield some important and unexpected insights on poverty orderings. Since the Foster *et al.* measures for $\alpha = 0$ and 1 correspond, respectively, to the headcount ratio and poverty gap ratio, the ordering condition for the headcount ratio is simply first order stochastic dominance and the ordering condition for the poverty gap ratio is simply second degree stochastic dominance. In the literature, first degree stochastic dominance is also called anonymous Pareto dominance which is equivalent to rank dominance (Saposnik, 1981 and 1983). It is also well-known that second degree stochastic dominance is equivalent to generalized Lorenz dominance. Certainly, these equivalent conditions make the Foster *et al.* measures easier to apply in empirical investigations.

Second, the equivalence to stochastic dominance reveals an important interrelationship among poverty-line orderings by different members of the Foster *et al.* class. Since a lower degree stochastic dominance implies a higher degree stochastic dominance over $[0, \infty)$, poverty-line orderings by different members of the Foster *et al.* class are nested. That is, if two distributions can be unanimously ranked by the Foster *et al.* measure with α_1 then they can also be ranked by any other Foster *et al.* measure with α_2 in the same direction provided that $\alpha_2 \geqslant \alpha_1$. On the other hand, if a member of the Foster *et al.* class fails to rank between two distributions then all measures with lower values of α cannot rank them either. This implication of the proposition is very useful in empirical studies: if the dominance relation holds for some member of the Foster *et al.* class then there is no need to check measures with greater values of α; the direction of dominance will never be reversed.[9]

Third, this proposition reveals a somewhat surprising result on the headcount ratio and poverty gap ratio. It is well known that both the headcount ratio and the poverty gap ratio are deficient in that they fail to satisfy some of the key poverty axioms. For a fixed poverty line, the headcount ratio and the distribution-sensitive Foster *et al.* measures may indicate opposite conclusions on poverty comparisons. Here we find that if the poverty line is allowed to vary over $(0, \infty)$, such a contradiction will never arise and the drawback usually associated with the headcount ratio does not apply anymore. In fact, as demonstrated by Atkinson (1987), if two distributions can be unanimously ordered by the headcount ratio at all income levels then all other poverty measures satisfying *monotonicity* will rank distributions in the same way as the headcount ratio does. It is interesting to note that while the headcount ratio fails to satisfy *monotonicity*, its ordering condition, first degree stochastic dominance, possesses this property. Similarly while the poverty gap ratio violates *strong transfer*, its ordering condition, second degree stochastic dominance, satisfies it.

Fourth, Foster and Shorrocks (1988b) also considered a narrower income interval for the poverty line and established essentially the same result as

Proposition 3.1. The interval for the poverty line is $(0, z^{\max}]$ with $z^{\max} < \infty$. Since an income level arbitrarily close to ∞ clearly cannot be viewed as a reasonable poverty line, this extension is meaningful for empirical applications. The dominance condition, $F\mathbf{D}_\alpha G$ over $[0, z^{\max}]$, can also rank order more pairs of distributions than $F\mathbf{D}_\alpha G$ over $[0, \infty)$ and, hence, is more useful. All the implications discussed above are also carried over along this extension (rank dominance and generalized Lorenz dominance are now applied to the censored income distributions which set all incomes above $z^{\max}$ to $z^{\max}$). Following Foster and Shorrocks' extension, one might also want to, as Atkinson (1987) suggested, consider $[z^{\min}, z^{\max}]$, where $z^{\min} > 0$ is the minimum poverty line, as the interval for the poverty line. Although the result of Proposition 3.1 still holds, most of the aforementioned implications do not. For example, the poverty ordering by the headcount ratio may not imply the ordering by the poverty gap ratio because incomes below $z^{\min}$ are not considered in head counting but are used in computing the poverty gap ratio.

Last but not least is that the connection between poverty orderings and stochastic dominance provides important distributional characterizations for poverty orderings of the headcount ratio, the poverty gap ratio and the Foster *et al.* measure with $\alpha = 2$. In the literature, first, second and third degrees of stochastic dominance have been completely characterized.[10] When applying these characterizations to poverty orderings, we know that (1) F has no more poverty than G by the headcount ratio for all poverty lines in $(0, \infty)$ if and only if F can be obtained from G via a sequence of income increments, (2) F has no more poverty than G by the poverty gap ratio for all poverty lines in $(0, \infty)$ if and only if F can be obtained from G via a sequence of income increments and progressive transfers, and (3) F has no more poverty than G by the Foster *et al.* measure with $\alpha = 2$ for all poverty lines in $(0, \infty)$ if and only if F can be obtained from G via a sequence of income increments, progressive transfers and/or favorable composite transfers.[11] If the poverty line is limited to $[0, z^{\max}]$ then the characterizations hold for the distributions of F and G censored at $z^{\max}$.

3.2. *The Daltonian class of poverty measures*

Hagenaars (1987) viewed poverty as the counterpart of well-being. For an individual, the amount of disutility due to being poor can indicate the intensity of poverty. Naturally the aggregation of individual disutilities can serve as a poverty measure. Hagenaars (1987), in the tradition of Dalton (1920), proposed the following class of Daltonian poverty measures:

$$P(F;\ z) = A(z) \int_0^z [u(z) - u(x)]\ \mathrm{d}F(x)$$

where u is individual's utility function and $A(z)$ is a positive normalization factor and is a differentiable function of z.

The Daltonian class contains the Clark *et al.* second measure ($u(x) = x^\beta$ and $A(z) = \beta^{-1} z^{-\beta}$), the Watts measure ($u(x) = \ln x$ and $A(z) = 1$), the Chakravarty measure ($u(x)=x^{\beta}$ and $A(z)=z^{-\beta}$) and the CDS measure introduced in Section 2 ($u(x)=-e^{-\gamma x}$ and $A(z)=e^{\gamma z}$). In general, for each utility function there is a corresponding Daltonian poverty measure and the measure satisfies *focus*, *symmetry* and *replication invariance*. In order for a Daltonian poverty measure to satisfy *continuity*, *monotonicity*, *transfer* and *weak transfer sensitivity*, the utility function must be continuous, increasing, concave and has convex marginal utility. The normalization factor $A(z)$ must be properly specified to ensure the satisfaction of the axiom of increasing poverty line.

Recently, Foster and Jin (1998) characterized poverty-line orderings of the Daltonian class for all poverty lines in $(0, \infty)$. They noted that a Daltonian poverty measure is essentially the poverty gap ratio applied to utility distributions. Since the ordering condition for the poverty gap ratio is simply second degree stochastic dominance, the ordering condition for a Daltonian poverty measure with u is simply second degree stochastic dominance over the distribution of $u(x)$. Further because second degree stochastic dominance is equivalent to generalized Lorenz dominance, the ordering condition becomes generalized Lorenz dominance over the utility distributions. Thus their main result can be summarized as follows:[12]

PROPOSITION 3.2 (Foster and Jin, 1998). For two distributions $F\in\Psi$ and $G\in\Psi$ and a Daltonian poverty measure P with individual utility function $u(x)$, $P(F; z)\leqslant P(G; z)$ for all $z\in(0,\infty)$ if and only if $\int_0^r u[\xi_t(F)]\,\mathrm{d}t\geqslant\int_0^r u[\xi_t(G)]\,\mathrm{d}t$ for all $r\in[0,1]$.

This proposition thus characterizes poverty orderings of the Clark *et al.* second measure, the Watts measure, the Chakravarty measure and the CDS poverty measure. For example, for the Watts measure, the comparison can be carried out by using a two-step procedure: first transform income into utility, i.e., from x to $\ln x$, then compare the generalized Lorenz curves of the two distributions of $\ln x$.

Foster and Jin (1998) pointed out that the ordinary generalized Lorenz dominance implies this transformed generalized Lorenz dominance. That is, $\int_0^r \xi_t(F)\,\mathrm{d}t\geqslant\int_0^r \xi_t(G)\,\mathrm{d}t$ for all $r\in[0,1]$ implies $\int_0^r u[\xi_t(F)]\,\mathrm{d}t\geqslant\int_0^r u[\xi_t(G)]\,\mathrm{d}t$ for all $r\in[0,1]$ provided that $u(x)$ is an increasing and concave function of x.[13] If $u(x)$ is strictly concave in x, then the transformed generalized Lorenz condition can also rank some pairs of distributions that the ordinary generalized Lorenz condition fails. In this sense, the poverty-line ordering by a Daltonian poverty measure with a strictly concave utility function is more complete than second degree stochastic dominance. Foster and Jin (1998) further pointed out that the poverty-line ordering by a Daltonian poverty measure is also nested in that $\int_0^r u[\xi_t(F)]\,\mathrm{d}t\geqslant\int_0^r u[\xi_t(G)]\,\mathrm{d}t$ for all $r\in[0,1]$ always implies $\int_0^r v[\xi_t(F)]\,\mathrm{d}t\geqslant\int_0^r v[\xi_t(G)]\,\mathrm{d}t$ for all $r\in[0,1]$ if v is more risk averse than u in the sense of Pratt (1964).

Given the works of Foster and Shorrocks (1988a, 1988b) and Foster and Jin (1998), it seems natural to consider the poverty orderings by a generalized

Daltonian poverty measure

$$P(F;\ z) = A(z) \int_0^z [u(z) - u(x)]^\alpha \ \mathrm{d}F(x),$$

where α is a nonnegative integer as in the Foster *et al.* measure. In fact this measure can be regarded as a generalized Foster *et al.* measure which replaces x with $u(x)$ and z with $u(z)$. Clearly an analogous proposition to Proposition 3.1 can be obtained by applying Foster and Shorrocks' proof to utility distributions instead of income distributions. Thus the ordering condition for this generalized Foster *et al.* measure with α is simply $(\alpha + 1)$th stochastic dominance applied to the transformed distributions of $u(x)$. This transformed stochastic dominance is more complete than the ordinary stochastic dominance for all $\alpha \geqslant 1$. For $\alpha = 0$, first degree transformed stochastic dominance and first degree ordinary stochastic dominance are equivalent.

3.3. *The rank-based poverty measures*

While poverty-line orderings of all existent additively separable poverty measures have been completely characterized via stochastic dominance or transformed stochastic dominance, little has been known about the *exact* ordering conditions of rank-based poverty measures such as the Sen and Thon measures. Although the exact characterizations of these measures remain a topic for future research, it is useful to examine here whether or not stochastic dominance can serve as necessary or sufficient conditions.

Since all aforementioned rank-based poverty measures except the Takayama measure satisfy the monotonicity axiom, it follows that first degree stochastic dominance (or rank dominance) over $[0, z^{\max}]$ should serve as a sufficient condition for poverty-line orderings of these measures for all poverty lines in $(0, z^{\max}]$. This is true because rank dominance is characterized by a sequence of simple income increments. This result can also be visualized through the following graph which depicts first degree stochastic dominance (rank dominance) curve.

