

Wealth inequality, preference heterogeneity and macroeconomic volatility in two-sector economies*

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Abstract: *We explore the link between wealth inequality, preference heterogeneity and macroeconomic volatility in a two-sector neoclassical growth model. First we prove that, if agents have homogeneous preferences, when the absolute risk tolerance is a strictly convex (concave) function, sufficiently high (low) levels of wealth inequality may lead to endogenous fluctuations in the neighborhood of the steady state. Second, we consider the effects of preference heterogeneity when agents are homogeneous with respect to their wealth. We show that when the utility function belongs to the HARA class, sufficiently high levels of preference heterogeneity may lead to endogenous fluctuations in the neighborhood of the steady state if the elasticity of intertemporal substitution in consumption is greater than one.*

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Journal of Economic Literature Classification Numbers: D30, D50, D90, O41.

1 Introduction

The relationship between income or wealth inequality and macroeconomic volatility has been largely ignored by the theoretical literature.¹ The existence of a link between inequality and instability is, however, of great importance, particularly in relation to issues of public policy. Going back to at least the work of Musgrave [25], if not further, it is a common practice to dissociate macroeconomic stabilization policies and redistribution policies, their instruments being a priori different. But, if volatility and wealth inequality are correlated, it becomes fundamentally important to understand how they interact and when these two dimensions of public intervention need to be considered simultaneously.

In this paper we consider a deterministic model in which inequality and preference heterogeneity may affect the economy through the stability properties of the steady state and generate periodical solutions. We adopt a neoclassical two-sector growth model with a pure consumption good and a capital good. Labor is provided inelastically. Agents are heterogeneous with respect to their preferences, their share of the initial stock of capital and their labor endowments. We assume that markets are complete and agents face no borrowing constraints. Due to the structure of the model, individual characteristics and heterogeneity do not affect the steady state values of the aggregate variables. However, in its representative agent version, for some standard technologies and preferences the model may exhibit instability and fluctuations.²

An economy may be unequal or heterogeneous in several dimensions. We first consider wealth inequality under homogeneous preferences by assuming that the agents are heterogeneous only with respect to their share of the initial stock of capital and their labor endowments. Comparing distributions of individual wealth is possible using a variety of criteria.³ A standard approach is to use the principle of transfers: inequality does not increase when income is transferred from a rich to a poor individual. Formally, in this case distributions are ranked using their Lorentz curves or equivalently using

¹On the empirical side, building on a comparison between Latin American and OECD economies, Breen and Garcia-Peñalosa [9] obtain a positive relationship between a country's volatility in its growth rate and income inequality.

²See Benhabib and Nishimura [3].

³See for instance Cowell [11].

second-order stochastic dominance. Rothschild and Stiglitz [28] consider a notion similar to second order stochastic dominance in which attention is restricted to continuous and concave functions (instead of non-decreasing and concave). We adopt this last ordering.

Our main result is that under some conditions on the technologies and provided preferences do not have hyperbolic absolute risk aversion, wealth inequality may be positively or negatively correlated to macroeconomic volatility depending on the concavity of the inverse of the agents' absolute risk aversion, called *absolute risk tolerance*. In particular, we find that when the consumption good is capital intensive at the steady state and the absolute risk tolerance is a convex function of consumption, there exist open sets of economies for which sufficiently high levels of wealth inequality lead to non-monotone dynamics in the neighborhood of the steady state. In the opposite case, non-monotone optimal paths are associated with low levels of wealth inequality.

In a second step, we consider preference heterogeneity under the same conditions on the technologies and the assumption that agents have identical initial wealth. In order to simplify the formulation, we assume that the instantaneous utility function of each consumer is defined with respect to a vector of individual parameters and we consider that heterogeneity across agents only concerns one parameter, the others being fixed. Using for instance CES preferences, some heterogeneity of the elasticity of intertemporal substitution in consumption can be introduced. We thus show that when the consumption good is capital intensive at the steady state and the utility function belongs to the HARA class, there exist open sets of economies for which sufficiently high levels of preference heterogeneity lead to non-monotone dynamics in the neighborhood of the steady state if the elasticity of intertemporal substitution in consumption is greater than one. When the utility function is not specified analytically, we also derive the same kind of conclusions based on the curvature of the absolute risk tolerance with respect to the individual characteristic.

In general, the usual axioms on preferences do not restrict the concavity properties of the absolute risk tolerance. Furthermore, as direct empirical investigation of the properties of preferences is usually impossible, only indirect model dependent evidence can be collected. As reviewed by Gollier [17], asset theory provides most of the indirect empirical evidence and suggests

that the absolute risk tolerance is not linear. The other set of indications is provided by household intertemporal behavior. Evidence there suggests that households do not behave as single consumers. Within the framework provided by models of collective behavior, this fact implies the non-linearity of the absolute risk tolerance (or the inefficiency of the allocation).⁴

Our analysis is standard for economies with heterogenous agents (see Ghiglino [12]). Wealth inequality affects the “social” utility function independently of the presence of heterogeneity in preferences. The first welfare theorem allows us to focus on the properties of the Pareto optimal allocations. These are obtained as solutions to a planner’s problem characterized by a social utility function depending on the welfare weights. In the model, these weights are continuous functions of the initial conditions. Consequently, the local dynamic properties of the general equilibrium model with heterogeneous agents and those of the planner’s problem with the welfare weights fixed at their steady state values are identical. Decentralization of these equilibria only occurs at a second stage of our analysis where we characterize the effect of agent’s heterogeneity on dynamics.

To our knowledge, the present paper provides the first analysis of the effects of taste heterogeneities and wealth inequalities on macroeconomic volatility in an intertemporal general equilibrium model. It is related to Ghiglino and Olszak [14] and Ghiglino [13] that consider a specific model drawn from Boldrin and Deneckere [5] in which technology belongs to a very small class. Assuming that the investment good sector is characterized by a Leontief technology while it is CES in the consumption good sector, these papers analyze only the effects of wealth inequality on local stability. Moreover, the model choices strongly limit the extent of the analysis of the relation between volatility, inequality and heterogeneity.

A few papers in the literature deal with the impact of wealth inequality or other kind of heterogeneities on the occurrence of local indeterminacy in models with externalities. Ghiglino and Olszak-Duquenne [15] is based on Example 2.1 in Boldrin and Rustichini [7] in which a labor augmenting externality is introduced in a two-sector growth model with a Leontief investment sector. The result that local indeterminacy is related to inequality suffers the same weaknesses as in the papers cited above. Herrendorf *et al.* [19] analyze an overlapping generations model with heterogeneously pro-

⁴See, e.g. Vermulen [33] or Mazzocco [21].

ductive agents. Heterogeneity is shown to reduce indeterminacy but general equilibrium effects are eliminated by the assumption that prices are “fixed”. Finally, Ghiglino and Sorger [16] looks at a continuous time growth model with a productive externality, endogenous labour supply, logarithmic preferences and two types of agents. They show that the initial distribution in individual wealth may have dramatic effects as driving the economy to a poverty trap or generate indeterminacy. However, no results pertaining to the effect of wealth inequality on indeterminacy is obtained because of the possible lack of continuity of the welfare weights.

The paper is organized as follows: In Section 2 the model is introduced while the equilibria are defined in Section 3. Section 4 focuses on the relationship between endowment distributions and the existence of endogenous fluctuations. The occurrence of macroeconomic volatility is related to wealth inequality in section 5 while it is related to preference heterogeneity in Section 6. Section 7 provides some comments on global dynamics. Section 8 presents a CES example and Section 9 concludes. Most of the proofs are gathered into a final Appendix.

2 The model

2.1 Producers

The technological side is formalized as in standard optimal growth models. There are two produced goods, one consumption good y_0 and one capital good y . The consumption good cannot be used as capital so it is entirely consumed. The capital good cannot be consumed and partially depreciates in each period at a constant rate $\mu \in [0, 1]$. There are two inputs, capital and labor. Labor is inelastically supplied and its total amount is normalized to 1. Each good is produced with a standard constant returns to scale technology:

$$y_0 = f^0(k^0, l^0), \quad y = f^1(k^1, l^1)$$

with $k^0 + k^1 \leq k$, k being the total stock of capital, and $l^0 + l^1 \leq 1$.

Assumption 1. *Each production function $f^j : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$, $j = 0, 1$, is C^2 , increasing in each argument, concave, homogeneous of degree one and such that for any $x > 0$, $f_1^j(0, x) = f_2^j(x, 0) = +\infty$, $f_1^j(+\infty, x) = f_2^j(x, +\infty) = 0$.*

There are two representative firms, one for each sector. For any $t \geq 0$, we denote by w_t the wage rate, r_t the gross rental rate of capital and p_t the price of investment good, all in terms of the price of the consumption good. For any given (k, y) , profit maximization in each representative firm is equivalent to solving the following problem of optimal allocation of productive factors between the two sectors:

$$\begin{aligned}
T(k, y) = & \max_{(k^0, k^1, l^0, l^1)} f^0(k_t^0, l_t^0) \\
& s.t. \quad y \leq f^1(k_t^1, l_t^1) \\
& \quad k^0 + k^1 \leq k \\
& \quad l^0 + l^1 \leq 1 \\
& \quad k^0, k^1, l^0, l^1 \geq 0
\end{aligned} \tag{1}$$

The social production function $T(k, y)$ describes the frontier of the production possibility set and gives the maximal output of the consumption good. It is also easy to show that the first derivatives of the social production function give the rental rate of capital and the price of the investment good:

$$T_1(k, y) = r(k, y), \quad T_2(k, y) = -p(k, y), \tag{2}$$

Under Assumption 1, from constant returns to scale of the two technologies we get $w(k, y) = T(k, y) - r(k, y)k + p(k, y)y > 0$.

2.2 Consumers

There are n agents and the total population is constant over time. In each period consumers provide inelastically a constant amount of labor ω_i , $i = 1, \dots, n$, with $\sum_{i=1}^n \omega_i = 1$. At the initial period $t = 0$, each agent i is endowed with a fixed share θ_i of the initial stock of capital k_0 , with $\sum_{i=1}^n \theta_i = 1$. Let $(\theta_i, \omega_i)_{i=1}^n = (\theta, \omega)$. Consumer's preferences are characterized by a discounted utility function of the form

$$U^i(x^i) = \sum_{t=0}^{\infty} \delta^t u_i(x_{it})$$

where $\delta \in (0, 1)$ is the discount factor, x_{it} is the consumption of agent i at time t and x^i is its intertemporal consumption stream. Agents are therefore different with respect to their preferences and their initial wealth. Each instantaneous utility function satisfies the following basic restrictions:

Assumption 2. $u_i(x_i)$ is C^2 , such that $u'_i(x_i) > 0$, $u''_i(x_i) < 0$ for any $x_i > 0$, and satisfies the Inada condition $\lim_{x_i \rightarrow 0} u'_i(x_i) = +\infty$.