In Figure 1, the maximum-poverty is represented by the horizontal axis (OE), i.e. all incomes are zero; the minimum-poverty is represented by the line AC, i.e. all incomes are no lower than the poverty line z. The censored income quantile curve of any distribution will necessarily fall between them. Naturally, the difference between the minimum-poverty quantile curve (AC) and the censored quantile curve of distribution F can be used as a measure of poverty for F. It is easy to see that all existent poverty measures, including both additively separable and rank-based measures except the Takayama measure, are results of the different ways to characterize this difference. For example, recall that $r_z(F)$ is the proportion of the people in F whose incomes fall below the poverty line and $\tilde{\xi}_r(F)$ is the income quantile of distribution F censored at z (in Figure 1, $\tilde{\xi}_r$ consists of ξ_r for $r < r_z$ and the horizontal portion of AC $(=z)$ for $r \geqslant r_z$), the Foster *et al.* measure is $\int_0^1 [z - \tilde{\xi}_r]^\alpha \ \mathrm{d}r$ normalized by z^α (z can be regarded as the area of

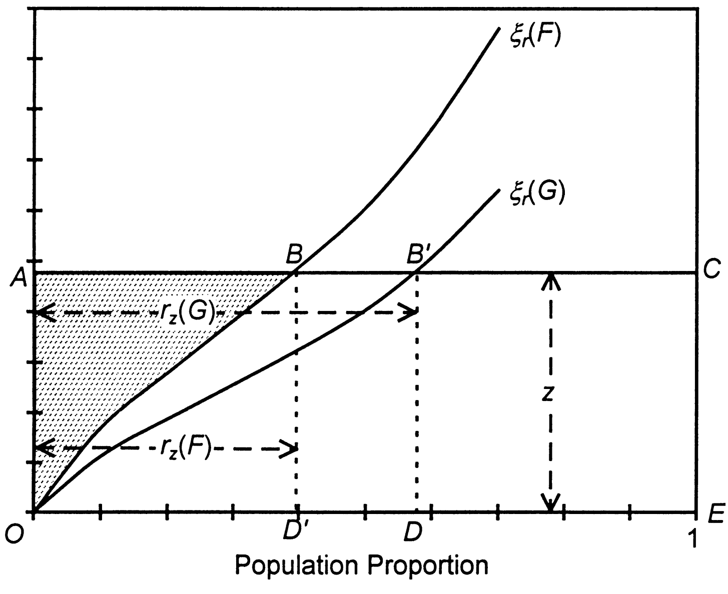

Figure 1. Income quartiles, rank dominance and poverty measures.

$OACE$); the Watts measure is $\int_0^1 [\ln z - \ln\tilde{\xi}_r]\, \mathrm{d}r$; the Sen measure is $2\int_0^1 [z-\tilde{\xi}_r][1-r/r_z]\, \mathrm{d}r$ normalized by z; the Thon (1983) class is $(2/(c-1))\int_0^1 [z-\tilde{\xi}_r][c/2-r]\, \mathrm{d}r$ normalized by z; the Kakwani measure is $(k+1)\int_0^1 [z-\tilde{\xi}_r][1-r/r_z]^k\, \mathrm{d}r$ normalized by z. The Takayama measure does not characterize the difference $z-\tilde{\xi}_r$, instead it characterizes the difference $\mu-\tilde{\xi}_r$ where μ is the mean income of the censored distribution. Thus for two distributions F and G, if F first degree dominates G then for any given poverty line z, $r_z(F) \leqslant r_z(G)$ and $\tilde{\xi}_r(F) \geqslant \tilde{\xi}_r(G)$ for all $r \in [0,1]$. Thus the difference $z-\tilde{\xi}_r$ (and $\ln z - \ln\tilde{\xi}_r$) of F is smaller than that of G and $1-r/r_z$ is also smaller for F than for G. It follows that all above measures except the Takayama measure will indicate that F has no more poverty than G.

One may argue that first degree stochastic dominance condition is ‘too’ sufficient for poverty-line orderings. This is indeed the case for the Thon poverty measure. Since the Thon measure satisfies the strong transfer axiom, second degree stochastic dominance can be used as a sufficient condition. This is because second degree stochastic dominance can be characterized via a sequence of income increments and progressive transfers. This is not quite so for the Sen measure. If the poverty line can take a value arbitrarily close to zero then second degree stochastic dominance may not even be a part of the sufficient ordering condition. To see these claims, we construct a generalized Lorenz curve for the censored distribution of F at z. Figure 2 displays this construction where the diagonal line OB represents the minimum-poverty line where all incomes are no lower than z and D stands for the point where income is censored. Thus line DC is parallel to

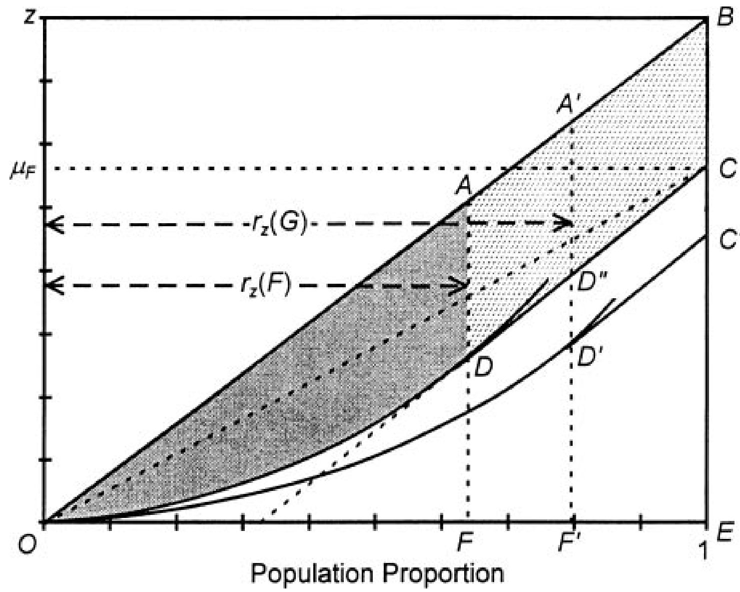

Figure 2. Censored generalized Lorenz dominance and the Sen and Thon poverty measures.

OB which has a slope of z. The length of BE is z and the length of CE is μ which is the mean income of the censored distribution. Finally the length of OF represents the proportion of the people below the poverty line z.

Now let us focus on the difference between the minimum-poverty line OB and the generalized Lorenz curve ODC. It is well-known that the area between the two curves (the shaded area) is related to the Thon measure (e.g. Jenkins and Lambert, 1998a). It is easy to show that $\int_0^1 [z - \tilde{\xi}_r][1 - r]\,\mathrm{d}r = \int_0^1 (rz - \int_0^r \tilde{\xi}_t\,\mathrm{d}t)\,\mathrm{d}r = \frac{1}{2}z - \int_0^1 \int_0^r \tilde{\xi}_t\,\mathrm{d}t\,\mathrm{d}r$ where $rz - \int_0^q \tilde{\xi}_t\,\mathrm{d}t$ is the distance between the line of minimum-poverty (OB) and the censored generalized Lorenz curve with $\int_0^1 \int_0^r \tilde{\xi}_t\,\mathrm{d}t\,\mathrm{d}r$ being the area underneath the generalized Lorenz curve. Note that the first term ($\frac{1}{2}z$) of the very right hand side is the area of OBE and the second term is the area (both darkly and lightly shaded) below the curve ODC. Thus the Thon measure of F is twice the shaded area in Figure 2 normalized by z. It follows that if the censored generalized Lorenz curve of F lies nowhere below that of G then F will have no more poverty than G by the Thon measure. Since generalized Lorenz dominance implies censored generalized Lorenz dominance, generalized Lorenz dominance is thus a sufficient condition for poverty orderings by the Thon measure for all poverty lines in $(0, \infty)$. If the poverty line is $(0, z^{\max}]$ then the sufficient condition is a generalized Lorenz dominance censored at $z^{\max}$.[14]

Generalized Lorenz dominance is also a sufficient condition for the Thon (1983) class of poverty measures. This is because the Thon measure with $c \geqslant 2$ can be expressed as $(2/(c-1)) \int_0^1 [z - \tilde{\xi}_r][\frac{1}{2}c - r]\,\mathrm{d}r = \frac{1}{2}z(c-1) - \{(\frac{1}{2}c - 1)\mu + \int_0^1 (\int_0^r \tilde{\xi}_t\,\mathrm{d}t)\,\mathrm{d}r\}$ and generalized Lorenz dominance implies the dominance of both μ and $\int_0^1 (\int_0^r \tilde{\xi}_t \, dt) \, dr$. Note that the Thon class can also be expressed as $\frac{1}{2}(z-\mu)(c-1)+\frac{1}{2}\mu-\int_0^1(\int_0^r \tilde{\xi}_t \, dt) \, dr$. Graphically, the first term is the area of OBC inflated by $(c-1)$, the second term is the area of OCE, and the third term is of course the area below the curve ODC.

Figure 2 also characterizes the Sen measure: it is the area of OAD (formed by lines OA, AD and curve OD) normalized by the area of OBE and the proportion of the people below the poverty line r_z.[15] This can be easily verified from the fact that the Sen measure can be expressed as $2(zr_z)^{-1}\int_0^{r_z}(rz-\int_0^r \tilde{\xi}_t \, dt) \, dr$.[16] From this graphical representation we can see that generalized Lorenz dominance (second degree stochastic dominance) alone is not sufficient to guarantee poverty-line ordering by the Sen measure. This is because the area of OAD in Figure 2 may not be smaller than that of $OA'D'$ even though F generalized Lorenz dominates G; the further normalization cannot guarantee it either.[17] However, if an additional first order condition at the poverty line is imposed then poverty ordering can be secured. The additional condition requires that the proportion of the poor people in F is not greater than that in G. Now the darkly shaded area of F is obviously not greater than that of G (the area of $OA'D'$) and the further normalization will not change the direction of the dominance.[18] Thus if the range for the poverty line is $[z^{\min}, z^{\max}]$, a sufficient ordering condition is a second degree stochastic dominance over $[0, z^{\min}]$ and a first degree dominance over $[z^{\min}, z^{\max}]$.[19] Clearly, if $z^{\min} \to 0$ then the second degree condition vanishes and the condition becomes first degree dominance over $[0, z^{\max}]$.

As for the Kakwani measure with $k>0$, although it can be expressed as a function of the darkly shaded area, i.e. $k(k+1)(zr_z)^{-1}\int_0^{r_z}(1-r/r_z)^{k-1}(rz-\int_0^r \tilde{\xi}_t \, dt) \, dr$, it is not clear from the graph whether or not second degree stochastic dominance can be used as a part of the ordering condition.[20] Thanks to a result of Howes (1993) which will be discussed in the next section, however, the dominance condition for the Sen measure can be shown to be a sufficient condition for the Kakwani measure. More generally, first degree stochastic dominance over $[z^{\min}, z^{\max}]$ and second degree stochastic dominance over $[0, z^{\max}]$ is a sufficient condition for poverty-line orderings of a poverty measure which satisfies *monotonicity* and *weak transfer* for all $z \in [z^{\min}, z^{\max}]$.[21]

The Takayama measure equals the ratio of the area between the dashed line OC (not the curved OC) and the generalized Lorenz curve (ODC) and the area of OCE. Clearly, generalized Lorenz dominance is not a part of the sufficient condition for poverty-line orderings of the Takayama measure.

4. Poverty-measure orderings

The concern for using different measures in ranking income distributions goes back to Sen's classical treatise on inequality measurement. Arguing that the notion of inequality is essentially incomplete, Sen (1973) preferred the approach of (strict) quasi-orderings in which a set of chosen measures are unanimous on inequality rankings. The best-known such a quasi-ordering tool is the Lorenz curve dominance which is characterized by all relative and symmetric inequality measures that also satisfy Pigou-Dalton's principle of transfers (Atkinson, 1970). Clearly, the completeness of quasi-orderings depends on the set of inequality measures considered. In general, the fewer measures are included in the set the more complete the ordering will be. For example, the inequality quasi-ordering for all relative measures satisfying the principle of transfers and the transfer sensitivity axiom (Shorrocks and Foster, 1987) or the principle of diminishing transfers (Kolm, 1976) is more complete than Lorenz dominance.[22] Since poverty measurement and inequality measurement are two closely connected areas, it is very natural for researchers to raise a similar concern in poverty measurement and to search for a dominance device similar to the Lorenz curve.

Foster (1984) raised the issue of considering multiple measures in poverty ranking. He showed that for discrete distributions with the same number of people and the same number of the poor, generalized Lorenz dominance among the poor is sufficient to entail that all symmetric poverty measures satisfying Sen's three axioms (*focus*, *monotonicity*, and *weak transfer*) will yield the same ranking. He also pointed out that the imposition of more restrictive axioms will reduce the number of admissable poverty measures and increase the completeness of orderings. Following Foster's initiation, several researchers have attempted to address poverty-measure orderings and derived ordering conditions for various classes of poverty measures. In this section, we review the literature on this topic and provide some extensions.

4.1. *Atkinson's stochastic dominance conditions*

In an important contribution, Atkinson (1987) revealed an important link between poverty orderings and stochastic dominance. His findings further consolidate the use of stochastic dominance as a tool in poverty rankings as suggested by Foster and Shorrocks (1988a, 1988b). In his Walras-Bowley lecture, Atkinson considered the 'diversity of judgements affecting all aspects of measuring poverty' and showed that 'dominance conditions may provide tools' to 'reach some degree of agreement' in poverty comparisons. The diversity of judgements that Atkinson considered includes both the level of the poverty line and the choice of poverty measures. Acknowledging the arbitrariness in selecting an appropriate poverty line, he specified a range $[z^{\min}, z^{\max}]$ with $z^{\min} > 0$ and $z^{\max} < \infty$ being the minimum and maximum poverty lines and let the poverty line vary within this range. Instead of using a single poverty measure as Foster and Shorrocks did, Atkinson considered all members of a given class of poverty measures. In particular, Atkinson focused on additively separable poverty measures and established ordering conditions for two classes Ξ_M and Ξ_{ST}. Recall that Ξ_M contains all poverty measures that are continuous and focused and satisfy *monotonicity* and that Ξ_{ST} is a subset of Ξ_M and members of which also satisfy the strong transfer axiom.

The following two propositions present Atkinson's main results.[23]

PROPOSITION 4.1 (Atkinson, 1987). Given two distributions $F \in \Psi$ and $G \in \Psi$, the necessary and sufficient condition for $P(F; z) \leqslant P(G; z)$ for all poverty measures $P \in \Xi_M$ and all poverty lines $z \in [z^{\min}, z^{\max}]$ is $F\mathbf{D}_1 G$ over $[0, z^{\max}]$.

PROPOSITION 4.2 (Atkinson, 1987). Given two distributions $F \in \Psi$ and $G \in \Psi$, the necessary and sufficient condition for $P(F; z) \leqslant P(G; z)$ for all poverty measures $P \in \Xi_{ST}$ and all poverty lines $z \in [z^{\min}, z^{\max}]$ is $F\mathbf{D}_2 G$ over $[0, z^{\max}]$.

Perhaps the most significant implication of these propositions is that if dominance relationships hold then no individual poverty measure needs to be consulted in ranking income distributions. The dominance conditions are easy to implement statistically and have clear welfare interpretations. Over the last decade or so, Atkinson's dominance conditions have provided important tools for empirical poverty rankings and have had substantial influences on the subsequent research on poverty measurement. Because of the significance of these results, it is useful to examine them carefully and offer some comments.

First, as pointed out by Atkinson, the ordering conditions do not depend on where the minimum poverty line is drawn. In other words, the exact location of $z^{\min}$ does not affect poverty orderings if all poverty measures are checked. This is indeed a somewhat surprising result since one would imagine that more lines may mean less completeness in poverty rankings. An important implication of this property is that in checking poverty rankings by such a class of poverty measures, we do not need to worry about the lower bound poverty line, all we need to concern is where to draw the maximum possible poverty line.

Second, and related to the first, is that poverty orderings for either class at a given poverty line $z^{\max}$ imply poverty orderings of the same class at a lower poverty line. Thus a unanimous poverty ranking at a higher poverty line cannot be reversed at a lower poverty line; if no unanimous ranking can be reached at a given poverty line then no unanimous ranking is possible at any higher poverty line. The key reason for these two remarkable results is that *all* measures are required to rank poverty in the same way at $z^{\max}$ and these different measures emphasize different parts of the distribution over $[0, z^{\max}]$. For example, as demonstrated in the proofs of Atkinson (1987) and Zheng (1999), any small distributional difference between F and G over $[0, z^{\max}]$ can be used to construct a particular measure which belongs to Ξ_M and ranks distributions in the same direction as the distributional difference.

Third, the propositions suggest a unification between the two dimensions of poverty orderings: the poverty-measure orderings by all measures of Ξ_M and Ξ_{ST} and poverty-line orderings of the Foster *et al.* measures with $\alpha = 0$ and 1 for all poverty lines in $(0, z^{\max}]$. As pointed out before, although the headcount ratio does not satisfy *monotonicity*, the unanimous ranking of it for all poverty lines over $(0, z^{\max}]$ entails poverty-measure ordering of all measures satisfying *monotonicity* for any poverty line in $(0, z^{\max}]$. Also, although the poverty gap ratio does not satisfy *strong transfer*, the unanimous ranking of it for all poverty lines in $(0, z^{\max}]$ entails poverty-measure ordering of all measures satisfying *strong transfer* for any poverty line in $(0, z^{\max}]$. An implication of this property is that we do not need to check two separate conditions for these two poverty orderings; either one is sufficient for the other.

Fourth, since second degree stochastic dominance is more complete than first degree stochastic dominance in that the former can rank order more pairs of distributions than the latter, the poverty-measure ordering by Ξ_{ST} is more complete than the ordering by Ξ_M.[24] This may not be a surprise since Ξ_{ST} is a subset of Ξ_M and hence should order more, at least not fewer, pairs of distributions. As is well recognized, first degree stochastic dominance performs poorly in empirical rankings of distributions; a large proportion of distributions may fail to satisfy this condition. Hence from the perspective of empirical researchers and from that of policy makers, first degree stochastic dominance is a poor empirical tool. Second degree stochastic dominance, on the other hand, can rank distributions which first degree stochastic dominance fails to rank and thus is empirically more useful. Of course, one needs to realize that this increase in the completeness is hinged upon one's acceptance of the notion of *strong transfer*. Since *strong transfer* has been generally, though not universally, agreed to be an indispensable poverty axiom, second degree stochastic dominance may be regarded as a more pertinent ordering condition than first degree stochastic dominance in empirical applications.

Fifth, while Atkinson focused on poverty orderings of additively separable poverty measures, the dominance conditions can characterize poverty-measure orderings of broader classes of measures when additivity is not assumed. That is, first degree stochastic dominance over $[0, z^{\max}]$ is the necessary and sufficient condition for poverty orderings by all members of Φ_M and for all poverty lines in $[z^{\min}, z^{\max}]$; second degree stochastic dominance over $[0, z^{\max}]$ is the necessary and sufficient condition for poverty orderings by all members of Φ_{ST} and for all poverty lines in $[z^{\min}, z^{\max}]$. This extension can be justified as follows. The necessity follows immediately because Ξ_M and Ξ_{ST} are subsets of Φ_M and Φ_{ST}, respectively, and thus what is true for Φ must also be true for Ξ in the first place. The sufficiency follows from the distributional characterizations of first and second degrees of stochastic dominance. It is well known that F first degree stochastic dominates G over $[0, \infty)$ if and only if F can be obtained from G by a sequence of income increments, and that F second degree stochastic dominates G over $[0, \infty)$ if and only if F can be obtained from G by a sequence of income increments and progressive transfers. When stochastic dominance over $[0, z^{\max}]$ is concerned, Howes (1993) suggested that the same characterization applies to the censored distributions of F and G at $z^{\max}$. It follows that $F\mathbf{D}_1G$ over $[0, z^{\max}]$ guarantees poverty orderings of Φ_M and $F\mathbf{D}_2G$ over $[0, z^{\max}]$ guarantees poverty orderings of Φ_{ST}. It is useful to point out that to ensure second degree stochastic dominance as a sufficient condition, poverty measures must satisfy the strong transfer axiom since *weak transfer* does not allow the number of the poor to change as a result of the transfer. Thus Φ_{ST} includes the Thon class of measures but not the Sen and Kakwani measures; Φ_M encompasses all of them.

Finally, it is interesting to note that while second degree stochastic dominance is equivalent to generalized Lorenz dominance, they may graphically characterize different poverty measures: stochastic dominance leads to the Foster *et al.* measure with $\alpha = 2$ but generalized Lorenz dominance yields the Thon measure. Figure 3 displays second degree stochastic dominance; generalized Lorenz dominance is depicted in Figure 2. Clearly for either graphical device the area

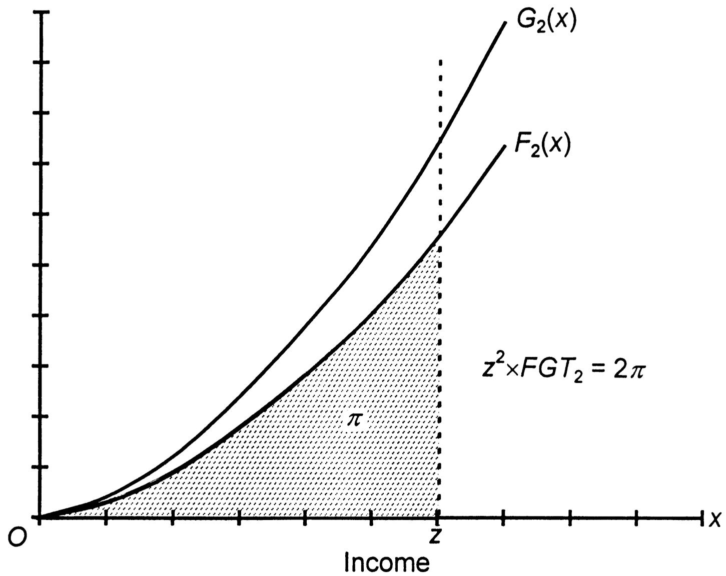

Figure 3. Second degree stochastic dominance and the Foster *et al.* poverty measure with $\alpha = 2$.

between the line of minimum-poverty and the dominance curve (up to the poverty line for stochastic dominance) is coherent with the dominance. Similar to the Gini coefficient as an inequality measure, these two areas can be regarded as poverty measures. From what we have discussed before, it is easy to see that twice the area between second degree stochastic dominance curve and the horizontal line (which corresponds to the minimum poverty) up to z is the Foster *et al.* measure with $\alpha = 2$ multiplied by z^2,[25] and twice the area between the censored generalized Lorenz curve and the diagonal line is the Thon measure multiplied by z.[26] Both first degree stochastic dominance and censored rank dominance graphically depict the same measure: the poverty gap ratio.

4.2. *Poverty orderings of measures satisfying the weak transfer axiom*

As demonstrated in the previous section, second degree stochastic dominance is not sufficient for the ranking of the Sen measure; a sufficient condition requires an additional first degree condition at the poverty line. The reason for first degree stochastic dominance to come into the picture is because the Sen measure does not satisfy the strong transfer axiom; it only satisfies *weak transfer*. Given this, one may be interested in characterizing poverty orderings for the class of poverty measures which includes the Sen and Kakwani measures. These measures satisfy *weak transfer* rather than *strong transfer*. The class of these measures is Φ_{WT} and its additively separable subset is Ξ_{WT}.

Howes (1993) provided such a characterization. He showed that the sufficient condition for the Sen measure turns out to be both necessary and sufficient for the class Φ_{WT}. Formally, his result can be stated as[27]

PROPOSITION 4.3 (Howes, 1993). Given two distributions $F \in \Psi$ and $G \in \Psi$, the necessary and sufficient condition for $P(F; z) \leqslant P(G; z)$ for all poverty measures $P \in \Phi_{WT}$ and all poverty lines $z \in [z^{\min}, z^{\max}]$ is $F\mathbf{D}_1 G$ over $[z^{\min}, z^{\max}]$ and $F\mathbf{D}_2 G$ over $[0, z^{\min}]$.