In a decentralized economy, an agent i maximizes his intertemporal utility function subject to a single budget constraint

$$\sum_{t=0}^{\infty} R_t x_{it} = \sum_{t=0}^{\infty} R_t w_t \omega_i + \theta_i r_0 k_0 \quad \text{with } i = 1, \dots, n.$$

where the price of the consumption good has been normalized to one at each period $t \geq 0$, and the discount factors R_t are defined as:

$$R_t = \prod_{\tau=0}^t \frac{1}{1 + d_\tau}$$

with d_t the common interest rate which satisfies $d_0 = [r_0 - p_{-1}]/p_{-1}$ and $d_t = [r_t + (1 - \mu)p_t - p_{t-1}]/p_{t-1}$ for any $t \geq 1$.⁵

3 Competitive equilibria

From the first welfare theorem, we know that every competitive equilibrium obtained in the decentralized economy is a Pareto optimal allocation. Let

$$\Delta = \left\{ \eta_1, \dots, \eta_n \mid \eta_i \geq 0 \text{ and } \sum_{i=1}^n \eta_i = 1 \right\}$$

be the unit simplex of \mathbb{R}^n . A Pareto optimal allocation is a solution to the planner's problem for a given vector of welfare weights $\eta = (\eta_1, \dots, \eta_n) \in \Delta$:

$$\begin{aligned} \max_{\{x_{it}, y_t\}_{t \geq 0}} & \sum_{i=1}^n \eta_i \sum_{t=0}^{\infty} \delta^t u_i(x_{it}) \\ \text{s.t.} & \sum_{i=1}^n x_{it} = T(k_t, y_t) \\ & k_{t+1} = y_t + (1 - \mu)k_t \\ & k_0 \text{ given,} \end{aligned}$$

⁵This equation reflects the absence of arbitrage opportunities in a perfect foresight equilibrium. It is also called the *portfolio equilibrium condition* (see Becker and Boyd [1]). The difference between the equation evaluated at time $t = 0$ and $t \geq 1$ comes from the fact that at the initial date there is no residual capital coming from the previous period and in some sense we have $k_0 = y_{-1}$.

The solution to the above program depends on the vector η and on k_0 . The set of Pareto optima is obtained when η spans Δ . A given competitive equilibrium is obtained for a η such that the associated allocations saturate the budget constraint of all the consumers. Note also that the solutions are interior as soon as $\omega_i \neq 0$ or $\theta_i \neq 0$ for $i = 1, \dots, n$.

Let u be a social utility function such that for $\eta = (\eta_1, \dots, \eta_n) \in \Delta$

$$u(x_t) = \max_{\{x_{it}\}_{t \geq 0}} \sum_{i=1}^n \eta_i u_i(x_{it}) \quad (3)$$

$$s.t. \quad \sum_{i=1}^n x_{it} = x_t$$

We may then define the indirect utility function as follows

$$V(k_t, k_{t+1}) = u(T(k_t, k_{t+1} - (1 - \mu)k_t)) \quad (4)$$

Consider now the social production function $T(k_t, y_t)$ with $y_t = k_{t+1} - (1 - \mu)k_t$. We easily derive that $T(k_t, y_t) = 0$ if and only if $k_{t+1} = f^1(k_t, 1) + (1 - \mu)k_t \equiv g(k_t)$. Moreover from Assumption 1 we get $g'(0) = +\infty$ and $g'(+\infty) = 1 - \mu$ so that there exists $\bar{k} > 0$ such that $y_t + (1 - \mu)k_t > k_t$ when $k_t < \bar{k}$ while $y_t + (1 - \mu)k_t < k_t$ when $k_t > \bar{k}$. It follows that it is not possible to maintain stocks over \bar{k} . We may therefore define the set of admissible paths as

$$\mathcal{D} = \{(k_t, k_{t+1}) \in \mathbb{R}_+^2 \mid 0 \leq k_t \leq \bar{k}, (1 - \mu)k_t \leq k_{t+1} \leq f^1(k_t, 1) + (1 - \mu)k_t\}$$

It is easy to show that \mathcal{D} is a compact, convex set. It follows that the planner's problem is equivalent to

$$\max_{\{k_t\}_{t \geq 0}} \sum_{t=0}^{+\infty} \delta^t V(k_t, k_{t+1}) \quad (5)$$

$$s.t. \quad (k_t, k_{t+1}) \in \mathcal{D}$$

$$k_0 \text{ given}$$

Note that the solution depends on k_0 .

In the present framework it is a standard result that the set of interior Pareto optima is the set of $\{k_t\}_{t \geq 0}$ that are solutions to the following system of Euler equations

$$V_2(k_t, k_{t+1}) + \delta V_1(k_{t+1}, k_{t+2}) = 0 \quad (6)$$

and that satisfy the transversality condition

$$\lim_{t \rightarrow +\infty} \delta^t k_t V_1(k_t, k_{t+1}) = 0$$

Notice that using (2) and (4), the Euler equation becomes:

$$u'(x_t)T_2(k_t, y_t) + \delta u'(x_{t+1}) [T_1(k_{t+1}, y_{t+1}) - (1 - \mu)T_2(k_{t+1}, y_{t+1})] = 0 \quad (7)$$

An interior aggregate steady state is a sequence $(k_t, y_t) = (k^*, \mu k^*)$, $\forall t \geq 0$, that solves the Euler equation. We derive from (7) and the first order conditions associated with program (1):

$$-\frac{T_1(k^*, \mu k^*)}{T_2(k^*, \mu k^*)} \equiv f_1^1(k_1(k^*, \mu k^*), l_1(k^*, \mu k^*)) = \frac{1}{\delta} - (1 - \mu) \quad (8)$$

The steady state is thus given by the modified golden rule and, as shown for instance in Becker and Tsyganov [2], we get:

Lemma 1. *Under Assumption 1, for any $\delta \in (0, 1)$, there exists a unique steady state k^* solution of (8).*

The aggregate consumption x^* can also be obtained as

$$x^* = T(k^*, \mu k^*) \equiv T^*$$

Notice that at the steady state, aggregate capital and consumption depend only on the characteristics of technologies through the social production function $T(k, y)$.

Near the steady state the behavior of the dynamic system is equivalent to the behavior of the linearized system. The dynamic properties of the steady state are then related to the eigenvalues of the matrix associated with the linearized system. Denote $T_{ij}^* = T_{ij}(k^*, \mu k^*)$, $V_{ij}^* = V_{ij}(k^*, k^*)$, $i, j = 1, 2$, the second order derivatives of the social production function and the indirect utility function evaluated at the steady state. We easily derive the following characteristic polynomial

$$\mathcal{P}(\lambda) = \lambda^2 \delta V_{12}^* + \lambda (\delta V_{11}^* + V_{22}^*) + V_{12}^* = 0 \quad (9)$$

As shown in Benhabib and Nishimura [3], and denoting $a_{00} = l_0/y_0$, $a_{10} = k_0/y_0$, $a_{01} = l_1/y$, $a_{11} = k_1/y$ the capital and labor coefficients in each sector, it is easy to get

$$T_{12}(k, y) = -T_{11}(k, y)b(k, y) \quad (10)$$

where

$$b(k, y) = a_{01} \left(\frac{a_{11}}{a_{01}} - \frac{a_{10}}{a_{00}} \right) \quad (11)$$

is the relative capital intensity difference across sectors. We get $b(k, y) > (<)0$ if and only if the investment (consumption) good is capital intensive. Similarly, we have

$$T_{22}(k, y) = T_{11}(k, y)b(k, y)^2 \quad (12)$$

It follows therefore that the stability property of the steady state will depend on $b = b(k^*, \mu k^*)$, the second order derivative with respect to k of the social production function T_{11}^* , and the first and second order derivatives of the instantaneous utility function, all evaluated at the steady state.

Definition 1. Let u be the social utility function, $u : R_+ \rightarrow R$, and $\rho(x) = -u'(x)/u''(x) > 0$ be the inverse of the social absolute risk aversion, also called social absolute risk tolerance.

Based on all these expressions, and denoting $\vartheta = [1 - \delta(1 - \mu)]^{-1}$, we easily compute from the definition of the indirect utility function⁶

$$\begin{aligned} V_{11}^* &= u''(x^*)(\vartheta T_1^*)^2 + u'(x^*)T_{11}^* [1 + (1 - \mu)b]^2 \\ V_{12}^* &= -u''(x^*)\delta(\vartheta T_1^*)^2 - u'(x^*)T_{11}^* b [1 + (1 - \mu)b] \\ V_{22}^* &= u''(x^*)\delta^2(\vartheta T_1^*)^2 + u'(x^*)T_{11}^* b^2 \end{aligned}$$

We may now introduce the following elasticities of the consumption good's output and the rental rate with respect to the capital stock, all evaluated at the steady state

$$\varepsilon_{ck} = T_{11}^* k^* / T_1^* > 0, \quad \varepsilon_{rk} = -T_{11}^* k^* / T_1^* > 0 \quad (13)$$

Substituting these expressions into V_{ij}^* and recalling that $T^* = x^*$ yield to the following characteristic polynomial:

$$\begin{aligned} \frac{-\mathcal{P}(\lambda)}{u''(x^*)(T_1^*)^2} &= \lambda^2 \delta \left\{ \delta \vartheta^2 + \rho(x^*) \frac{\varepsilon_{rk}}{\varepsilon_{ck} x^*} b [1 + (1 - \mu)b] \right\} \\ &- \lambda \left\{ \delta(1 + \delta) \vartheta^2 + \rho(x^*) \frac{\varepsilon_{rk}}{\varepsilon_{ck} x^*} [b^2 + \delta [1 + (1 - \mu)b]^2] \right\} \\ &+ \delta \vartheta^2 + \rho(x^*) \frac{\varepsilon_{rk}}{\varepsilon_{ck} x^*} b [1 + (1 - \mu)b] = 0 \end{aligned} \quad (14)$$

For given discount factor and technology parameters, the characteristic roots depend on $\rho(x)$. Note that $\rho(x)$ close to zero indicates a high degree of curvature of the utility function.

⁶At the steady state, the Euler equation (7) becomes $T_2^* + \delta [T_1^* - (1 - \mu)T_2^*] = 0$ and may be equivalently rewritten as $-T_2^* = \delta \vartheta T_1^*$. It follows that $T_1^* - (1 - \mu)T_2^* = \vartheta T_1^*$.

4 Endogenous fluctuations and volatility

The steady state value of individual consumption depends on the initial distribution of capital and labor. The exact relationship is provided by the following Lemma.

Lemma 2. *Under Assumptions 1 and 2, the individual allocations evaluated at the steady state k^* are*

$$x_i^*(\theta_i, \omega_i) = \omega_i [x^* - (1 - \delta)\vartheta T_1^* k^*] + (1 - \delta)\vartheta T_1^* k^* \theta_i > 0$$

where $x^* = T(k^*, \mu k^*) \equiv T^*$.

In the present general equilibrium model, the social utility function depends on the welfare weights. Furthermore, these depend on the equilibrium allocations which in turn depend on the initial conditions and on the distribution of individual endowments. When the welfare weights are continuous functions of the initial capital stock, the dynamic properties of the competitive equilibrium with equilibrium welfare weights can be analyzed from the planner's problem defined in terms of the social utility function with fixed welfare weights. Indeed, local stability means that with initial conditions slightly away from the steady state, the optimal path still converges to it. The following Lemma gives a sufficient condition for continuity.

Lemma 3. *Under Assumptions 1 and 2, the welfare weights $(\eta_1, \dots, \eta_n) \in \Delta$ are continuous functions of the initial capital stock and labor endowments if for any $(k_t, k_{t+1}) \in \text{int}\mathcal{D}$:*

$$T_2(k_t, k_{t+1} - (1 - \mu)k_t) + b(k_t, k_{t+1} - (1 - \mu)k_t) T_1(k_t, k_{t+1} - (1 - \mu)k_t) \neq 0 \quad (15)$$

Notice that this continuity property may be ensured under a simple condition on the capital intensity difference:

Corollary 1. *Under Assumptions 1 and 2, the welfare weights $(\eta_1, \dots, \eta_n) \in \Delta$ are continuous functions of the initial capital stock and labor endowments if for any $(k_t, k_{t+1}) \in \text{int}\mathcal{D}$, the consumption good is capital intensive, i.e. $b(k_t, k_{t+1} - (1 - \mu)k_t) \leq 0$.*

The polynomial (14) clearly shows that the characteristic roots and therefore the stability properties of the steady state are functions of the

social absolute risk tolerance $\rho(x)$. We then need to show that this curvature index can also be expressed as a continuous function of the individual shares of capital and labor endowments.

Lemma 4. *Under Assumptions 1 and 2, let condition (15) hold. Then the social absolute risk tolerance is a continuous function of the initial capital stock and labor endowments given by*

$$\rho(x_t) = - \sum_{i=1}^n \frac{u'_i(x_{it}(\theta_i, \omega_i))}{u''_i(x_{it}(\theta_i, \omega_i))} \quad (16)$$

As a direct consequence of the last two Lemmas we may finally derive that the dynamic properties of the competitive equilibrium with equilibrium welfare weights can be analyzed from the planner's problem defined in terms of the social utility function with fixed welfare weights.

Lemma 5. *Under Assumptions 1 and 2, let condition (15) hold. Then the stability properties of the steady state of the general equilibrium model with equilibrium weights and of the growth model with fixed welfare weights are equivalent.*

In the sequel we consider capital intensity configurations under which condition (15) is satisfied and pursue our analysis of the equilibrium path when the welfare weights are fixed at their steady state values. Our objective is to derive some relationship between the dynamical properties of the equilibrium path and the degrees of inequality and heterogeneity in the economy. The degree of wealth inequality refers to the distribution of capital and labor endowments while the degree of preference heterogeneity refers to the differences in the utility functions across agents.

Lemma 4 shows that the social absolute risk tolerance $\rho(x)$ is obtained as the sum of the individual absolute risk tolerance $\rho_i(x_i)$ and consequently is a continuous function of initial capital and labor endowments. This property allows us to capture the link between wealth inequality and macroeconomic volatility through the value of $\rho(x)$. The effect of preference heterogeneity on social absolute risk tolerance is more difficult to describe and will be provided later in the paper. We now focus on the relationship between inequality and macroeconomic volatility.

We first give conditions on the capital intensity difference b for which stability does not depend on inequality and heterogeneity.

Proposition 1. *Under Assumptions 1 and 2, for any $\rho(x^*) > 0$, the steady state is saddle-point stable with monotone convergence in the following cases:*

1. *when the investment good is capital intensive ($b > 0$);*
2. *when the consumption good is capital intensive with $b \leq -1/(1 - \mu)$.*

Therefore, the distribution of shares and/or labor endowments does not affect stability.

Remark: Proposition 1 shows that when the investment good is capital intensive, the steady state of the planner's problem is always saddle-point stable. The local equivalence with the decentralized general equilibrium problem is obtained since condition (15) is satisfied at the steady state. Indeed, we have shown in the proof of Proposition 1 that $b - \delta\vartheta < 0$ and thus $T_2(k^*, \mu k^*) + bT_1(k^*, \mu k^*) = T_1(k^*, \mu k^*)(b - \delta\vartheta) \neq 0$.

In order to deal with the case in which wealth inequality matters, we need to define the interval of admissible values for the social absolute risk tolerance which depends on the distribution of shares and/or labor endowments in the economy. Let indeed

$$\underline{\rho} = \min_{(\theta, \omega)} \rho(x^*), \quad \bar{\rho} = \max_{(\theta, \omega)} \rho(x^*)$$

We have $\rho(x^*) \in (\underline{\rho}, \bar{\rho})$, and these bounds are usually easy to obtain. Consider for instance the case in which agents have identical preferences. The most equal configuration is obtained if every individual owns the same proportion of the total wealth, i.e. if $\omega_i = \theta_i = 1/n$. Each agent i then consumes at the steady state the same amount given by

$$x_i^* = x^*/n$$

On the contrary, the most unequal wealth distribution is obtained if one individual, say $i = 1$, owns the total wealth of the economy while all the others do not own anything, i.e. $\omega_1 = \theta_1 = 1$ and $\omega_i = \theta_i = 0$ for any $i \neq 1$. Agent 1 then consumes at the steady state the whole production of consumption good

$$x_1^* = x^*$$

We may then define the following expressions

$$\hat{\rho}(x^*) = n\rho(x^*/n), \quad \tilde{\rho}(x^*) = (n-1)\rho(0) + \rho(x^*) \quad (17)$$

If we assume that $\rho(x)$ is convex or concave, we get

$$\underline{\rho} = \min\{\hat{\rho}(x^*), \tilde{\rho}(x^*)\}, \quad \bar{\rho} = \max\{\hat{\rho}(x^*), \tilde{\rho}(x^*)\}$$

and these bounds correspond to the two situations of uniformity (equality) and full inequality (one agent consumes everything).

We also need to introduce two critical values for $\rho(x^*)$:

$$\rho_c = -\frac{\delta\vartheta^2\varepsilon_{ck}x^*}{b[1+(1-\mu)b]\varepsilon_{rk}}, \quad \rho_f = -\frac{2\delta(1+\delta)\vartheta^2\varepsilon_{ck}x^*}{[1+(2-\mu)b][\delta+[1+(1-\mu)\delta]b]\varepsilon_{rk}} \quad (18)$$

which allows us to determine respectively the sign of $\mathcal{P}(0)$ and $\mathcal{P}(-1)$. The following two Propositions are the main results of this section. The first one corresponds to the case with $\rho_c > 0$ and $\rho_f < 0$ and deals with the existence of damped fluctuations.

Proposition 2. *Under Assumptions 1 and 2, let the consumption good be capital intensive with $b \in (-1/(1-\mu), -1/(2-\mu)] \cup [-\delta/(1+\delta(1-\mu)), 0)$.*

1. *If $\rho(x^*) > \rho_c$, the steady state is saddle-point stable with oscillations.*
2. *If $\rho(x^*) < \rho_c$, the steady state is saddle-point stable with monotone convergence.*

The correlation between wealth inequality and the existence of damped fluctuations requires that $\rho_c \in (\underline{\rho}, \bar{\rho})$. If on the contrary $\rho_c \notin (\underline{\rho}, \bar{\rho})$, then for any $\rho(x^*) \in [\underline{\rho}, \bar{\rho}]$, the steady state is either saddle-point stable with monotone convergence if $\rho_c > \bar{\rho}$ or saddle-point stable with oscillations if $\rho_c < \underline{\rho}$.

The following Proposition, which corresponds to the case with $0 < \rho_c < \rho_f$, focuses on the existence of persistent oscillations.

Proposition 3. *Under Assumptions 1 and 2, let the consumption good be capital intensive with $b \in (-1/(2-\mu), -\delta/(1+\delta(1-\mu)))$.*

1. *If $\rho(x^*) > \rho_f$, the steady state is locally unstable with oscillating divergence.*
2. *If $\rho(x^*) \in (\rho_c, \rho_f)$, the steady state is saddle-point stable with oscillating convergence. Moreover, ρ_f is a flip bifurcation value and there generically exist period-two cycles, in a right (or left) neighborhood of ρ_f , which are saddle-point stable (or unstable).*
3. *If $\rho(x^*) < \rho_c$, the steady state is saddle-point stable with monotone convergence.*

Notice that $\rho_f \in (\underline{\rho}, \bar{\rho})$ is a necessary condition for the existence of a correlation between wealth inequality and persistent oscillations. However, even if $\rho_f < \underline{\rho}$ so that the steady state is locally unstable for any $\rho(x^*) \in [\underline{\rho}, \bar{\rho}]$,

period-two cycles may still exist. Indeed Mitra and Nishimura [23] show that in a two-sector optimal growth model, period-two cycles may persist for values of the discount factor “far” from the bifurcation value.⁷

Propositions 2 and 3 show that the existence of endogenous fluctuations requires a capital intensive consumption good. The intuition for this result, initially provided by Benhabib and Nishimura (1985), may be summarized as follows. Consider an instantaneous increase in the capital stock k_t . This results in two opposing forces:

- Since the consumption good is more capital intensive than the investment good, the trade-off in production becomes more favorable to the consumption good. The Rybczinsky theorem thus implies a decrease of the output of the capital good y_t . This tends to lower the investment and the capital stock in the next period k_{t+1} .

- In the next period the decrease of k_{t+1} implies again through the Rybczinsky effect an increase of the output of the capital good y_{t+1} . Indeed, the decrease of k_{t+1} improves the trade-off in production in favor of the investment good which is relatively less intensive in capital. Therefore this tends to increase the investment and the capital stock in period $t + 2$, k_{t+2} .

However the properties of preference also matter. The existence of persistent cycles requires first that the agents accept the fluctuations in their consumption levels. This requires that the elasticity of intertemporal substitution in consumption is large enough, i.e. the relative absolute risk tolerance is low enough. But endogenous fluctuations require also that the oscillations in relative prices must not present intertemporal arbitrage opportunities. For instance, possible gains from postponing consumption from periods when the marginal rate of transformation between consumption and investment is high to periods when it is low must not be worth it. This configuration is obtained provided that the discount factor is low enough.

These intuitions show that there are simple conditions under which wealth inequality matters for the occurrence of macroeconomic volatility. On the technological side, the consumption good has to be capital intensive. On the preference side, we will show in the following how wealth inequality influences the local stability properties through its effect on $\rho(x)$.

⁷As initially proved in Benhabib and Nishimura [3], period-two cycles may occur if the discount factor is less than 1 and crosses some bifurcation value.

5 Wealth inequality and macroeconomic volatility

We establish now the link between macroeconomic volatility and wealth inequality. We assume that agents are homogeneous with respect to their preferences, i.e. $u_i(x) = v(x)$, but heterogeneous with respect to their shares of initial capital and/or labor endowments (θ_i, ω_i) . Along the steady state the two-dimensional distribution of (θ_i, ω_i) gives rise to the distribution of individual wealth w_i . Indeed, from Lemma 2 and its proof we obtain that $w_i = \phi\omega_i + \psi\theta_i$ with ϕ and ψ two constants which depend on the steady state values of aggregate consumption and capital.⁸ Our definition of inequality will then refer to the distribution of individual wealth.

There are several possible formal definitions to characterize the level of inequality of a distribution. The empirical literature often uses the Gini index. This quantity is related to second order stochastic dominance as it represents the surface between the 45 degree line and the normalized (by the mean) Lorenz curve. The Gini index provides a complete ranking of distributions which is equivalent to second order stochastic dominance for the class of distributions with Lorenz curves that do not cross. On the contrary, this ranking is not complete for distributions with crossing Lorenz curves that cannot be compared. We choose the following definition, which is based on a notion similar to stochastic dominance, that is easier to apply in our framework (see Rothschild and Stiglitz [12]).