This result is in fact closely related to Foster's initial inquiry of poverty rankings by all measures satisfying Sen's three axioms. When other standard axioms such as *replication invariance* are further assumed, Foster's result can be stated as: for a given $z \in [z^{\min}, z^{\max}]$, if $F(z) = G(z)$ then the necessary and sufficient condition for $P(F; z) \leqslant P(G; z)$ for all poverty measures $P \in \Phi_{WT}$ is $F\mathbf{D}_2 G$ over $[0, z]$. In this sense, Proposition 4.3 generalizes Foster's result to the situation where $F(z) < G(z)$.

To prove Foster's result, one needs to consider distributions F and G censored at z. Clearly second degree stochastic dominance over $[0, z]$ implies dominance between the censored distributions. Since the two censored distributions are identical at z and above by definition, no distributional adjustments are needed at z and beyond. It follows that a sequence of progressive transfers and simple increments which do not change the number of the poor are what we need to obtain the censored distribution of F from the censored distribution of G.

If $F(z) < G(z)$ instead of $F(z) = G(z)$ is assumed, then we can always construct a new distribution G' from G by a sequence of income increments such that $F(z) = G'(z)$ and the second degree dominance relation between $F(z)$ and $G(z)$ is retained. Figure 4 shows that this is possible. After this operation, the problem becomes what Foster addressed.[28]

In proving the necessity, Howes relied critically on the weak definitions of both *monotonicity* and *weak transfer* (see also footnote 23) which only require the poverty level not to go up as a result of an income increment to the poor or a progressive transfer. Zheng (1999) provided a proof when both axioms are defined in the strict form as Sen (1976) initially formulated.

Presumably this 'mixed' ordering condition should be between the first degree dominance condition and the second degree dominance condition in terms of the completeness of ranking distributions. This seems to be reasonable because Φ_{ST} is included in Φ_{WT} which, in turn, is included in Φ_M. However, the 'mixed' ordering condition may not be more complete than the first degree dominance condition if $z^{\min} \to 0$. Thus, unlike both first and second degree conditions where the determination of $z^{\min}$ is unimportant, the value that $z^{\min}$ takes in the mixed condition is critical. As a consequence, poverty orderings of Φ_{WT} at $z^{\max}$ may not be preserved at a lower poverty line.

Is it desirable to have such a 'mixed' ranking tool? The justification for using this condition lies critically upon the maintaining of *weak transfer* in lieu of *strong transfer*. Sen originally proposed *strong transfer* in his 1976 paper, but later found that the measure he proposed violated the axiom and hence maintained only *weak*

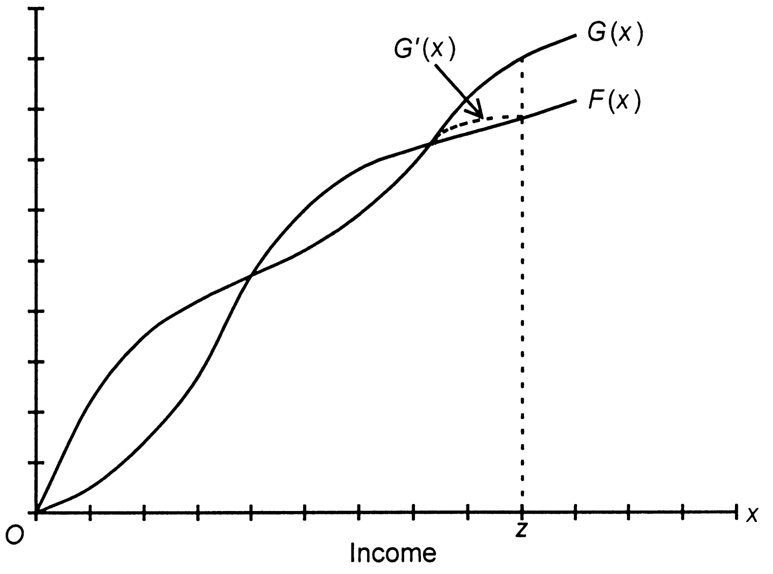

Figure 4. Mixed stochastic dominance.

transfer (Sen, 1981). He viewed *strong transfer* as a perfectly suitable requirement for an inequality measure but less compelling as an axiom for a poverty measure. Sen's view, however, has been subsequently challenged by other researchers.[29] By now it is well recognized in the poverty measurement literature that *strong transfer* is a reasonable poverty axiom and should be well kept. The desirability of requiring *strong transfer* can be further consolidated from Donaldson and Weymark's (1986) finding that *weak transfer*, *monotonicity* and *continuity* imply *strong transfer*. Since *monotonicity* and *continuity* are generally agreed to be desirable poverty axioms, anyone who accepts *weak transfer* ought to accept *strong transfer*. It follows that there is no compelling reason to consider poverty orderings by poverty measures satisfying *weak transfer* and, hence, both first degree and second degree dominance conditions are more pertinent to empirical poverty rankings than this 'mixed' condition.

4.3. *An alternative characterization of the ordering of distribution-sensitive poverty measures*

Spencer and Fisher (1992) considered the criterion for ranking one distribution as having more 'hardship' than another, and established a dominance condition of 'absolute rotated Lorenz curve' which is 'a reverse-mirror imagine of the Lorenz curve reflected through the diagonal 45 degree line'. The words 'hardship' and 'poverty' are synonymous and individual hardship is measured as an increasing and convex function of the poverty gap $u = z - x^*$ where x^* is income variable x

censored at z. The aggregated hardship of income distribution F is defined as

$$H(F;\ z) = \int_0^z h(z - x^*)\ \mathrm{d}F(x)$$

The absolute rotated Lorenz curve is constructed by plotting the normalized cumulation of the largest $100r\%$ of poverty gaps, $\int_0^r (z - \tilde{\xi}_t)\mathrm{d}t$, paired with the population share r for all $r \in [0, 1]$. Figure 5 plots such an absolute rotated Lorenz curve. In the graph, the horizontal axis represents population share and the vertical axis indicates the corresponding normalized cumulation of poverty gaps. The absolute rotated Lorenz curve is concave to a certain point and then becomes a flat line. Utilizing a result due to Yitzhaki (1990), Spencer and Fisher (1992) stated that:

PROPOSITION 4.4 (Spencer and Fisher, 1992). Given two distributions $F \in \Psi$ and $G \in \Psi$, the necessary and sufficient condition for $H(F;\ z) \leqslant H(G;\ z)$ for all increasing and convex functions h is that the absolute rotated Lorenz curve of F lies nowhere above that of G.

Since for a given population share $r \in (0, 1)$, the absolute rotated Lorenz curve ordinate is $\int_0^r (z - \tilde{\xi}_t)\ \mathrm{d}t = rz - \int_0^r \tilde{\xi}_t\ \mathrm{d}t$ and $\int_0^r \tilde{\xi}_t\ \mathrm{d}t$ is the censored generalized Lorenz ordinate corresponding to r, the absolute rotated Lorenz curve ordinate simply equals the distance, in Figure 3, between the line of minimum-poverty and the generalized Lorenz curve. It follows that the absolute rotated Lorenz curve of F lies nowhere above that of G if and only if F censored generalized Lorenz dominates G.[30]

Jenkins and Lambert (1997, 1998a and 1998b) and Shorrocks (1995, 1998) further elaborated this new dominance device and provided some interesting observations. Jenkins and Lambert noticed that all commonly used poverty measures can be expressed as hardship functions. They also pointed out that the absolute rotated Lorenz curve can visually depict three aspects of poverty of a distribution: the incidence of poverty, the intensity of poverty, and the inequality of incomes among the poor. In Figure 5, the population share from which the curve becomes flat is the headcount ratio (incidence of poverty), the maximum height of the curve represents the poverty gap ($z - \mu$ and μ is the mean income of the censored distribution), finally the curvature of the curve between the origin and the headcount ratio indicates the degree of inequality among the poor. This discovery is of particular interest given that Sen (1976) motivated his search for a better poverty measure by persuasively arguing that a poverty measure should reflect these three aspects of poverty. Because of this nice feature, Jenkins and Lambert (1997) renamed the curve as the TIP (Three I's of Poverty) curve.

Jenkins and Lambert also showed that the TIP curve dominance is equivalent to the censored generalized Lorenz dominance and hence can be used to check poverty orderings by all poverty measures in Φ_{ST}. Somewhat more importantly, they applied the TIP curve dominance to poverty orderings with different but fixed poverty lines. To do this they considered poverty measures of poverty gap

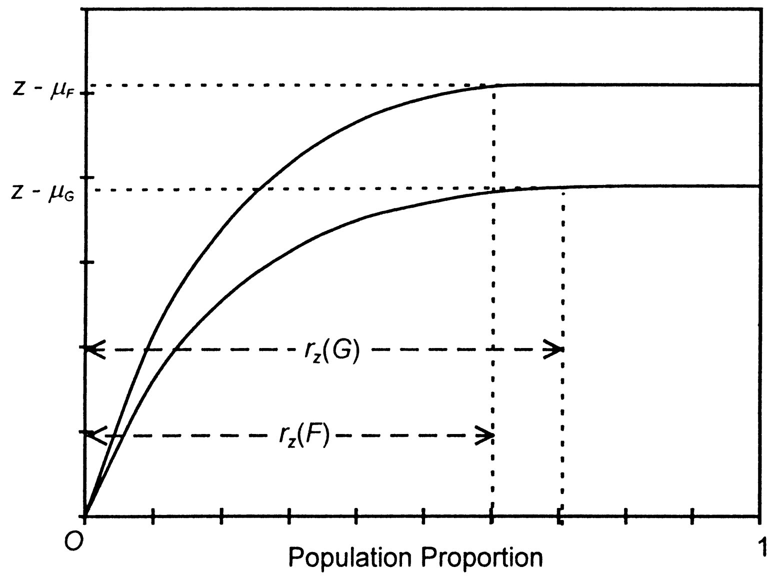

Figure 5. Absolute rotated Lorenz curve dominance and the three 'I's of poverty.

ratios, $(z - x^*)/z$, and the TIP curve is also scaled down by z. This class of poverty measures contains essentially the so-called relative poverty measures. A relative poverty measure does not indicate a change in the poverty level if all incomes of a distribution and the poverty line are simultaneously doubled. Clearly, not all poverty measures are relative. For two distributions F and G, if all incomes are transformed via $(z - x^*)/z$, then the poverty lines of both distributions become the same (=1) and subsequently poverty comparison can be carried out as usual.[31] They further showed that if there is a gap everywhere (except at the origin) between the two TIP curves of the normalized distributions (by their respective poverty lines), then the conclusion of poverty ranking may hold for a range of poverty lines under certain conditions. They characterized the situations where this extension can take place and described the exact extent to which this variation in the line can be.[32] In a follow-up paper, they (1998b) showed that when two TIP curves cross once unanimous poverty ordering is still possible for a class of poverty measures satisfying some more restrictive axioms. This result in effect can be further consolidated as we will see below in Propositions 4.5 and 4.6.

Shorrocks (1998) viewed the absolute rotated Lorenz curve as an application of a more general ordering condition for deprivation and named the corresponding dominance curve as 'deprivation profile'. He characterized deprivation profile dominance as a tool in ranking 'bads' such as wage discrimination (Jenkins, 1994), unemployment duration (Shorrocks, 1993) and poverty. He showed that deprivation profile dominance is the necessary and sufficient condition for deprivation ranking by all deprivation indices which are strictly monotonic and equality-preferring. For a poverty measure these requirements are *monotonicity* and *strong transfer*, respectively. When applying deprivation profile to poverty rankings, Shorrocks (1995) showed that the area beneath the profile is what he called the 'modified Sen measure'. In fact for a discrete distribution this new measure is a member of the class that Thon (1983) introduced. For a continuous distribution, it is simply the Thon (1979) measure.