Definition 2. *Assume that there are N types of consumers ordered according to the steady state individual wealth, i.e. $w_i < w_j$ for $i < j$.⁹ Let $N_i(J)$ be the number of consumers of type i in economy J and let $\mathcal{N}(J)$ be the corresponding distribution. Furthermore, assume that the mean of the distribution $\sum_{i=1}^N N_i(J)w_i/n$ is independent of J . Then Economy A is said to be no more unequal than Economy B if $\sum_{i=1}^N N_i(A)f(w_i) \leq \sum_{i=1}^N N_i(B)f(w_i)$ for all continuous convex functions f .¹⁰ This case is denoted $\mathcal{N}(A) \preceq_I \mathcal{N}(B)$.*

The intuition behind Definition 2 is that a spread in the distribution of a consumer's type decreases or increases the expected value of a function $f(x)$

⁸ $w_i = \sum_{t=0}^{\infty} \delta^t x_i^*(\theta_i, \omega_i) = \frac{1}{1-\delta}(\omega_i(x^* - (1-\delta)\vartheta T_1^* k^*) + (1-\delta)\vartheta T_1^* k^* \theta_i)$.

⁹When considering the effect of a redistribution, N is at most equal to $2n$ as there are n types in the initial configuration and n types in the final configuration.

¹⁰Equivalently, Economy A is said to be no more unequal than Economy B if $\sum_{i=1}^N N_i(A)f(w_i) \geq \sum_{i=1}^N N_i(B)f(w_i)$ for all continuous concave functions f .

depending on whether f is concave or convex. Rothschild and Stiglitz [28] have shown the equivalence of \preceq_I with a class of intuitive notions of spread. In particular, they show that $\mathcal{N}(A) \preceq_I \mathcal{N}(B)$ implies that $\mathcal{N}(B)$ has more weight in the tails than $\mathcal{N}(A)$. Finally, in equilibrium, $\sum_{i=1}^N N_i(J)x_i$ is equal to x^* regardless of the distribution because of market clearing. Therefore, assuming that $\sum_{i=1}^N N_i(J)w_i/n$ is independent of J is not restrictive because $w_i = x_i/(1 - \delta)$.

The fact that wealth inequality may have an effect on macroeconomic volatility is a consequence of Propositions 2 and 3. In order to simplify the formulation, we introduce two different types of assumption concerning the capital intensity difference that are based on Propositions 2 and 3.

Assumption 3. *The consumption good is capital intensive with*

i) $b \in (-1/(1 - \mu), -1/(2 - \mu)) \cup [-\delta/(1 + \delta(1 - \mu)), 0)$ and $\rho_c \in (\underline{\rho}, \bar{\rho})$,

or

ii) or $b \in (-1/(2 - \mu), -\delta/(1 + \delta(1 - \mu)))$ and $\rho_f > \bar{\rho} > \rho_c > \underline{\rho}$.

Assumption 4. *The consumption good is capital intensive with $b \in (-1/(2 - \mu), -\delta/(1 + \delta(1 - \mu)))$ and $\rho_f \in (\underline{\rho}, \bar{\rho})$.*

Assumption 3 is linked to the existence of damped fluctuations while Assumption 4 concerns the occurrence of persistent fluctuations. Both are associated with a capital intensive consumption good. Such restrictions appear to be compatible with recent empirical evidences. Building on aggregate Input-Output tables, it has been proved by Takahashi *et al.* [32] that over the last 30 years, the OECD countries are characterized by a more capital intensive consumption sector than the investment sector.

Remark: As shown in Proposition 1, if $b > 0$, or $b \leq -1/(1 - \mu)$, wealth inequality does not have any effect on macroeconomic volatility.

To analyze the effect of wealth inequality on the occurrence of persistent endogenous fluctuations, we introduce the following assumption:

Assumption 5. *In case 1. of Proposition 3, the flip bifurcation generates saddle-point stable period-two cycles in a right neighborhood of ρ_f .*

Although this assumption concerns non trivial restrictions on the non-linear part of the Euler equation, a number of robust examples of saddle-point

stable period-two cycles have been provided by Boldrin and Deneckere [5] and Mitra and Nishimura [23].

Proposition 4. *Let Assumptions 1 and 2 hold, together with either Assumption 3, or Assumptions 4 and 5. Let the individual absolute risk tolerance $\rho(x_i)$ be a strictly convex (resp. concave) function. Then sufficiently high (resp. low) levels of wealth inequality lead to endogenous fluctuations in a neighborhood of the steady state. More precisely, there exists a distribution $\mathcal{N}(0)$ such that one of the following cases holds:*

i) If Assumption 3 holds, the steady state is saddle-point stable with monotone convergence for any economy J with $\mathcal{N}(J) \preceq_I \mathcal{N}(0)$ (resp. $\mathcal{N}(0) \preceq_I \mathcal{N}(J)$) and is saddle-point stable with oscillations otherwise.

ii) If Assumptions 4 and 5 hold, the steady state is saddle-point stable for any economy J with $\mathcal{N}(J) \preceq_I \mathcal{N}(0)$ (resp. $\mathcal{N}(0) \preceq_I \mathcal{N}(J)$) and is unstable otherwise. Moreover, there generically exist period-two cycles, in a right (resp. left) neighborhood of $\mathcal{N}(0)$, which are saddle-point stable.

Remark: A limitation of Proposition 4 concerns the fact that Assumptions 3 and 4 involve endogenous variables.¹¹ However, we prove in Section 8 below that there exist non-empty open sets of economies for which these Assumptions can be satisfied.

The curvature of the absolute risk tolerance involves the third and fourth order derivatives of the utility functions which are not limited by the standard assumptions on preferences. The traditional theory of precautionary saving requires the third derivative to be positive while the fourth derivative is unconstrained. Furthermore, a sufficient condition for the expected wealth accumulation to be increasing with the earning risks is that $v'(x)v'''(x)/(v''(x))^2$ is a constant k with $k > 0$. The HARA class, which includes the CARA and CRRA specifications, has this property.¹² The following result states that in this case inequality is neutral.

Corollary 2. *Assume that individual preferences are represented by a utility function of the HARA class, i.e.*

¹¹This problem is standard in the optimal growth literature (see for instance Benhabib and Nishimura [3]). General existence results are usually complemented with examples of economies characterized by CES technologies and HARA preferences.

¹²See Carroll and Kimball [10].

$$v(x) = \frac{1-\sigma}{\sigma} \left(\frac{ax}{1-\sigma} + e \right)^\sigma$$

with $a > 0, e \geq 0$ and $\sigma > 0$ as parameters. Then wealth inequality plays no role on the occurrence of macroeconomic volatility.

Remark: This neutrality result is easily explained first by the linearity of the absolute risk tolerance with respect to consumption, and second by the fact that the aggregate consumption level is independent from the initial distribution of wealth. In a two-sector model with two consumption goods, the investment good being also consumed, the aggregate consumption level depends on the initial distribution of wealth. Building on this property and using the Gini index to evaluate the degree of inequality in the economy, Bosi and Seegmuller [8] derive a non-neutrality result even with CES preferences: they show that more wealth inequality implies more macroeconomic volatility.

Recent week indirect evidence in support of the concavity of absolute risk tolerance (see Gollier [17], Guiso and Paiella [18]) together with Proposition 4 suggest that a sufficient high level of wealth inequality leads to a monotone behavior of the optimal path. Such a conclusion is in line with the main result of Herrendorf *et al.* [19]. Considering an overlapping generations model with heterogeneously productive agents, they show that sufficient heterogeneity reduces the scope of expectations-driven fluctuations and favors the monotone convergence of the equilibrium path toward a unique steady state.

6 Preference heterogeneity and macroeconomic volatility

In this section we establish the link between macroeconomic volatility and preference heterogeneity. We thus assume that agents are homogeneous with respect to their wealth, i.e. $\omega_i = \theta_i = 1/n$, but heterogeneous with respect to their utility function. As a consequence of Lemma 2, all agents consume the same amount at the steady state, i.e. $x_i^* = x^*/n$. We assume that preferences are characterized by a single parameter. We denote the instantaneous utility function of each consumer i as

$$u_i(x_i) = u(x_i; \sigma_i) \tag{19}$$

with $\sigma_i \in \mathbb{R}$. A typical example is given by the HARA utility function of Corollary 2 with $e = 0$. In this class, an agent i can be identified by his individual elasticity of intertemporal substitution in consumption given by $1/(1 - \sigma_i)$. The analysis could be extended to functions depending on any number of parameters. In this case, the present study carries through if heterogeneity only concerns one parameter, the others being fixed. We complete Assumption 2 by the following

Assumption 6. $u(x_i; \sigma_i)$ is C^2 over $\mathbb{R}_+ \times \mathbb{R}$. Moreover, for any given $\sigma_i \in \mathbb{R}$, $u_1(x_i; \sigma_i) > 0$, $u_{11}(x_i; \sigma_i) < 0$ for all $x_i > 0$, and satisfies the Inada condition $\lim_{x_i \rightarrow 0} u_1(x_i; \sigma_i) = +\infty$.

The spread in the parameters σ_i then characterizes the level of heterogeneity of the economy. Indeed, the agents can be distributed on the real line according to their σ_i . Using again the methodology of Rothschild and Stiglitz [28] to characterize the degree of heterogeneity we may adapt Definition 2 as follows:

Definition 3. Assume that there are N types of consumers ordered according to the preference parameter, i.e. $\sigma_i < \sigma_j$ for $i < j$. Let $N_i(J)$ be the number of consumers of type i in economy J and let $\mathcal{N}(J)$ be the corresponding distribution. Furthermore, assume that the mean of the distribution $\sum_{i=1}^N N_i(J)\sigma_i/n$ is independent of J . Then Economy A is said to be no more heterogeneous than Economy B if $\sum_{i=1}^N N_i(A)f(\sigma_i) \leq \sum_{i=1}^N N_i(B)f(\sigma_i)$ for all continuous convex functions f .¹³ This case is denoted $\mathcal{N}(A) \preceq_I \mathcal{N}(B)$.

As previously, a spread in the distribution of a consumer's type decreases or increases the expected value of a function $f(\sigma)$ depending on whether f is concave or convex. When focusing on the role of wealth inequality, comparing distributions with the same mean is not restrictive. In the case of preference heterogeneity this assumption is much less natural and strictly speaking restrictive. However, considering that $\sum_{i=1}^N N_i(J)\sigma_i \equiv m_\sigma$ is independent of J is an artefact which is only introduced to provide a unified analysis and does not imply a strong distortion for the measure of the degree of heterogeneity in the economy. Notice also that second order stochastic

¹³Equivalently, Economy A is said to be no more heterogeneous than Economy B if $\sum_{i=1}^N N_i(A)f(\sigma_i) \geq \sum_{i=1}^N N_i(B)f(\sigma_i)$ for all continuous concave functions f .

dominance can in some cases be generalized to the class of distributions with different mean.

We may define the individual absolute risk tolerance which now depends explicitly on the individual characteristic σ_i :

$$\rho_i(x_i) = \rho(x_i; \sigma_i) = -\frac{u_1(x_i; \sigma_i)}{u_{11}(x_i; \sigma_i)}$$

Since at the steady state $x_i^* = x^*/n$, we get $\rho(x_i^*; \sigma_i) = \rho(x^*/n; \sigma_i)$, and the social absolute risk tolerance becomes

$$\rho(x^*; \sigma_1, \dots, \sigma_n) = \sum_{i=1}^n \rho(x^*/n; \sigma_i)$$

In order to deal with the case in which preference heterogeneity matters, we need to define the interval of admissible values for the social absolute risk tolerance which depends now on the distribution of the individual characteristic σ_i . For a given number n of agents and a given mean m_σ , considering the set Σ_{m_σ} of distributions with mean m_σ , we get $\rho(x^*) \in (\underline{\rho}, \bar{\rho})$ with

$$\underline{\rho} = \min_{\Sigma_{m_\sigma}} \rho(x^*), \quad \bar{\rho} = \max_{\Sigma_{m_\sigma}} \rho(x^*)$$

Notice that we can derive an implicit support for the distributions in Σ_{m_σ} which is such that $\sigma_i \in [0, nm_\sigma]$.

Proceeding in a similar way as in Section 4, these bounds are easy to compute. The most homogeneous configuration is obtained if every individual characteristic σ_i is equal to the mean m_σ . On the contrary the most heterogeneous configuration is obtained if one individual, say $i = 1$, has a characteristic equal to nm_σ while all the others are characterized by $\sigma_i = 0$, $i \neq 1$. We may then define the following expressions

$$\hat{\rho}(x^*) = n\rho(x^*/n; m_\sigma), \quad \tilde{\rho}(x^*) = (n-1)\rho(x^*; 0) + \rho(x^*; nm_\sigma) \quad (20)$$

If we assume that $\rho(x)$ is convex or concave, we get

$$\underline{\rho} = \min\{\hat{\rho}(x^*), \tilde{\rho}(x^*)\}, \quad \bar{\rho} = \max\{\hat{\rho}(x^*), \tilde{\rho}(x^*)\}$$

and these bounds correspond to the two situations of uniformity (homogeneity) and full heterogeneity.

As shown in Lemma 1, modifying the distribution of the σ_i does not affect the value of the aggregate stationary consumption x^* . Using the same argument as for Proposition 4, we derive results concerning the effect of preference heterogeneity on macroeconomic volatility depending on whether the absolute risk tolerance $\rho(x^*/n; \sigma_i)$ is a convex or concave function of σ_i .

Proposition 5. *Let Assumptions 1 and 6 hold, together with either Assumption 3, or Assumptions 4 and 5. Let the individual absolute risk tolerance $\rho(x^*/n; \sigma_i)$ be a strictly convex (resp. concave) function of σ_i . Then, sufficiently high levels (resp. low) of preference heterogeneity lead to endogenous fluctuations in the neighborhood of the steady state. More precisely, there exists a distribution $\mathcal{N}(0)$ such that one of the following cases holds:*

i) Under Assumption 3 the steady state is saddle-point stable with monotone convergence for any economy J with $\mathcal{N}(J) \preceq_I \mathcal{N}(0)$ (resp. $\mathcal{N}(0) \preceq_I \mathcal{N}(J)$) and is saddle-point stable with oscillations otherwise.

ii) Under Assumptions 4 and 5, the steady state is saddle-point stable for any economy J with $\mathcal{N}(J) \preceq_I \mathcal{N}(0)$ (resp. $\mathcal{N}(0) \preceq_I \mathcal{N}(J)$) and is unstable otherwise. Moreover, there generically exist period-two cycles, in a right (resp. left) neighborhood of $\mathcal{N}(0)$, which are saddle-point stable.

Contrary to what occurs with wealth inequality, when standard HARA preferences are considered, we easily derive clear-cut conclusions:

Corollary 3. *Assume that individual preferences are represented by a utility function of the HARA class, i.e.*

$$u(x_i; \sigma_i) = \frac{1-\sigma_i}{\sigma_i} \left(\frac{ax_i}{1-\sigma_i} + e \right)^{\sigma_i}$$

with $a > 0, e \geq 0$ and $\sigma_i > 0$. Then sufficiently high levels of preference heterogeneity lead to endogenous fluctuations in the neighborhood of the steady state if and only if $\sigma_i \in (0, 1)$.

Notice that if $e = 0$, the preferences are characterized by a standard CES utility function with an elasticity of intertemporal substitution given by $1/(1 - \sigma_i)$. Corollary 3 then implies that a higher level of preference heterogeneity implies the existence of macroeconomic volatility if and only if the elasticity of intertemporal substitution in consumption is greater than 1.

Considering a general non-specified utility function, Proposition 5 provides the same kind of characterization as Proposition 4, which is based on partial derivatives of the utility function that are not covered by the usual fundamental axioms on preferences.

Remark: If we consider both wealth inequality and preference heterogeneity, Propositions 4 and 5 imply that when the individual absolute risk tolerance $\rho(x_i; \sigma_i)$ is jointly convex (or concave) with respect to $(x_i; \sigma_i)$,

the effects on macroeconomic volatility derived from a joint increase of inequality and heterogeneity are mutually amplified. On the contrary, if the curvatures of $\rho(x_i; \sigma_i)$ with respect to x_i and σ_i are opposite, the intuition suggests that the final effects on macroeconomic volatility will depend on the dominant curvature.

7 Some comments on global dynamics

Propositions 1 to 5 provide local stability results. Benhabib and Nishimura [3] consider a representative-agent two-sector optimal growth model and provide global results. Their main conclusions depend on the sign of the cross derivative $V_{12}(k_t, k_{t+1})$. Assuming a linear utility function, they prove that if $b(k_t, y_t) \in (-1/(1-\mu), 0)$, i.e. $V_{12}(k_t, k_{t+1}) < 0$, over the interior of the set of admissible capital stocks \mathcal{D} , then for any initial capital stock k_0 , the optimal path either converges to a steady state or to a period-two cycle (see Theorem 3 p. 296).¹⁴ A similar result is more difficult to obtain in our framework. We have

$$V_{12}(k_t, k_{t+1}) = u''(x_t) \left\{ T_2(k_t, y_t) [T_1(k_t, y_t) - (1-\mu)T_2(k_t, y_t)] \right. \\ \left. + \rho(x_t) T_{11}(k_t, y_t) b(k_t, y_t) [1 + (1-\mu)b(k_t, y_t)] \right\} \quad (21)$$

with $y_t = k_{t+1} - (1-\mu)k_t$. Even if we assume that $b(k_t, y_t) \in (-1/(1-\mu), 0)$ over the interior of \mathcal{D} , the sign of $V_{12}(k_t, k_{t+1})$ may change from negative to positive when $\rho(x_t)$ increases. It is therefore necessary to introduce an additional condition to get global conclusions:

Proposition 6. *Under Assumptions 1 and 2, let the consumption good be capital intensive over the interior of the set \mathcal{D} with $b(k_t, y_t) \in (-1/(1-\mu), 0)$. If the following condition holds*

$$\rho(x_t) \neq -\frac{T_2(k_t, y_t)[T_1(k_t, y_t) - (1-\mu)T_2(k_t, y_t)]}{T_{11}(k_t, y_t)b(k_t, y_t)[1 + (1-\mu)b(k_t, y_t)]} \quad (22)$$

then from any initial capital stock k_0 , the optimal path converges either to the steady state k^ with oscillations or to a cycle of period two.*

If condition (22) does not hold, we may expect the existence of more complex optimal paths. We know indeed since Boldrin and Montrucchio [6]

¹⁴The occurrence of period-two cycles is obtained if there exists a bifurcation value $\delta^* \in (0, 1)$ such that $b(k^*, \mu k^*) \in (-1/(2-\mu), -\delta^*/(1+\delta^*(1-\mu)))$.

that period-three cycles and thus chaotic optimal paths may occur in two-sector models if the sign of $V_{12}(k_t, k_{t+1})$ is not constant.¹⁵ In our framework, if $b(k_t, y_t) \in (-1/(2 - \mu), -\delta/(1 + \delta(1 - \mu)))$ and the absolute risk tolerance $\rho(x)$ increases sufficiently, the sign of $V_{12}(k_t, k_{t+1})$ reverses at least one time and we may conjecture the existence of a cascade of period-doubling bifurcations leading to a-periodic and chaotic paths. Of course, such an assertion needs to be proved. But it suggests a possible stronger relationship between wealth inequality, preference heterogeneity and macroeconomic volatility.

8 An example

Following Nishimura *et al.* [26], we consider a two-sector economy with CES technologies such that

$$y_0 = \left(\alpha_0 l_0^{1-1/\gamma} + \alpha_1 k_0^{1-1/\gamma} \right)^{\gamma/(\gamma-1)}, \quad y_1 = \left(\beta_0 l_1^{1-1/\gamma} + \beta_1 k_1^{1-1/\gamma} \right)^{\gamma/(\gamma-1)}$$

with $\alpha_0 + \alpha_1 = \beta_0 + \beta_1 = 1$. Both sectors are characterized by the same elasticity of capital-labor substitution $\gamma > 0$. We also assume that the capital stock fully depreciates within one period, i.e. $\mu = 1$.

Assumption 7. $\gamma > \left(1 + \frac{\ln \beta_1}{\ln \delta} \right)^{-1} \equiv \hat{\gamma} \in (0, 1)$

This restriction is quite standard with CES technologies. It is well-known indeed that when the elasticity of capital/labor substitution is less than 1, the Inada conditions are not satisfied and corner solutions cannot be a priori ruled out. Throughout this section we will therefore assume that $\gamma > \hat{\gamma}$.

Under Assumption 7, existence and uniqueness of the steady state in capital k^* and consumption x^* are easily derived. Moreover, it is straightforward to show that the investment (consumption) good sector is capital intensive if and only if $\beta_1/\beta_0 > (<) \alpha_1/\alpha_0$.¹⁶

8.1 The effects of wealth inequality

We introduce a specification of preferences which includes several special cases such as the CRRA formulation. We assume that agents have the same

¹⁵See also Sorger [30, 31], Mitra [22], Nishimura and Yano [27], Mitra and Sorger [24].

¹⁶The proof of these facts is given in Appendix 10.10.

preferences and as in Gollier [17], we consider that the utility function of agent i is such that its first derivative satisfies

$$u'(x_i) = \exp\left(-r \frac{x_i^{1-\sigma}}{1-\sigma}\right)$$

with $r, \sigma > 0$.¹⁷ The individual absolute risk tolerance is then

$$\rho(x_i) = \frac{x_i^\sigma}{r} \quad (23)$$

It follows that r measures the degree of relative risk aversion evaluated at consumption level $x_i = 1$. The empirical evidence favors a value of risk aversion close to 2, so following Gollier [17] we let $r = 2$. Moreover, the absolute risk tolerance is strictly concave if $\sigma \in (0, 1)$ and strictly convex if $\sigma > 1$. Notice that the CRRA formulation is obtained when $\sigma = 1$.