4.4. *Poverty orderings for more restrictive classes of measures*

Atkinson's two dominance conditions indicate that poverty orderings can have more power by limiting the admissible poverty measures and moving from first degree stochastic dominance to second degree stochastic dominance. He also suggested that one can extend this to third or even higher degrees of stochastic dominance. Since even second degree stochastic dominance or generalized Lorenz dominance can be very incomplete in ranking distributions, it is useful, not only from the perspective of theorists but also from that of empirical researchers, that such a possibility be exploited and the limit of such extensions be analyzed. The appeal and possibility of using third degree stochastic dominance as a ranking tool have also been echoed in inequality ordering literature. For example, Shorrocks and Foster (1987) and Foster and Shorrocks (1988c) characterized third degree stochastic dominance as a tool for ranking equal-mean distributions by the class of transfer-sensitive inequality measures. They also suggested that 'the inequality orderings may be further strengthened by considering higher degrees of stochastic dominance' (1988c). Fishburn and Willig (1984) established a general relationship between stochastic dominance and inequality rankings of equal-mean distributions.

Recently, Zheng (1999) extended Atkinson's conditions to (mixed) third degree stochastic dominance. Following Atkinson (1987), Zheng also considered additively separable poverty measures although the results hold for the broader classes of measures when additivity is relaxed. The class of poverty measures is Ξ_{WTS} in which each measure satisfies *weak transfer sensitivity* in addition to other axioms specified for measures in Ξ_{ST}. The weak transfer sensitivity axiom requires that the effect of poverty-reduction of a progressive transfer be reduced as the transfer moves up along the income scale. In terms of an additively separable poverty measure, this axiom requires $\partial^3 p(x,z)/\partial x^3 < 0$ for all $x \in [0, z)$. This (mixed) third degree dominance condition can be stated as follows:[33]

PROPOSITION 4.5 (Zheng, 1999). Given two distributions $F \in \Psi$ and $G \in \Psi$, the necessary and sufficient condition for $P(F; z) \leqslant P(G; z)$ for all poverty measures $P \in \Xi_{WTS}$ and all poverty lines $z \in [z^{\min}, z^{\max}]$ is $F\mathbf{D}_2 G$ over $[z^{\min}, z^{\max}]$ and $F\mathbf{D}_3 G$ over $[0, z^{\max}]$.

This result shows that, unlike the Atkinson conditions, the ordering condition for Ξ_{WTS} is a mixture of second and third degrees of stochastic dominance. In this sense, the ordering condition resembles that of Ξ_{WT} as established by Howes

(1993). The reason for having both levels of stochastic dominance in the ordering condition is because *weak transfer sensitivity*, like *weak transfer*, limits all transfers within the poverty domain and no one crosses the poverty line as a result of income transfers. Thus the situation at the poverty line is not addressed by the axiom of weak transfer sensitivity, it is covered only by *strong transfer* which leads to the second degree condition.[34]

Now let's examine the ordering condition for the class of weak transfer-sensitive measures. Clearly for a single fixed poverty line $z^{\max}$, the condition can rank more pairs of distributions than second degree stochastic dominance. If a range of poverty line $[z^{\min}, z^{\max}]$ is considered, however, the additional power of the mixed condition beyond the second degree condition depends critically upon the location of $z^{\min}$. If, in the extreme, the lower bound poverty line $z^{\min}$ is arbitrarily close to zero, this mixed condition collapses to that of second degree stochastic dominance. In this limiting case, the imposition of *weak transfer sensitivity* and, hence, the smaller set of poverty measures, does not rank more pairs of distributions than the class of Ξ_{ST}.

A natural attempt to increase the power of poverty orderings beyond Atkinson's second degree condition, regardless of the lower-bound poverty line, is to remove the second degree requirement from the ordering condition. As a consequence, the ordering condition becomes $F\mathbf{D}_3G$ over $[0, z^{\max}]$. Clearly this condition can have substantially more power than the second degree condition $F\mathbf{D}_2G$ over $[0, z^{\max}]$; it also possesses all the nice properties that both Atkinson's conditions enjoy.[35] The use of $\mathbf{D}_3$ alone as an ordering tool has been suggested by several people including Ravallion (1994). Characterizing $\mathbf{D}_1$, $\mathbf{D}_2$ and $\mathbf{D}_3$ as, respectively, the dominance of 'poverty incidence', 'poverty deficit' and 'poverty severity' curves of income distributions, Ravallion suggested that when $\mathbf{D}_2$ 'is inconclusive, one can further restrict the range of admissible poverty measures' and 'if one is content to rely solely on distribution-sensitive poverty measures', then $\mathbf{D}_3$ 'can be tested'. The independent use of $\mathbf{D}_3$ as an ordering tool in poverty comparisons is both theoretically appealing and empirically important and, hence, it is very useful to explore the underlying axiomatic requirements and the corresponding classes of poverty measures.

Zheng (1999) showed that no class of poverty measures grouped by existent poverty axioms can characterize $\mathbf{D}_3$; additional poverty axioms have to be introduced. The new poverty axiom is a stronger version of the weak transfer sensitivity axiom — the strong transfer sensitivity axiom — it implies *weak transfer sensitivity* but not vice versa. While *weak transfer sensitivity* does not allow people to cross the poverty line as a result of transfers, *strong transfer sensitivity* holds even when the number of the poor changes. To some extent, the difference between *weak transfer sensitivity* and *strong transfer sensitivity* is similar to that between *weak transfer* and *strong transfer*. Zheng also showed that for a poverty measure $P(F; z) = \int_0^z p(x, z)\, \mathrm{d}F(x)$, *strong transfer sensitivity* amounts to requiring a stronger continuity axiom — the strong continuity axiom — than what is normally specified: it requires $\partial p(x, z)/\partial x$ to be a continuous function of x everywhere (even at the poverty line z). Denoting Ξ_{STS} as the class of poverty measures satisfying *strong transfer sensitivity*, we have

PROPOSITION 4.6 (Zheng, 1999). Given two distributions $F \in \Psi$ and $G \in \Psi$, the necessary and sufficient condition for $P(F; z) \leqslant P(G; z)$ for all poverty measures $P \in \Xi_{STS}$ and all poverty lines $z \in [z^{\min}, z^{\max}]$ is $F\mathbf{D}_3 G$ over $[0, z^{\max}]$.

Can we justify the use of $\mathbf{D}_3$ alone as a ranking tool of poverty? Or equivalently can we justify *strong transfer sensitivity* and *strong continuity* as reasonable poverty axioms? Here views may differ. Unlike the justification of *strong transfer* where a strong case can be made in favor of *strong transfer* over *weak transfer*, there is no compelling reason for having either *strong transfer sensitivity* or *strong continuity*. As pointed out in Zheng (1999) *strong transfer sensitivity* requires a poverty measure to be more sensitive to income redistribution within the poor than an equal-distance and equal-amount redistribution between the poor and nonpoor. Although this requirement may be reasonable under certain circumstances, it is an open question whether the axiom can be generally and unconditionally justified if *focus* is considered as an indispensable requirement. Without the requirement of *strong transfer sensitivity* or *strong continuity*, as pointed out by Zheng (1999), the only situation in which $\mathbf{D}_3$ alone can be used as a poverty ordering criterion is when $z^{\min} = z^{\max} \to \infty$.

The results presented in this subsection also hold when the restriction of additivity on poverty measure is relaxed. That is $F\mathbf{D}_2 G$ over $[z^{\min}, z^{\max}]$ and $F\mathbf{D}_3 G$ over $[0, z^{\max}]$ is the necessary and sufficient condition for $P(F; z) \leqslant P(G; z)$ for all poverty measures $P \in \Phi_{WTS}$ and all poverty lines $z \in [z^{\min}, z^{\max}]$, and $F\mathbf{D}_3 G$ over $[0, z^{\max}]$ is the necessary and sufficient condition for $P(F; z) \leqslant P(G; z)$ for all poverty measures $P \in \Phi_{STS}$ and all poverty lines $z \in [z^{\min}, z^{\max}]$. The proofs of these extensions are similar to those for the Atkinson dominance conditions as provided by Howes (1993).

4.5. *Higher degrees of stochastic dominance and poverty orderings*

The review so far has demonstrated the importance of stochastic dominance in characterizing orderings of various classes of poverty measures. It is a natural attempt to further explore the relationship between stochastic dominance and poverty orderings.

In inequality measurement, Fishburn and Willig (1984) showed a one-to-one equivalence between *k*th degree stochastic dominance and inequality orderings of equal-mean distributions by all measures satisfying the 'principle of *Tk*'. 'Principle of *T*2' is simply Pigou-Dalton's principle of transfers and 'principle of *T*3' is Kolm's principle of diminishing transfers. In general, the 'principle of *Tk*' says that 'a beneficial transfer at lower income levels, paired with its inverse at uniformly higher levels, yields a transfer with positive net social benefit' and hence less inequality. Thistle (1994) also investigated inequality orderings and stochastic dominance to a higher degree. In particular, he explored the implication of infinite degree stochastic dominance on inequality orderings and inequality measures.

The axiomatic requirements for generating poverty orderings to higher degrees are similar to Fishburn and Willig's principle. Note that *weak transfer sensitivity* also amounts to requiring that poverty be reduced as a result of a favorable composite transfer which is restricted to the poverty domain. A favorable composite transfer, as defined by Shorrocks and Foster (1987), consists of a pair of mean-variance-preserving transfers (a progressive transfer at lower income levels and a regressive transfer at higher income levels). For *weak transfer sensitivity*, such a favorable composite transfer does not change poverty status of any individual. Naturally a next higher level sensitivity axiom (in weak form) would require that poverty reduction due to a favorable composite transfer be inversely related to income levels. Since for an additively separable poverty measure P, *weak transfer sensitivity* requires $\partial^3 p(x,z)/\partial x^3 < 0$ for $x \in [0,z)$, it is easy to see that this higher level sensitivity axiom amounts to require $\partial^4 p(x,z)/\partial x^4 > 0$ for $x \in [0,z)$. One can certainly extend this requirement to an even higher level, although the interpretation may become less transparent. The notion of this extension is however very clear: income redistribution among the bottom poor weights more and more in determining the overall poverty level, as more restrictive sensitivity axioms are imposed. Mathematically, this extension amounts to assuming that the derivative of $p(x,z)$ alternates its sign all the way to the desired degree. That is, $(-1)^i \partial^i p(x,z)/\partial x^i > 0$ for $x \in (0,z)$ and $i = 1, 2, \ldots n$ ($n \geqslant 3$). If we denote this general class as Ξ_n, i.e.

$$\Xi_n := \{P \in \Xi_{ST} \mid (-1)^i \partial^i p(x,z)/\partial x^i > 0 \text{ for } x \in (0,z) \text{ and } i = 3, 4, \ldots n.\},$$

then we can generalize Proposition 4.5 to any degree of stochastic dominance. The following proposition can be verified in a way similar to Proposition 4.5.

PROPOSITION 4.7. Given two distributions $F \in \Psi$ and $G \in \Psi$, the necessary and sufficient condition for $P(F;\, z) \leqslant P(G;\, z)$ for all poverty measures $P \in \Xi_n$ ($n \geqslant 3$) and all poverty lines $z \in [z^{\min}, z^{\max}]$ is $F\mathbf{D}_i G$ over $[z^{\min}, z^{\max}]$ for $i = 2, 3, \ldots, n$ and $F\mathbf{D}_n G$ over $[0, z^{\max}]$.

Clearly the same implications for Proposition 4.5 hold for Proposition 4.7. In particular, if $z^{\min}$ is arbitrarily close to zero then the mixed nth degree dominance condition collapses to second degree dominance. This implies that in this situation no matter how restrictive is the class of poverty measures, poverty orderings cannot have more power than the class of Ξ_{ST}. If we adopt a stronger version poverty axiom such as *strong transfer sensitivity* which 'ignores' the poverty line, then the nth ($n \geqslant 3$) degree sensitivity axiom would require that $(-1)^i \partial^i p(x,z)/\partial x^i > 0$ for $x \in [0,z)$, $i = 1, 2, \ldots n$, and $\partial^{i-2} p(x,z)/\partial x^{i-2}$ be continuous in x everywhere for $i = 3, 4, \ldots n$. If poverty measures are further restricted from Ξ_n to those satisfying such a stronger axiom then $\mathbf{D}_n$ alone can be used as an ordering criterion.

Before moving onto next topic, it is useful to examine poverty measures that are coherent with the nth degree mixed ordering condition and $\mathbf{D}_n$. Of all additively separable poverty measures, the Watts measure, the Clark *et al.* second measure, the Chakravarty measure and the CDS measure can be easily verified to be coherent with the mixed dominance condition of any degree. In fact, if we define infinite degree mixed stochastic dominance as nth degree mixed stochastic dominance when $n \to \infty$, then all these measures are coherent with infinite degree mixed stochastic dominance. Notice further that all these measures are the Dalton-type. It is not difficult to see that all Daltonian poverty measures are sufficient to characterize poverty orderings of the classes Ξ and Φ. In what follows we examine the Daltonian poverty measures that are consistent with infinite degree mixed stochastic dominance.

A Daltonian poverty measure is $P(F; z) = A(z) \int_0^z [u(z) - u(x)]\, \mathrm{d}F(x)$. For P to be consistent with infinite degree mixed stochastic dominance, the necessary and sufficient condition is $(-1)^n u^{(n)}(x) < 0$ for $n = 1, 2, ..., \infty$. Denote $\upsilon(x) = u'(x)$ then $\upsilon(x)$ is the so-called completely monotone function, i.e. $(-1)^n \upsilon^{(n)}(x) > 0$ for $n = 1, 2, ..., \infty$. Using the Bernstein's theorem (see Widder, 1941, p. 160), we know that $\upsilon(x)$ can be written as a Laplace transform of a nonnegative function $f(x)$. That is

$$\upsilon(x) = \int_0^\infty f(t) e^{-tx}\, \mathrm{d}t$$

where $f(t) \geqslant 0$. Thus $u(x)$ can be expressed as $u(x) = \int \upsilon(x)\, \mathrm{d}x + C$ where C is a constant. It is easy to verify that for the Watts measure, $f(t) = 1$; for the Clark *et al.* second measure, $f(t) = t^{-\beta}/\Gamma(1-\beta)$ where $\Gamma(\cdot)$ is the Gamma function; and for the CDS function, $f(t) = \delta(t - \gamma)$ which takes values 0 if $t \neq \gamma$ and ∞ if $t = \gamma$.[36]

In fact, infinite degree mixed stochastic dominance can be further characterized by a subclass of the Daltonian class that we just identified. Note that $\upsilon(x)$ is actually a weighted sum of $e^{-\gamma x}$ for various nonnegative values of γ (see also Fishburn and Willig, 1984), the class of poverty measures corresponding to $\{-e^{-\gamma x}\}$ would be suffice to characterize infinite degree mixed stochastic dominance. This subclass is precisely the CDS class that we introduced earlier in this review.

Finally let us examine poverty measures that are coherent with $\mathbf{D}_n$. Since this type of measures need to satisfy the condition that $\partial^{i-2} p(x, z)/\partial x^{i-2}$ is continuous in x everywhere for $i = 3, 4, ...n$ besides that $(-1)^i \partial^i p(x, z)/\partial x^i > 0$ for $x \in (0, z)$. None of the above Daltonian measures can be coherent with $\mathbf{D}_n$. The only existent measure that is coherent with $\mathbf{D}_n$ is the Foster *et al.* measure $P(F; z) = \int_0^z (1 - x/z)^{n-1}\, \mathrm{d}F(x)$. But other Daltonian measures can be transformed to satisfy those strong sensitivity axioms. For example, while the Watts measure $P(F; z) = \int_0^z (\ln z - \ln x)\, \mathrm{d}F(x)$ does not satisfy *strong transfer sensitivity*, its variation $P(F; z) = \int_0^z (\ln z - \ln x)^2\, \mathrm{d}F(x)$ does satisfy the axiom. When $n \to \infty$, Fishburn (1976) showed that no nontrivial utility function is 'congruent' with $\mathbf{D}_n$. It follows that the set of poverty measures that are coherent with $\mathbf{D}_\infty$ is empty.

4.6. *Poverty aversion, distribution-sensitivity and poverty orderings*

In a recent paper, Zheng (2000b) considered poverty orderings for a class of additively separable measures which have minimum distribution-sensitivity or poverty aversion. The established condition is similar to Meyer's (1977) second degree stochastic dominance with respect to a function or Foster and Jin's (1998) transformed generalized Lorenz dominance.

The measure of distribution-sensitivity for an additively separable poverty measure $P(F; z) = \int_0^z p(x, z)\, \mathrm{d}F(x)$ is defined as

$$s_p(x, z) = -\frac{\partial^2 p(x, z)}{\partial x^2} \bigg/ \frac{\partial p(x, z)}{\partial x}$$

and reflects the relative decrease in poverty as a result of a progressive transfer. Zheng (2000b) also pointed out that such a measure can be used to gauge poverty aversion — a concept which has been repeatedly used but a former definition has not hitherto been given.[37] Thus distribution-sensitivity and poverty aversion can be regarded as synonymous. Following Pratt (1964), it can be easily shown that a poverty measure is more distribution-sensitive than another if and only if the former is an increasing convex function of the latter. For a given reference poverty measure with individual deprivation function $q(x, z)$, we can define a class of poverty measures which are more distribution-sensitive than that of $q(x, z)$ as follows:

$$\Xi(q) := \{P \in \Xi_M \mid s_p(x, z) \geqslant s_q(x, z) \text{ for all } x \in [0, z]\}.$$

That is, for each poverty measure in $\Xi(q)$, its poverty deprivation function $p(x, z)$ is an increasing convex function of $q(x, z)$. It is useful to note that the usual distribution-sensitive class Ξ_{ST} is a special case of $\Xi(q)$ with $q = z - x$.

Using the methods of Meyer (1977) and Atkinson (1970), we can prove the following ordering condition of $\Xi(q)$ for a given poverty line z.

> PROPOSITION 4.8. Given two distributions $F \in \Psi$ and $G \in \Psi$, the necessary and sufficient condition for $P(F; z) \leqslant P(G; z)$ for all poverty measures $P \in \Xi(q)$ with a given poverty line z is $\int_0^y F(x)\, \mathrm{d}q(x, z) \leqslant \int_0^y G(x)\, \mathrm{d}q(x, z)$ for all $y \in [0, z]$ or, equivalently, $\int_0^r q(\tilde{\xi}_t(F), z)\, \mathrm{d}t \geqslant \int_0^r q(\tilde{\xi}_t(G), z)\, \mathrm{d}t$ for all $r \in [0, 1]$ where $\tilde{\xi}_t(F)$ and $\tilde{\xi}_t(G)$ are, respectively, the quantile functions of F and G censored at z.

This proposition reveals that the ordering condition is a generalized absolute rotated Lorenz dominance. Here the dominance curve is obtained by cumulating individual poverty deprivation $q(\tilde{\xi}_t, z)$, instead of $z - \tilde{\xi}_t$, from largest to smallest. Thus the dominance condition can be visualized via a device similar to Jenkins and Lambert's TIP curve. Unlike the absolute rotated Lorenz curve which mirrors a generalized Lorenz curve, this generalized absolute rotated Lorenz curve may not have such an equivalence unless $q(x, z)$ is of the Daltonian type.

For a function $q(x, z)$ which is strictly convex in x, the corresponding dominance condition is stronger than the generalized Lorenz dominance in the sense that the former condition can rank order more distributions than the latter condition. Thus this new dominance condition may have substantially more power than Atkinson's second degree stochastic dominance condition. The additional power of the new dominance condition over Atkinson's second degree condition can be illustrated via the Kolm three-person simplex. This is done in Figure 6 where the minimum distribution-sensitive measure is the Watts measure and the additional power of poverty orderings is represented by the darkly shaded area (the maximum poverty line is assumed to be 12 for simplicity).

Unlike the Atkinson conditions where poverty orderings at a given z imply poverty orderings at a lower poverty line $\tilde{z}$ ($\tilde{z} < z$), this generalized ordering condition may not possess this property. As shown in Zheng (2000b), this property holds if and only if $q(x, \tilde{z})$ is a convex function of $q(x, z)$ for $\tilde{z} < z$. Fortunately, for all conceivable poverty measures $q(x, z)$, this requirement is satisfied and hence the property may be retained.

The condition presented in Proposition 4.8 can also be used to compare the ordering power of various classes of minimum distribution-sensitive poverty measures. It is easy to compute that, for $x \in (0, z]$, the distribution-sensitivity is $1/x$ for the Watts measure, $(1-\beta)/x$ for the Clark *et al.* measure, $(\alpha - 1)/(z - x)$ for the Foster *et al.* measure, and γ for the CDS poverty measure (this is why this measure is called constant distribution-sensitivity measure). Thus the poverty ordering with the Clark *et al.* second measure as the minimum distribution-sensitive measure

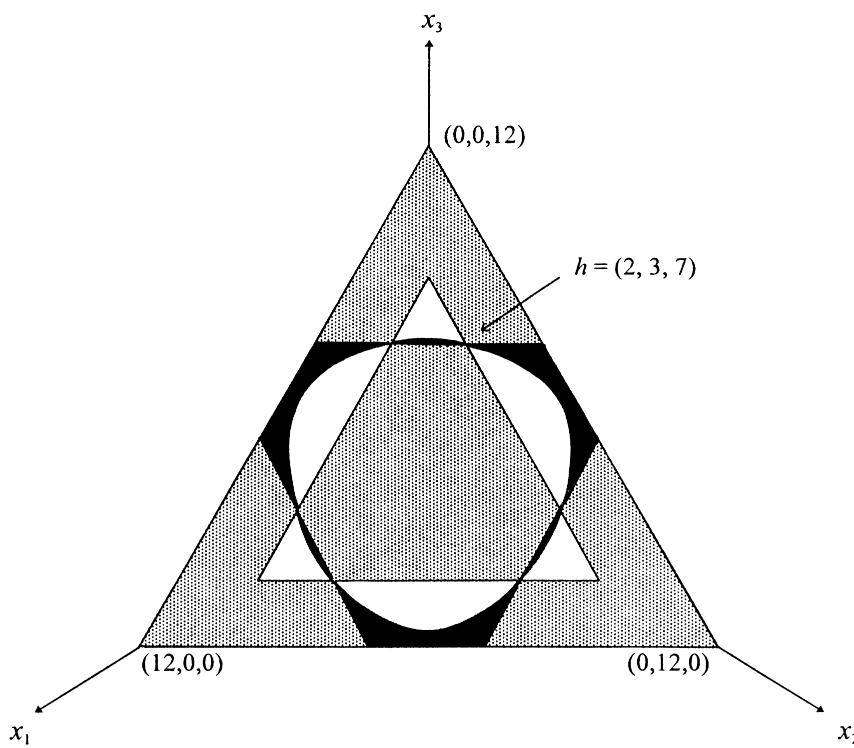

Figure 6. Poverty orderings with minimum distribution-sensitive measures ($q(x, z) = \ln z - \ln x$)).

becomes more complete as β becomes smaller: the Clark *et al.* measure with a negative β is more complete than the Watts measure which, in turn, is more complete than the Chakravarty measure which takes only positive values of β. For the Foster *et al.* measure, the ordering power increases as α increases. There is no general comparison of ordering power between the Clark *et al.* second measure and the Foster *et al.* measure since neither one is more distribution-sensitive than the other for all $x \in (0, z]$ except when α or β take extreme values. There is also no direct power comparison of the CDS measure with either the Foster *et al.* or the Clark *et al.* second measures.