In this framework, we start considering the case in which the distribution of initial shares and/or labor endowments does not affect stability. In a CES economy, Proposition 1 allows us to get a global stability result:

Corollary 4. *Under Assumption 7, if $\beta_1/\beta_0 > \alpha_1/\alpha_0$, then k^* is saddle-point stable with monotone convergence for any $\rho(x^*) > 0$.*

In order to show that the local stability of the steady state is modified when the degree of wealth inequality is increased, we provide illustrations for each of the configurations identified in the previous sections. We thus have to define the interval of admissible values $(\underline{\rho}, \bar{\rho})$ for the absolute risk tolerance. As shown in Section 4, we get from (17)

$$\hat{\rho}(x^*) = \frac{n}{2} \left(\frac{x^*}{n}\right)^\sigma, \quad \tilde{\rho}(x^*) = \frac{x^{*\sigma}}{2}$$

If $\sigma > 1$, $\rho(x^*)$ increases with the level of inequality so that $\underline{\rho} = \hat{\rho}(x^*)$ and $\bar{\rho} = \tilde{\rho}(x^*)$. On the contrary, if $\sigma < 1$, $\rho(x^*)$ decreases with the level of inequality so that $\underline{\rho} = \tilde{\rho}(x^*)$ and $\bar{\rho} = \hat{\rho}(x^*)$.

In the following results, we define an economy as a 4-uple of parameters $(\alpha_1, \beta_1, \gamma, \sigma)$. The next Corollary deals with the correlation between wealth inequality and the existence of damped oscillations.¹⁸

¹⁷If $\sigma \in (0, 1)$, the Inada condition in Assumption 6 does not hold as $\lim_{x_i \rightarrow 0} u_1(x_i; \sigma_i) = 1$, and corner solutions with $x_{it} = 0$ for some $t > 0$ cannot be a priori ruled out. However, as shown in Lemma 2 and Appendix 10.10, the individual consumption levels at the steady state x_i^* do not depend on preferences and, under Assumption 1, are interior as long as $\lim_{x_i \rightarrow 0} u_1(x_i; \sigma_i) \neq 0$. It follows that for any $\sigma > 0$, $x_{it} \neq 0$ in the neighborhood of x_i^* .

¹⁸In this case $b \in (-\infty, -1] \cup [-\delta, 0)$, i.e. $(\alpha_1, \beta_1, \gamma)$ satisfy Assumption 7, $\beta_1/\beta_0 < \alpha_1/\alpha_0$ and $\alpha_1\beta_0/(\alpha_0\beta_1) \in (0, (1 + (\delta^{\gamma-1}/\gamma\beta_1)^{-\gamma})^{1/\gamma}] \cup [(1 + (\delta\beta_1)^{-\gamma})^{1/\gamma}, +\infty)$.

Corollary 5. *i) There exist a non-empty set of values of n and an open set of economies $(\alpha_1, \beta_1, \gamma, \sigma)$ with $\sigma > 1$, for which a sufficiently high (low) level of wealth inequality leads to the existence of damped fluctuations (monotone convergence) in the neighborhood of the steady state.*

ii) There exist a non-empty set of values of n and an open set of economies $(\alpha_1, \beta_1, \gamma, \sigma)$ with $\sigma < 1$, for which a sufficiently low (high) level of wealth inequality leads to the existence of damped fluctuations (monotone convergence) in the neighborhood of the steady state.

The following Corollary deals with the correlation between wealth inequality and the existence of persistent fluctuations.¹⁹

Corollary 6. *i) There exist a non-empty set of values of n and an open set of economies $(\alpha_1, \beta_1, \gamma, \sigma)$ with $\sigma > 1$, for which a sufficiently high level of wealth inequality leads to the existence of period-two cycles in the neighborhood of the steady state.*

ii) There exist a non-empty set of values of n and an open set of economies $(\alpha_1, \beta_1, \gamma, \sigma)$ with $\sigma < 1$, for which a sufficiently low level of wealth inequality leads to the existence of period-two cycles in the neighborhood of the steady state.

Persistent fluctuations, i.e. the occurrence of saddle-point stable period-two cycles, is obtained if the flip bifurcation is super-critical, i.e. if Assumption 5 holds. In such a case, and considering for instance i) in Corollary 6, a low level of wealth inequality is associated to the existence of damped fluctuations in the neighborhood of the steady state, and an increase of wealth inequality will lead to the occurrence of persistent fluctuations. As already mentioned previously, Assumption 5 corresponds to non-trivial conditions on the non-linear part of the Euler equation. However, a number of robust examples provided in the literature on optimal growth show that these conditions are usually satisfied.²⁰

¹⁹In this case $b \in (-1, -\delta)$, i.e. $(\alpha_1, \beta_1, \gamma)$ satisfy Assumption 7, $\beta_1/\beta_0 < \alpha_1/\alpha_0$ with $\alpha_1\beta_0/(\alpha_0\beta_1) \in ([1 + (\delta^{(\gamma-1)/\gamma}\beta_1)^{-\gamma}]^{1/\gamma}, [1 + (\delta\beta_1)^{-\gamma}]^{1/\gamma})$.

²⁰See for instance Boldrin and Deneckere [5] and Mitra and Nishimura [23].

8.2 The effects of preference heterogeneity

As in Section 8.1, we assume that the first derivative of the utility function of agent i satisfies

$$u_1(x_i; \sigma_i) = \exp\left(-\frac{x_i^{1-\sigma_i}}{2(1-\sigma_i)}\right)$$

with $\sigma_i > 0$. Considering the configuration with wealth equality in which $\omega_i = \theta_i = 1/n$ and $x_i^* = x^*/n$, the individual absolute risk tolerance at the steady state is then

$$\rho(x^*/n; \sigma_i) = \frac{1}{2} \left(\frac{x^*}{n}\right)^{\sigma_i} \quad (24)$$

It follows that $\rho(x^*/n; \sigma_i)$ is a convex function of σ_i since

$$\rho_{22}(x^*/n; \sigma_i) = (\ln(x^*/n))^2 \frac{(x^*/n)^{\sigma_i}}{2} > 0$$

As shown in Section 6, and considering that the individual characteristic is distributed over a segment $[0, nm_\sigma]$ with m_σ the mean of the distribution, we derive from (20) the interval of admissible values for $\rho(x^*)$:

$$\underline{\rho} = \hat{\rho}(x^*) = \frac{n}{2} \left(\frac{x^*}{n}\right)^{m_\sigma}, \quad \bar{\rho} = \tilde{\rho}(x^*) = \frac{n-1}{2} + \frac{1}{2} \left(\frac{x^*}{n}\right)^{nm_\sigma} \quad (25)$$

We finally define an economy as a 4-uple $(\alpha_1, \beta_1, \gamma, m_\sigma)$ and we get the following clear-cut result:

Corollary 7. *i) There exist a non-empty set of values of n and an open set of economies $(\alpha_1, \beta_1, \gamma, m_\sigma)$ for which a sufficiently high (low) level of wealth inequality leads to the existence of damped fluctuations (monotone convergence) in the neighborhood of the steady state.*

ii) There exist a non-empty set of values of n and an open set of economies $(\alpha_1, \beta_1, \gamma, m_\sigma)$ for which a sufficiently high level of wealth inequality leads to the existence of period-two cycles in the neighborhood of the steady state.

We may finally consider both wealth inequality and preference heterogeneity. We have just proved that, when agents have homogeneous initial wealth, preference heterogeneity favors the occurrence of macroeconomic volatility. If $\sigma > 1$, Section 8.1 indicates that when agents have homogeneous preferences wealth inequality favors macroeconomic volatility. The intuition would then suggest that both effects are mutually amplified when wealth inequality and preference heterogeneity are considered jointly. However it is easy to derive that the individual absolute risk tolerance $\rho(x_i; \sigma_i)$ is not jointly convex with respect to $(x_i; \sigma_i)$ since its Hessian matrix has a negative determinant when $\sigma_i > 1$.

9 Conclusion

The present paper identifies the conditions on technology and consumer's preferences such that wealth inequality and preference heterogeneity are positively or negatively correlated with macroeconomic volatility. Assuming only wealth inequality, we first show that when the consumption good is capital intensive at the steady state, the impact of wealth inequality depends on the concavity of the absolute risk tolerance. Assuming on the contrary that only preference heterogeneity occurs, the same kind of conclusions are obtained even with preferences belonging to the HARA class. Using a CES utility function, we show that more heterogeneity implies the existence of macroeconomic volatility if and only if the elasticity of intertemporal substitution in consumption is larger than one.

The properties of the absolute risk tolerance are difficult to obtain, except when preferences are analytically specified as for the HARA class. But they also play a crucial role in asset pricing theory and some effort is being devoted to find direct empirical evidence and indirect, model dependent, evidence. These findings, as well as indirect evidence obtained from the collective household model, suggest that the absolute risk tolerance is not linear. According to our results, this evidence confirms that wealth inequality and agent's heterogeneity matter in the existence of fluctuations.

It would be interesting to generalize the present analysis to endogenous growth models with multiple balanced growth rates. In this case it would be possible to relate differences in the growth rates across countries to the level of inequality. Indeed, in endogenous growth models characterized by several balanced growth rates, wealth and income inequality may affect the stability of these and therefore affect the effective growth experience. However, the main difficulty comes from the fact that endogenous growth models may lack the required continuity property of the welfare weights.

10 Appendix

10.1 Proof of Lemma 2

The first order conditions corresponding to the individual program are

$$\begin{aligned}\delta^t u'_i(x_{it}) &= \pi_i R_t \quad \forall t \geq 0 \text{ and } i = 1, \dots, n \\ \sum_{t=0}^{\infty} R_t x_{it} &= \sum_{t=0}^{\infty} R_t w_t \omega_i + \theta_i r_0 k_0\end{aligned}$$

where π_i is the Lagrange multiplier associated with the intertemporal budget constraint. From (2) and (7) we conclude that the interest rate satisfies

$$1 + d_t = \frac{r_t + (1-\mu)p_t}{p_{t-1}} = -\frac{T_1(k_t, y_t) - (1-\mu)T_2(k_t, y_t)}{T_2(k_{t-1}, y_{t-1})}$$

for any $t \geq 1$ and $1 + d_0 = r_0/p_{-1}$ for $t = 0$. The Euler equation (7) evaluated at a steady state $x_{it} = x_i^*$ gives $1 + d^* = \delta^{-1}$ and thus $R_t = \delta^t$. Recall that from (2) we also get $T_1^* = r^*$, $T_2^* = -p^*$ and $w^* = x^* - r^*k^* + p^*\mu k^*$. The budget constraint evaluated along the stationary path with $k_t = k^*$ for all $t \geq 0$ and $p_{-1} = p^*$ becomes

$$x_i^* = w^*\omega_i + (1 - \delta)\theta_i \frac{p^*}{\delta} k^*$$

with $w^* = x^* - (1 - \delta)\vartheta T_1^* k^* > 0$, $p^* = \delta\vartheta T_1^*$ and the result follows. \square

10.2 Proof of lemma 3

In a one-sector economy with heterogeneous agents, Kehoe, Levine and Romer [20] show that the welfare weights are continuous functions of the initial capital stock. This continuity property happens to be satisfied because the value function of the planner's problem (5) is C^2 . However, in a multisector economy such a property is much more difficult to obtain. Santos [29] shows that one of the sufficient conditions is to assume strong concavity for the indirect utility function $V(k_t, k_{t+1})$ (see Assumption B and Theorem 2.2 in Santos [29]). Strong concavity implies that the Hessian matrix of $V(k_t, k_{t+1})$ is always non-singular and negative-definite. In other words, the smallest eigenvalues in absolute value of the Hessian matrix needs to be strictly positive over the domain of definition of $V(k_t, k_{t+1})$. In our two-sector model, recall that the indirect utility function

$$V(k_t, k_{t+1}) = u(T(k_t, k_{t+1}) - (1 - \mu)k_t)$$

is defined over the compact, convex set \mathcal{D} . We know that T is concave non-strictly so that its Hessian matrix is singular, which means $|H_T| = T_{11}T_{22} - T_{12}^2 = 0$ for any $(k_t, k_{t+1}) \in \mathcal{D}$.²¹ The Hessian matrix of V is then

$$\begin{aligned} H_V(k_t, k_{t+1}) &= u' \begin{pmatrix} 1 & -(1-\mu) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -(1-\mu) & 1 \end{pmatrix} \\ &+ u'' \begin{pmatrix} T_1 - (1-\mu)T_2 \\ T_2 \end{pmatrix} \begin{pmatrix} T_1 - (1-\mu)T_2 & T_2 \end{pmatrix} \end{aligned}$$

²¹See for instance Benhabib and Nishimura [4].