5. Summary and conclusions

This paper has reviewed the literature of poverty orderings. Poverty orderings, or partial poverty orderings to be exact, require unanimous rankings of distributions by a class of poverty measures or a set of poverty lines. The need to consider multiple poverty measures and multiple poverty lines arises inevitably from the arbitrariness embedded in choosing a poverty measure and a poverty line for poverty comparisons. Since Foster (1984) raised the issue of poverty orderings, many researchers have contributed to this topic. These contributions established various conditions under which distributions can be rank ordered by a class of poverty measures and/or a range of poverty lines. The objective of this review has been to put together these researches, analyze the interrelationship among various ordering conditions and investigate the possibility of increasing the completeness of poverty orderings.

We first surveyed poverty-line orderings — the rankings of distributions by an individual poverty measure for a range of poverty lines. The ordering conditions are related to stochastic dominance which are based upon the comparisons of the *c.d.f.*s of income distributions. For the Foster *et al.* class, a bilateral relationship between the poverty-line ordering by a member of the class and stochastic dominance is established. For a Daltonian poverty measure, second degree stochastic dominance of the transformed income (or utility) is characterized as the necessary and sufficient ordering condition; second degree stochastic dominance implies these transformed dominances. For other measures such as the Sen and Thon measures, stochastic dominance relations (or mixtures of them) are shown only to be sufficient conditions. Although the exact ordering conditions for these measures can be identified mathematically, it remains a topic for future research to interpret these conditions and characterize them graphically.

We then reviewed poverty-measure orderings — the rankings of distributions by a class of measures with a fixed poverty line. The ordering conditions again are connected with stochastic dominance relations. While these conditions were originally established for additively separable poverty measures, they also characterize poverty orderings when the restriction of additivity on a poverty measure is relaxed. We also extended the relationship between poverty orderings and stochastic dominance to any higher degree and identified axiomatic requirements on poverty measures for such an extension. For a single and fixed poverty line, a higher degree dominance condition can increase the power of poverty orderings. For a variable poverty line, however, the increase in the power may be reduced as the lower bound line decreases; when the lower bound line is arbitrarily close to zero any higher degree stochastic dominance condition collapses to Atkinson's second degree dominance condition. A natural attempt to break such an 'impossibility' situation is to remove all lower degree dominance requirements at the poverty line, but this amounts to introducing stronger axioms than what are normally specified and these stronger axioms may not be unconditionally justified. An alternative approach to move beyond second degree stochastic dominance is to impose a minimum distribution-sensitivity requirement on the set of poverty measures and such an attempt can indeed increase the completeness of orderings.

One can certainly investigate poverty orderings of other classes of measures by considering different sets of poverty axioms. For example, following Vickson (1975, 1977), we can consider a class of additively separable poverty measures that exhibit decreasing poverty aversion, i.e. $s(x, z)$ is a decreasing function of x, and establish an ordering condition similar to that of Vickson. Also following Meyer (1977a) we may consider poverty orderings by a class of additively separable poverty measures possessing both minimum distribution-sensitivity and maximum distribution-sensitivity, i.e. all measures in the class are bounded both from above and from below by two pre-specified poverty measures.

While the classes of relative inequality measures and absolute inequality measures characterize different ordering conditions (Lorenz dominance and absolute Lorenz dominance, respectively, for all measures satisfying the principle of transfers), the corresponding classes of poverty measures do not. For example, both (relative and absolute) classes of distribution-sensitive poverty measures characterize generalized Lorenz dominance as their ordering condition for a common but variable poverty line. While there is no poverty measure except a monotonic transformation of the headcount ratio that can be both relative and absolute (Zheng, 1994), the above result indicates that both requirements are perfectly compatible in poverty orderings. Recently, Mitra and Ok (1995) considered poverty orderings for a class of compromise poverty measures (they are absolute and homogenous of degree 1 in all incomes and the poverty line) with a fixed poverty line. This class of poverty measures is essentially that of the Foster *et al.* measures (without being normalized by z^{α}). They showed that generalized Lorenz dominance is a sufficient condition for this class with $\alpha \geqslant 1$. They also identified a weaker sufficient condition but a necessary condition is yet to be found.

Before closing this review, it is necessary to remind readers that we did not review poverty orderings in terms of equivalence scales. Since in poverty comparisons, an equivalence scale is used to adjust the difference in family composition, the use of different scales can clearly lead to different poverty orderings. Atkinson (1992) considered poverty orderings for a class of additively separable poverty measures with multiple equivalence scales. He assumed that the poverty line for a single person is given and fixed but the lines for other types of families are left undetermined and may vary according to an ordinal rule: the line of a larger family is always greater than that of a smaller family. Based upon this setting, Atkinson (1992) was able to derive a sequence of ordering conditions. Jenkins and Lambert (1993) further generalized Atkinson's conditions and Bradbury (1997) investigated poverty comparisons with a set of bounded equivalence scales. Clearly, Atkinson's assumptions are not completely satisfactory and thus, given the importance of equivalence scales in poverty studies, future work on this topic would be very useful.

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Notes

1. See, for example, Seidl (1988), Zheng (1993) and Foster and Sen (1997).
2. In this review we adopt the weak definition of the poor (Donaldson and Weymark, 1986). That is, a person is poor if and only if his income is strictly less than the poverty line. This definition not only is consistent with the perception of poverty but also enables us to use censored income distributions (Zheng, 1997).
3. In this paper, we do not explicitly deal with the class of ethical poverty measures such as those proposed by Blackorby and Donaldson (1980). As explained by Zheng (1997) and others, the properties that such a measure may possess depend largely upon the specification of the utility function. Also members of this class may become either additively separable measures or rank-based measures when appropriate restrictions are imposed.
4. As shown by Foster and Shorrocks (1991), the desirability of this class of poverty measures can be justified by the subgroup consistency axiom; the axiom requires that an increase in a subgroup's poverty increase the overall poverty.
5. Chaubey (1994) recently pointed out that Takayama's derivation is flawed. The correct measure should be a multiplication of the measure that Takayama (1979) derived. This correction, however, does not affect the properties that the Takayama measure violates.
6. In this paper, we are interested in poverty orderings not poverty decomposition per se and, thus, axioms such as decomposability and subgroup consistency will not be directly dealt with. See Foster and Shorrocks (1991) for detailed discussions on decomposability and subgroup consistency of a poverty measure. It is also useful to point out that not all poverty axioms listed above are independent. For example, *strong transfer* implies *monotonicity*. See Zheng (1997) for more discussions and a list of the core axioms.
7. Examples of additively separable poverty measures in Ξ_{WT} but not in Ξ_{ST} can be easily constructed. For example, the measure with $p(x,z)=(a-x/z)^{\alpha}$ for $\alpha \geqslant 2$ and $a>1$ satisfies *weak transfer* but not *strong transfer* and hence belongs to Ξ_{WT}. The poverty measure constructed by Bourguignon and Fields (1997) also belongs to this class.

8. For example, Atkinson (1987) considered weak orderings, Howes (1993) used semi-strict orderings, and Zheng (1999) adopted strict orderings.
9. Thus empirically it may be meaningful to find the minimum α value at which the dominance relation holds.
10. Rothschild and Stiglitz (1970) and Davies and Hoy (1995) dealt with continuous distributions while Fields and Fei (1978), Shorrocks and Foster (1987) and Foster and Shorrocks (1988b) addressed discrete distributions. The terminologies they used are also different. For example, a mean-preserving spread in continuous distributions is called a regressive transfer in discrete distributions; a mean-variance-preserving transformation (Davies and Hoy, 1995) is the same as a favorable composite transfer (Shorrocks and Foster, 1987). In this paper, to be consistent with the poverty measurement literature, we use terminologies for discrete distributions although we deal with continuous distributions.
11. A favorable composite transfer, as defined by Shorrocks and Foster (1987), consists of a pair of mean-variance-preserving transfers: a progressive transfer at lower income levels and a regressive transfer at higher income levels.
12. Since for a utility function $u(x)$, $P(F; z) - P(G; z) = \int_0^z [F(x) - G(x)]\, \mathrm{d}u(x)$, thus the necessary and sufficient condition for $P(F; z) \leqslant P(G; z)$ for all $z \in (0, \infty)$ is $\int_0^z [F(x) - G(x)]\, \mathrm{d}u(x) \leqslant 0$ for all $z \in (0, \infty)$. This condition is what Meyer (1977b) characterized as 'second degree stochastic dominance with respect to function $u(x)$', as pointed out by Foster and Jin (1998).
13. The characterization of the transformed generalized Lorenz dominance is a bit different from that of the ordinary generalized Lorenz dominance; the latter uses income increments and mean-preserving transfers while the former uses utility increments and utility-preserving transfers. For example, the transformed generalized Lorenz dominance with $u(x) = \ln x$ is characterized by income (utility) increments and geometric-mean-preserving transfers. Since an arithmetic-mean-preserving progressive transfer can be regarded as an income increase and a geometric-mean-preserving progressive transfer, it follows that ordinary generalized Lorenz dominance necessarily implies transformed generalized Lorenz dominance.
14. This conclusion can also be verified alternatively and more directly as follows. Consider two distributions F and G and denote the Thon measure as $T(F; z)$. Following the definition of $T(F; z)$, it is easy to show that $T(F; z) - T(G; z) = (2z)^{-1} \int_0^z [F(x) - G(x)][2 - F(x) - G(x)]\, \mathrm{d}x$. Since $[2 - F(x) - G(x)]$ is nonnegative and decreasing in x, then by the second mean-value theorem of integration, we obtain that there exists a value $\theta \in (0, z)$ such that $T(F; z) - T(G; z) = z^{-1} \int_0^\theta [F(x) - G(x)]\, \mathrm{d}x$. It follows that $T(F; z) - T(G; z) \leqslant 0$ if and only if $\int_0^\theta F(x)\, \mathrm{d}x \leqslant \int_0^\theta G(x)\, \mathrm{d}x$. Clearly second degree stochastic dominance entails this conclusion. It is useful to note from the above derivation that, mathematically, the necessary and sufficient condition for $T(F; z) - T(G; z) \leqslant 0$ for all $z \in (0, z^{\max}]$ is $\int_0^z [1 - F(x)]^2\, \mathrm{d}x \geqslant \int_0^z [1 - G(x)]^2\, \mathrm{d}x$ for all $z \in (0, z^{\max}]$. It is unclear whether or not this condition has a simple graphical characterization.
15. Sen also graphically illustrated this measure in his 1976 paper. His graphical description would be the same as ours if his graph is scaled up by the mean income of the distribution. Note that Sen's motivation of the graphical illustration, unlike ours here, is to show that his poverty measure is 'a translation of the Gini coefficient from the measurement of inequality to that of poverty'. The sufficient condition presented in this paper was first identified in Zheng (1993, p. 67).
16. This can also be obtained from the relationship between the Thon measure and the Sen measure as given in Zheng (1997, p. 146, eq. (3.9)). Denote the Sen measure for

distribution F as $S(F;\, z)$, the Thon measure as $T(F;\, z)$ and the income gap ratio as $I(F;\, z)$, the relationship is $S(F;\, z) = \{T(F;\, z) - 2r_z I(F;\, z)[1 - r_z]\}/r_z$. In Figure 2, $r_z I(F;\, z)[1 - r_z]$ is simply the area of $ABCD$ divided by z. Thus $S(F;\, z)$ is twice the darkly shaded area normalized by both z and r_z.