Considering (10) and (12), the determinant of $H_V(k_t, k_{t+1})$ is

$$\begin{aligned}
|H_V| &= (u')^2 |H_T| + (u'')^2 \left| \begin{pmatrix} T_1 - (1-\mu)T_2 & \\ & T_2 \end{pmatrix} \begin{pmatrix} T_1 - (1-\mu)T_2 & T_2 \\ & \end{pmatrix} \right| \\
&+ u'u'' \begin{pmatrix} -T_2 & T_1 \end{pmatrix} H_T \begin{pmatrix} -T_2 \\ T_1 \end{pmatrix} \\
&= u'u'' T_{11} [T_2^2 + 2bT_1T_2 + b^2T_1^2] = u'u'' T_{11} (T_2 + bT_1)^2 \geq 0
\end{aligned}$$

Strict concavity of the social utility function $u(x)$ is a necessary condition for the Hessian matrix of V to be non singular. Such a property easily follows from Assumption 2. Moreover, if over the interior of the set \mathcal{D} we have $T_2(k_t, k_{t+1} - (1-\mu)k_t) + b(k_t, k_{t+1} - (1-\mu)k_t)T_1(k_t, k_{t+1} - (1-\mu)k_t) \neq 0$ then $|H_V| > 0$ and the value function of the planner's problem (5) is C^2 . \square

10.3 Proof of Lemma 4

Without loss of generality assume there are three types of consumers. Let n_i be the number of agents of type $i = 1, 2, 3$ with $n_1 + n_2 + n_3 = n$. It is easy to show that all agents of the same type are given the same Pareto weight.²² The social utility function is thus defined by

$$u(x) = \max \eta_1 n_1 u_1(x_1) + \eta_2 n_2 u_2(x_2) + \eta_3 n_3 u_3((x - n_1 x_1 - n_2 x_2)/n_3)$$

with $\eta_i \geq 0$ and $\eta_1 + \eta_2 + \eta_3 = 1$. The first and second order derivatives of the social utility function can be related to the derivatives of the individual utility function of the agents. Indeed, the first order conditions associated with program (3) give

$$\begin{aligned}
\Psi^1(x_1, x_2, x; \eta_1, \eta_2) &= \eta_1 n_1 u_1'(x_1) - \eta_3 n_1 u_3' \left(\frac{x - n_1 x_1 - n_2 x_2}{n_3} \right) = 0 \\
\Psi^2(x_1, x_2, x; \eta_1, \eta_2) &= \eta_2 n_2 u_2'(x_2) - \eta_3 n_2 u_3' \left(\frac{x - n_1 x_1 - n_2 x_2}{n_3} \right) = 0
\end{aligned}$$

Then the following expressions are easily obtained

²²The first order conditions associated with the maximization program (3) that defines the social utility function give $\eta_i = \lambda/u_i'(x_i)$ with $\lambda \geq 0$ the Lagrange multiplier associated with the resources constraint. Consider then two agents j and k , $j \neq k$, of the same type, i.e. such that $u_j = u_k$, $\omega_j = \omega_k$ and $\theta_j = \theta_k$. It follows that x_j and x_k are solutions of two optimizations problems with the same utility function, the same initial resources and the same budget constraint. Therefore we obviously derive $x_j = x_k$ and thus $\eta_j = \eta_k$.

$$u'(x) = \eta_3 u'_3 \left(\frac{x - n_1 x_1 - n_2 x_2}{n_3} \right) = \eta_1 u'_1(x_1) = \eta_2 u'_2(x_2)$$

$$u''(x) = \eta_1 u''_1(x_1) \frac{\partial x_1}{\partial x}$$

where x represents the aggregate consumption. The implicit function theorem applied to Ψ allows us to express x_1 as a function of x . In matrix form we can write

$$\begin{pmatrix} \frac{\partial x_1}{\partial x} \\ \frac{\partial x_2}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial \Psi^1}{\partial x_1} & \frac{\partial \Psi^1}{\partial x_2} \\ \frac{\partial \Psi^2}{\partial x_1} & \frac{\partial \Psi^2}{\partial x_2} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial \Psi^1}{\partial x} \\ \frac{\partial \Psi^2}{\partial x} \end{pmatrix}$$

Some straightforward computations give

$$\frac{\partial x_1}{\partial x} = \frac{\eta_2 \eta_3 u''_2(x_2) u''_3(x_3)}{\eta_1 \eta_2 n_3 u''_1(x_1) u''_2(x_2) + \eta_1 \eta_3 n_2 u''_1(x_1) u''_3(x_3) + \eta_2 \eta_3 n_1 u''_2(x_2) u''_3(x_3)}$$

Expression (16) then follows from the definition of $\rho(x)$. The continuity property of $\rho(x)$ follows from the fact that when condition (15) holds, the optimal consumption at time t , x_t , is a continuous function of the initial capital stock and labor endowments.²³ \square

10.4 Proof of Proposition 1

From the characteristic polynomial we derive the discriminant

$$\Delta = \left\{ \delta \vartheta^2 (1 + \sqrt{\delta})^2 + \rho(x^*) \frac{\varepsilon_{rk}}{\varepsilon_{ck} x^*} \left[b + \sqrt{\delta} [1 + (1 - \mu)b] \right]^2 \right\} \\ \times \left\{ \delta \vartheta^2 (1 - \sqrt{\delta})^2 + \rho(x^*) \frac{\varepsilon_{rk}}{\varepsilon_{ck} x^*} \left[b - \sqrt{\delta} [1 + (1 - \mu)b] \right]^2 \right\} \geq 0$$

Therefore the characteristic roots are real. Moreover we get

$$\mathcal{P}(0) = \delta \vartheta^2 + \rho(x^*) \frac{\varepsilon_{rk}}{\varepsilon_{ck} x^*} b [1 + (1 - \mu)b]$$

$$\mathcal{P}(1) = -\rho(x^*) \frac{\varepsilon_{rk}}{\varepsilon_{ck} x^*} (1 - \mu b) (\delta - \vartheta^{-1} b)$$

$$\mathcal{P}(-1) = 2\delta(1 + \delta)\vartheta^2 + \rho(x^*) \frac{\varepsilon_{rk}}{\varepsilon_{ck} x^*} [1 + (2 - \mu)b] \left[\delta + [1 + (1 - \mu)\delta]b \right]$$

The equality between prices and costs at the steady state gives

$$\begin{pmatrix} w & r \end{pmatrix} \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = \begin{pmatrix} 1 & p \end{pmatrix}$$

and thus $wa_{01} + ra_{11} = p$. When evaluated at the steady state, the Euler equation (7) rewrites as $p = \delta \vartheta r$. We then obtain after substitution in the previous equation

$$\delta r (\vartheta - \delta^{-1} a_{11}) = wa_{01} > 0$$

²³A similar result has been obtained by Wilson [34].

Prices positivity implies $\vartheta - \delta^{-1}a_{11} > 0$. From equation (11), we get

$$\delta - \vartheta^{-1}b = \frac{a_{00}(\delta - \vartheta^{-1}a_{11}) + \vartheta^{-1}a_{10}a_{01}}{a_{00}} > 0$$

Then we have $b < \delta\vartheta$ which entails $b < 1/\mu$ and thus $\mathcal{P}(1) < 0$ for any $\rho(x^*) > 0$. Moreover we get $\mathcal{P}(0) > 0$ for any $\rho(x^*) > 0$ if and only if $b > 0$ or $b \in (-\infty, -1/(1 - \mu)]$. In these cases, we have $\mathcal{P}(1)\mathcal{P}(0) < 0$ so that there exists one characteristic root into $(0, 1)$ while the other is greater than 1 since the product of characteristic roots is equal to $1/\delta$. Therefore the optimal path monotonically converges to the steady state. \square

10.5 Proof of Proposition 2

Recall that $\mathcal{P}(1) < 0$ for any $\rho(x^*) > 0$. When $b > -1/(1 - \mu)$, $\mathcal{P}(0) < 0$ if and only if $\rho(x^*) > \rho_c$. Moreover, $\mathcal{P}(-1) > 0$ for any $\rho(x^*) > 0$ when $b \in (-\infty, -1/(2 - \mu)] \cup [-\delta/(1 + \delta(1 - \mu)), 0)$. Since $-1/(1 - \mu) < -1/(2 - \mu)$, let us assume that $b \in (-1/(1 - \mu), -1/(2 - \mu)] \cup [-\delta/(1 + \delta(1 - \mu)), 0)$. We conclude that when $\rho(x^*) > \rho_c$, $\mathcal{P}(0)\mathcal{P}(-1) < 0$ so that there exists one characteristic root into $(-1, 0)$ while the other is less than -1 . It follows that the optimal path converges to the steady state with oscillations. On the contrary, when $\rho(x^*) < \rho_c$, $\mathcal{P}(0)\mathcal{P}(1) < 0$ so that there exists one characteristic root into $(0, 1)$ while the other is greater than 1. It follows that the optimal path monotonically converges to the steady state. \square

10.6 Proof of Proposition 3

When $b \in (-1/(2 - \mu), -\delta/(1 + \delta(1 - \mu)))$, we have already proved that $\mathcal{P}(0) < 0$ if and only if $\rho(x^*) > \rho_c$. Moreover, $\mathcal{P}(-1) > 0$ if and only if $\rho(x^*) < \rho_f$. Straightforward computations clearly show that since $b \in (-1/(2 - \mu), -\delta/(1 + \delta(1 - \mu)))$, we have $\rho_c < \rho_f$. We then get:

If $\rho(x^*) > \rho_f$, the steady state is locally unstable with oscillations.