17. This darkly shaded area without being normalized by r_z can also be regarded as a poverty measure. This measure is a variation of the Sen measure and is $2\int_0^z [F(z) - F(x)](1 - x/z)\, \mathrm{d}F(x)$. For a discrete and increasingly ordered distribution $(x_1, x_2, ..., x_n)$, this modified Sen measure is $2[(n+1)nz]^{-1} \sum_{i=1}^{q} (z - x_i)(q + 1 - i)$ where q is the number of the poor.
18. This last claim can be verified as follows. Clearly, censored generalized Lorenz dominance of F over G implies that $\{$the area of $OA'D''\}/r_z(G) \leqslant \{$the area of $OA'D'\}/r_z(G)$. Since DC is parallel to OB, it is easy to show that $\{$the area of $OAD\}/r_z(F) \leqslant \{$the area of $OA'D''\}/r_z(G)$. Thus, $\{$the area of $OAD\}/r_z(F) \leqslant \{$the area of $OA'D'\}/r_z(G)$.
19. Similar to that of the Thon measure, this sufficiency can also be shown directly as follows. Consider two distributions F and G and denote the Sen measure as $S(F;\, z)$, it is easy to verify that for a $z \in [z^{\min}, z^{\max}]$, $S(G;\, z) - S(F;\, z) = \int_0^z \{F^2(x)/r_z(F) - G^2(x)/r_z(G) + 2(G(x) - F(x)\}\, \mathrm{d}x$. If $r_z(F) \leqslant r_z(G)$, then $S(G;\, z) - S(F;\, z) \geqslant \int_0^z \{F^2(x)/r_z(G) - G^2(x)/r_z(G) + 2(G(x) - F(x)\}\, \mathrm{d}x = [r_z(G)]^{-1} \int_0^z [G(x) - F(x)][2r_z(G) - F(x) - G(x)]\, \mathrm{d}x$. Since $[2r_z(G) - F(x) - G(x)]$ is nonnegative and decreasing in x for $x < z$, then by the second mean-value theorem of integration, we obtain that there exists a value $\theta \in (0, z)$ such that $S(G;\, z) - S(F;\, z) \geqslant 2r_z(G) \int_0^\theta [G(x) - F(x)]\, \mathrm{d}x$. It follows that second degree stochastic dominance $\int_0^\theta G(x)\, \mathrm{d}x \geqslant \int_0^\theta F(x)]\, \mathrm{d}x$ for all $\theta \in [0, z]$ entails the conclusion. Also mathematically the necessary and sufficient condition for $S(F;\, z) \leqslant S(G;\, z)$ for all $z \in [z^{\min}, z^{\max}]$ is $\int_0^z F(x)[2 - F(x)/r_z(F)]\, \mathrm{d}x \leqslant \int_0^z G(x)[2 - G(x)/r_z(G)]\, \mathrm{d}x$ for all $z \in [z^{\min}, z^{\max}]$.
20. A variation of the Kakwani measure, $k(k+1)z^{-1} \int_0^{r_z} (1 - r/r_z)^{k-1} (rz - \int_0^r \tilde{\xi}_t\, \mathrm{d}t)\, \mathrm{d}r$, which is the Kakwani measure without being normalized by r_z, can be regarded as the weighted average of the darkly shaded area with the weight being $(1 - r/r_z)^{k-1}$. Clearly for $k \geqslant 1$, first degree dominance at z and second degree dominance over $[0, z]$ will ensure poverty ranking of this modified Kakwani measure at the poverty line z.
21. For the Clark *et al.* first measure, this condition is also a bit too sufficient in the sense that a much weaker sufficient condition can be identified. Clearly for $\alpha = 1$, the measure becomes the poverty gap ratio and, hence, second degree stochastic dominances over $[0, z^{\max}]$ serves as the necessary and sufficient condition. For an integer $\alpha > 1$, since the measure is simply a product of the headcount ratio and the Foster *et al.* measure, a natural sufficient condition would be $F\mathbf{D}_1 G$ over $[z^{\min}, z^{\max}]$ and $F\mathbf{D}_{\alpha+1} G$ over $[0, z^{\max}]$. Clearly for $\alpha \geqslant 2$, this condition is weaker than the mixed ordering condition for the Sen measure.
22. Davies and Hoy (1995) characterized inequality orderings for this more restrictive class as crossing Lorenz curves which satisfy a set of conditions based on cumulative coefficients of variation. Recently Foster and Sen (1997) and Formby *et al.* (2000) described a dominance method based on distributions normalized by their respective mean incomes. Formby *et al.* (2000) termed this approach normalized stochastic dominance (NSD) to distinguish it from the ordinary stochastic dominance. Second degree NSD is equivalent to Lorenz dominance and third degree NSD is equivalent to the Davies and Hoy's approach.
23. It is useful to note that Atkinson defined all poverty axioms in their weak form. That is, for an additively separable poverty measure P and any $x \in [0, z)$, *monotonicity*

require $\partial p(x,z)/\partial x \leqslant 0$ instead of $\partial p(x,z)/\partial x < 0$ as is generally assumed in the literature, and *strong transfer* additionally requires $\partial^2 p(x,z)/\partial x^2 \geqslant 0$ instead of $\partial^2 p(x,z)/\partial x^2 > 0$. Sen's initial definitions are all in strict form. If the weak forms are accepted then the headcount ratio would satisfy both *monotonicity* and *strong transfer*, and this is clearly in disagreement with Sen's criticisms of the measure. Although the use of these weak axioms does not weaken the ordering conditions, the proofs are no longer valid since the counter-examples that Atkinson used do not satisfy the required axioms. Zheng (1999) provided proofs for the Atkinson's theorems when poverty axioms are defined in their strict versions.

24. Using Kolm's three-person simplex, Zheng (2000a) characterized the additional power that the second degree dominance condition gains over the first degree dominance condition. The illustration considers both a fixed poverty line and a variable poverty line.
25. If the income distribution is normalized by the poverty line z, i.e. x is transformed into $\tilde{x} = x/z$ and the poverty line become $\tilde{z} = 1$, the Foster *et al.* measure with $\alpha = 2$ is simply twice the area between second degree stochastic dominance curve and the horizontal line up to $\tilde{x} = 1$.
26. Formby, Smith and Zheng (1999) recently showed that for normalized stochastic dominance (NSD), which characterizes inequality orderings, the area between second degree NSD curve and the line of equal distribution represents a member of the generalized entropy class, $\frac{1}{2}$(coefficient of variation)^2. That is, the coefficient of variation equals the square root of twice the area.
27. Zheng (1999) showed that the mixed dominance condition can be fully characterized by the additively separable subset of Φ_{WT}, which is Ξ_{WT}.
28. Foster (1984) wondered about this extension as he noted that '(i)t would be interesting to extend this ordering to distributions having different numbers of poor persons'. Proposition 4.3 shows that this extension is legitimate if the distribution with higher generalized Lorenz curve within the poor also happens to have fewer poor people. Jenkins and Lambert (1998a) noted that the potential for extending Foster's result to the case where the distribution with higher generalized Lorenz curve happens to have more poor people is 'limited'. Proposition 4.3 reveals that if all poverty measures of the Sen type are considered then such an extension is not possible; the first degree requirement of the ordering condition cannot be relaxed.
29. Also see Zheng (1997, p. 133) on this point.
30. Note that (uncensored) generalized Lorenz dominance implies censored generalized Lorenz dominance at any value z and, hence, implies the dominance of absolute rotated Lorenz curves. For each z, the censored generalized Lorenz curve is obtained from the uncensored generalized Lorenz curve by replacing the curve between r_z and 1 with its tangent line which has a slope z. Thus the corresponding censored generalized Lorenz curve for different values of z can be constructed from the same curve by using different tangent lines. In contrast, the absolute rotated Lorenz curve needs to be completely reconstructed for each z.
31. For a relative poverty measure P, $P(F; z_F) = P(F'; 1)$ where F' is the *c.d.f.* of x/z_F. Thus for two different poverty lines z_F and z_G, Jenkins and Lambert's result can also be stated as $z_F^{-1} \int_0^{yz_F} F(t)\,\mathrm{d}t \leqslant z_G^{-1} \int_0^{yz_G} G(t)\,\mathrm{d}t$ for all $y \in [0,1]$. This is because for all relative poverty measures, $P(F; z_F) \leqslant P(G; z_G)$ is equivalent to $P(F'; 1) \leqslant P(G'; 1)$. Thus by Atkinson's second degree condition, $P(F; z_F) \leqslant P(G; z_G)$ if and only if $\int_0^y F'(t)\,\mathrm{d}t \leqslant \int_0^y G'(t)\,\mathrm{d}t$ for all $y \in [0,1]$. The derivation uses the relation $F'(t) = F(tz_F)$. Both the condition and derivation here are very similar to Formby *et al.*'s (2000) characterization of inequality orderings when means are unequal.

While Jenkins and Lambert considered relative poverty measures when poverty lines differ. It is also possible to consider absolute poverty measures. A poverty measure is absolute if it is unaffected by an equal-amount increment to all incomes and the poverty line. Thus if P is an absolute measure, $P(F; z_F) = P(F'; 0)$ where F' is the *c.d.f.* of $x - z_F$. It follows that $P(F; z_F) \leqslant P(G; z_G)$ for all absolute poverty measures P if and only if $\int_0^{z_F - y} F(t)\,\mathrm{d}t \leqslant \int_0^{z_G - y} G(t)\,\mathrm{d}t$ for all $y \in [0, \max(z_F, z_G)]$. When poverty lines are different, Jenkins and Lambert's TIP curve dominance only implies poverty orderings by all absolute poverty measures. It is also useful to point out that when poverty lines are the same (but may vary), the TIP curve dominance implies poverty orderings for all poverty measures including the relative, absolute and intermediate subclasses. In this sense, the difference between relativity and absoluteness in poverty orderings is much less pronounced than that in inequality orderings (see, for example, Moyes, 1987 and Bossert and Pfingsten, 1990).

32. The intuition for this interesting result is that as the poverty line increases, the normalized TIP curve will be shifted down. If there is a gap everywhere between two TIP curves, then the higher TIP curve can always moves down until part of the curve overlaps or touches the lower TIP curve (except at the origin). Of course if two TIP curves touch anywhere besides the origin then there is no possibility to have Jenkins and Lambert's extension.
33. This proposition was initially presented in Zheng (1993, Proposition 3.5, pp. 68–69).
34. See Zheng (1999) for a more detailed exposition on this point.
35. The additional power of the mixed dominance condition of Proposition 4.5 and $\mathbf{D}_3$ over the Atkinson second degree dominance condition has also been graphically illustrated by Zheng (2000a) using the Kolm three-person simplex.
36. Thistle (1994) characterized the Atkinson family of inequality measures using the $f(t)$ function for the Clark *et al.* second measure of poverty.
37. A notable exception is Jenkins and Lambert (1998b) who used this measure implicitly as a measure of poverty aversion and established a theorem using poverty aversion at the poverty line. Zheng (2000b) explored the normative meaning of this measure as a measure of poverty aversion.

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