If $\rho(x^*) \in (\rho_c, \rho_f)$, the steady state is saddle-point stable with oscillating convergence. When $\rho(x^*)$ crosses ρ_f , a flip bifurcation occurs since one characteristic root crosses -1 with positive speed, i.e.

$$\frac{\partial \mathcal{P}(-1)}{\partial \rho(x^*)} = \frac{\varepsilon_{rk}}{\varepsilon_{ck}x^*} [1 + (2 - \mu)b] \left[\delta + [1 + (1 - \mu)\delta]b \right] < 0$$

Finally, if $\rho(x^*) < \rho_c$, the steady state is saddle-point stable with monotone convergence. \square

10.7 Proof of Proposition 4

Let $u_i(x) = v(x)$, $i = 1, \dots, N$ be the subscript indicating the type of agents and N_i be the number of agents of type i with $\sum_{i=1}^N N_i = n$. Lemma 4 gives

$$\rho(x^*) = - \sum_{i=1}^N N_i \frac{v'(x_i^*)}{v''(x_i^*)}$$

Recalling that the individual wealth is given by $w_i = x_i/(1 - \delta)$, let $f(x_i) = f(x_i/(1 - \delta)) \equiv \rho(x_i)$. If $\rho(x_i) = -v'(x_i)/v''(x_i)$ is a convex function, Definition 2 implies that B is more unequal than A if and only if $\sum_{i=1}^N N_i(A)\rho(x_i) < \sum_{i=1}^N N_i(B)\rho(x_i)$. If we define $\rho(J)$ as the value of $\rho(x^*)$ associated with the distribution $\mathcal{N}(J)$, the previous condition becomes $\rho(A) < \rho(B)$. On the other hand, according to Propositions 2 and 3, an increase in ρ leads to the existence of endogenous fluctuations. Therefore, when the individual absolute risk tolerance $\rho(x)$ is a convex function, wealth inequality is positively correlated with macroeconomic volatility. The different subcases are derived from Propositions 2 and 3. The second part of the result is proved similarly. \square

10.8 Proof of Corollary 3

From the utility function we derive

$$\rho(x_i; \sigma_i) = \frac{x_i}{1-\sigma_i} + \frac{e}{a} \text{ and } \rho_{22}(x_i; \sigma_i) = \frac{2x_i}{(1-\sigma_i)^3}$$

Then $\rho(x_i; \sigma_i)$ is a convex function of σ_i if and only if $\sigma_i \in (0, 1)$. \square

10.9 Proof of Proposition 6

The result easily follows from the expression of $V_{12}(k_t, k_{t+1})$ given by (21) and Theorems 2 and 3 in Benhabib and Nishimura ([3], p. 293 and 296). \square

10.10 Computations for the CES example

As shown in Nishimura *et al.* [26], the steady state is given by:

$$k^* = \frac{\left(\frac{\alpha_1\beta_0}{\alpha_0\beta_1}\right)^\gamma \left(\frac{(\delta\beta_1)^{1-\gamma}-\beta_1}{\beta_0}\right)^{\frac{\gamma}{1-\gamma}}}{1-(\delta\beta_1)^\gamma \left(1-\left(\frac{\alpha_1\beta_0}{\alpha_0\beta_1}\right)^\gamma\right)}$$

$$x^* = \frac{[1-(\delta\beta_1)^\gamma] \left(\frac{\alpha_1\beta_0}{\alpha_0\beta_1}\right)^\gamma}{1-(\delta\beta_1)^\gamma \left(1-\left(\frac{\alpha_1\beta_0}{\alpha_0\beta_1}\right)^\gamma\right)} \left[\alpha_0 \left(\frac{\alpha_1\beta_0}{\alpha_0\beta_1}\right)^{1-\gamma} + \alpha_1 \frac{\beta_0}{(\delta\beta_1)^{1-\gamma}-\beta_1} \right]^{\frac{\gamma}{\gamma-1}}$$

since Assumption 7 implies $(\delta\beta_1)^{1-\gamma} > \beta_1$. Lemma 2 gives $x_i^*(\theta_i, \omega_i) = w^*\omega_i + (1-\delta)T_1^*k^*\theta_i > 0$ with $w^* = x^* - (1-\delta)T_1^*k^* > 0$ and

$$T_1^* = \alpha_1 \left[\alpha_0 \left(\frac{\alpha_1\beta_0}{\alpha_0\beta_1}\right)^{1-\gamma} \frac{(\delta\beta_1)^{1-\gamma}-\beta_1}{\beta_0} + \alpha_1 \right]^{\frac{1}{\gamma-1}}$$

At the steady state we get the capital intensity difference across sectors

$$b = (\delta\beta_1)^\gamma \left[1 - \left(\frac{\alpha_1\beta_0}{\alpha_0\beta_1}\right)^\gamma \right]$$

and the ratio of elasticities

$$\frac{\varepsilon_{ck}}{\varepsilon_{rk}} = \frac{\alpha_1\beta_0}{\alpha_0} \frac{\gamma((\delta\beta_1)^{1-\gamma}-\beta_1)b}{(\delta\beta_1)^{1-\gamma} \left(\frac{\alpha_1\beta_0}{\alpha_0\beta_1}\right)^{\gamma-1} [1-(\delta\beta_1)^\gamma][(\delta\beta_1)^{1-\gamma}-\beta_1]}$$

10.11 Proof of Corollary 5

We proceed numerically by finding parameters' values which satisfy

$$\alpha_1\beta_0/(\alpha_0\beta_1) \in (0, (1 + (\delta^{(\gamma-1)/\gamma}\beta_1)^{-\gamma})^{1/\gamma}] \cup [(1 + (\delta\beta_1)^{-\gamma})^{1/\gamma}, +\infty) \quad (26)$$

so that $\rho_f < 0$, and $\rho_c \in (\underline{\rho}, \bar{\rho})$.

i) Let $\alpha_1 = 0.45$, $\beta_1 = 0.2$, $\delta = 0.6$, $\gamma = 1.66$, $n = 25$ and $\sigma = 1.1$. Then, using the expressions given in Appendix 10.10, we find that $b \approx -0.18$, i.e. (26) is satisfied, $x^* \approx 0.4$, $\varepsilon_{ck}/\varepsilon_{rk} \approx 0.13$, $\rho_c \approx 0.1758$, $\underline{\rho} = \hat{\rho}(x^*) \approx 0.1338$ and $\bar{\rho} = \tilde{\rho}(x^*) \approx 0.1846$. The absolute risk tolerance being convex, Propositions 2 and 4 imply that a sufficiently high (low) level of wealth inequality leads to a value of $\rho(x^*)$ larger (lower) than ρ_c and thus to the existence of damped fluctuations (monotone convergence). By continuity there exists an open set of parameters' values close to the previous values such that the same result holds.

ii) Let $\alpha_1 = 0.45$, $\beta_1 = 0.2$, $\delta = 0.6$, $\gamma = 1.538$, $n = 25$ and $\sigma = 0.95$. Similarly, we find that $b \approx -0.1991$, i.e. (26) is satisfied, $x^* \approx 0.398$, $\varepsilon_{ck}/\varepsilon_{rk} \approx 0.191$, $\rho_c \approx 0.229$, $\underline{\rho} = \tilde{\rho}(x^*) \approx 0.2083$ and $\bar{\rho} = \hat{\rho}(x^*) \approx 0.2447$. The absolute risk tolerance being concave, Propositions 2 and 4 imply that a sufficiently low (high) level of wealth inequality leads to a value of $\rho(x^*)$

larger (lower) than ρ_c and thus to the existence of damped fluctuations (monotone convergence). By continuity there exists an open set of parameters' values close to the previous values such that the same result holds. \square

10.12 Proof of Corollary 6

We proceed numerically by finding parameters' values which satisfy

$$\alpha_1\beta_0/(\alpha_0\beta_1) \in ([1 + (\delta^{(\gamma-1)/\gamma}\beta_1)^{-\gamma}]^{1/\gamma}, [1 + (\delta\beta_1)^{-\gamma}]^{1/\gamma}) \quad (27)$$

with $\rho_f \in (\underline{\rho}, \bar{\rho})$ and $\rho_c \notin (\underline{\rho}, \bar{\rho})$.

i) Let $\alpha_1 = 0.85$, $\beta_1 = 0.2$, $\delta = 0.1$, $\gamma = 1.49$, $n = 25$ and $\sigma = 1.1$. Then, using the expressions given in Appendix 10.10, we find that $b \approx -0.3$, i.e. (27) is satisfied, $x^* \approx 0.1834$, $\varepsilon_{ck}/\varepsilon_{rk} \approx 0.22$, $\rho_c \approx 0.0133$, $\rho_f \approx 0.0627$, $\underline{\rho} = \hat{\rho}(x^*) \approx 0.0561$ and $\bar{\rho} = \tilde{\rho}(x^*) \approx 0.0774$. The absolute risk tolerance being convex, Propositions 3 and 4 imply that increasing the level of wealth inequality leads to increasing values of $\rho(x^*)$ that will cross the flip bifurcation value ρ_f , and thus imply the existence of period-two cycles. By continuity there exists an open set of parameters' values close to the previous values such that the same result holds.

ii) Let $\alpha_1 = 0.85$, $\beta_1 = 0.2$, $\delta = 0.1$, $\gamma = 1.38$, $n = 25$ and $\sigma = 0.95$. Then, using the expressions given in Appendix 10.10, we find that $b \approx -0.329$, i.e. (27) is satisfied, $x^* \approx 0.182$, $\varepsilon_{ck}/\varepsilon_{rk} \approx 0.435$, $\rho_c \approx 0.024$, $\rho_f \approx 0.1134$, $\underline{\rho} = \tilde{\rho}(x^*) \approx 0.099$ and $\bar{\rho} = \hat{\rho}(x^*) \approx 0.1163$. The absolute risk tolerance being concave, Propositions 3 and 4 imply that decreasing the level of wealth inequality leads to increasing values of $\rho(x^*)$ that will cross the flip bifurcation value ρ_f , and thus imply the existence of period-two cycles. By continuity there exists an open set of parameters' values close to the previous values such that the same result holds. \square

10.13 Proof of Corollary 7

i) We proceed as in the proof of Corollary 5i). Let $\alpha_1 = 0.45$, $\beta_1 = 0.2$, $\delta = 0.6$, $\gamma = 1.66$, $n = 25$ and $m_\sigma = 1.4$. Then, we find the same values for b , x^* , $\varepsilon_{ck}/\varepsilon_{rk}$ and ρ_c , together with $\underline{\rho} = \hat{\rho}(x^*) \approx 0.0388$ and $\bar{\rho} = \tilde{\rho}(x^*) \approx 12$. The absolute risk tolerance being convex with respect to σ , Propositions 2 and 4 imply that a sufficiently high (low) level of heterogeneity leads to a

value of $\rho(x^*)$ larger (lower) than ρ_c and thus to the existence of damped fluctuations (monotone convergence).

ii) We proceed as in the proof of Corollary 6i). Let $\alpha_1 = 0.85$, $\beta_1 = 0.2$, $\delta = 0.1$, $\gamma = 1.49$, $n = 20$ and $m_\sigma = 1.4$. Then, we find the same values for b , x^* , $\varepsilon_{ck}/\varepsilon_{rk}$, ρ_c and ρ_f , together with $\underline{\rho} = \hat{\rho}(x^*) \approx 0.014$ and $\bar{\rho} = \tilde{\rho}(x^*) \approx 9.5$. The absolute risk tolerance being convex, Propositions 3 and 4 imply that increasing the level of heterogeneity leads to increasing values of $\rho(x^*)$ that will cross the flip bifurcation value ρ_f and imply the existence of period-two cycles. \square

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