Indeterminacy with small externalities: the role of non-separable preferences*

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Abstract: *In this paper we consider a Ramsey one-sector model with non-separable homothetic preferences, endogenous labor and productive external effects arising from average capital and labor. We show that indeterminacy cannot arise when there are only capital externalities but that it does when there are only labor external effects. We prove that sunspot fluctuations are fully consistent with small market imperfections and realistic calibrations for the elasticity of capital-labor substitution (including the Cobb-Douglas specification) provided the elasticity of intertemporal substitution in consumption and the elasticity of the labor supply are large enough.*

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1 Introduction

In this paper we consider a Ramsey model with non-separable homothetic preferences, endogenous labor and productive external effects arising from average capital and labor. We show that indeterminacy cannot occur when there are only capital externalities but that it does when there are only labor external effects. We prove that sunspot fluctuations are fully consistent with small market imperfections and realistic calibrations for the elasticity of capital-labor substitution (including the Cobb-Douglas specification) provided the elasticity of intertemporal substitution in consumption and the elasticity of the labor supply are large enough.

Since the seminal contribution of Benhabib and Farmer (1994), the Ramsey (1928) one-sector growth model augmented to include endogenous labor supply and external effects has become a standard framework for the analysis of local indeterminacy and expectations-driven fluctuations based on the existence of sunspot equilibria.[1] Considering additively-separable preferences and a Cobb-Douglas technology with output externalities,[2] Benhabib and Farmer (1994) show the existence of local indeterminacy if the external effects are large enough to generate an increasing aggregate labor demand function with respect to wage. This conclusion has been widely criticized from the fact that strong increasing returns and a positively sloped aggregate labor demand function cannot be supported empirically. Moreover, Hintermaier (2003) proves that if the aggregate returns on capital and labor are restricted to be less than one, there is no concave separable utility function

[1] See the recent survey of Benhabib and Farmer (1999).

[2] As shown in Boldrin and Rustichini (1994), positive capital externalities alone do not provide any mechanism for the occurrence of a continuum of equilibria when inelastic labor is considered (see also Kehoe (1991)).

compatible with local indeterminacy when the technology is Cobb-Douglas. More recently, considering instead factor-specific external effects and general formulations for additively-separable preferences and technology, Pintus (2006) relaxes the conditions of Benhabib and Farmer. He shows that local indeterminacy arises under small labor externalities (i.e. a decreasing aggregate labor demand function) provided that the elasticity of capital-labor substitution is sufficiently larger than one, the elasticity of intertemporal substitution in consumption and the elasticity of the labor supply are large enough. The Cobb-Douglas specification for the technology is thus ruled out.

By assuming non-separable preferences, Bennett and Farmer (2000) look for conditions that make local indeterminacy consistent with a Cobb-Douglas technology augmented to include small output externalities. They consider a particular formulation for the utility function, as specified in King *et al.* (1988), which encompasses the additively-separable formulation of Benhabib and Farmer (1994). However, Hintermaier (2001, 2003) proves that when the technology is Cobb-Douglas with small output external effects, the restrictions for the concavity of the KPR utility function precludes the existence of local indeterminacy.[3] This conclusion has been extended to general technologies with factor-specific external effects by Pintus (2004). As a consequence, the existence of local indeterminacy with non-separable preferences appears to be less likely than with additively-separable utility functions.[4]

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[3]The main result of Hintermaier (2003) actually concerns general non-separable preferences which encompass the KPR formulation.

[4]Pelloni and Waldmann (1998) consider a KPR utility function within a one-sector endogenous growth model and show that local indeterminacy is consistent with a decreasing labor supply function. The apparent contradiction with the main result of Hintermaier (2003) is easily explained firstly by the assumption of constant aggregate returns to capital necessary to get endogenous growth, and secondly by the corresponding large amount of

Based on all these contributions, one question still remains open: is it possible to get local indeterminacy with a general technology including small factor-specific externalities when a non-separable utility function is considered ? We provide in this paper a positive answer to this question considering however a crucial simplifying assumption on preferences: we assume that the utility function is homogeneous of degree one with respect to consumption and leisure. This simplification allows to completely characterize preferences in terms of the elasticity of intertemporal substitution in consumption and the share of consumption within total utility.[5]

We first prove that local indeterminacy of equilibria cannot be generated when there are only capital externalities. As a consequence, we concentrate on the focal case where capital externalities are absent and show that local indeterminacy occurs with small external effects if the elasticity of intertemporal substitution in consumption and the elasticity of capital-labor substitution are large enough (the lower bounds for these elasticities tending to infinity as the labor externalities go to zero). Considering extremely low market imperfections then implies that capital and labor are more substitutable than in the usual Cobb-Douglas specification, but local indeterminacy appears to be compatible with standard calibrations for the structural parameters. We prove however that even with a Cobb-Douglas technology, locally indeterminate equilibria may also occur but require slightly larger labor externalities (still compatible with a negative slope of the aggregate labor demand function) and thus a lower elasticity of intertemporal substitution

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factor-specific externalities.

[5]Within the framework of an OLG model, we use in Lloyd-Braga *et al.* (2006) a similar assumption which allows to study the local indeterminacy of equilibria in terms of the share of young agents' consumption over the wage income.

in consumption. A striking result concerns the fact that the occurrence of local indeterminacy does not basically depend on the share of consumption into total utility. For any value of this share within the unit segment, the existence of sunspot fluctuations requires the same kind of restrictions. However, there is a trade-off between the value of the share and the lower bounds for the elasticity of intertemporal substitution in consumption and the elasticity of capital-labor substitution: when a lower share is considered, local indeterminacy occurs with a lower elasticity of intertemporal substitution in consumption but a larger elasticity of capital-labor substitution.

This paper is organized as follows: The next section sets up the basic model. In section 3 we prove the existence of a normalized steady state. Section 4 contains the derivation of the characteristic polynomial and presents the geometrical method used for the local dynamic analysis. In section 5 we present our main results on local indeterminacy with some numerical illustrations. Section 6 contains some concluding comments. All the proofs are gathered in a final appendix.

2 The model

2.1 The production structure

Consider a perfectly competitive economy in which the final output is produced using capital K and labor L. Although production takes place under constant returns to scale, we assume that each of the many firms benefits from positive externalities due to the contributions of the average levels of capital and labor, respectively $\bar{K}$ and $\bar{L}$. Capital external effects are usually interpreted as coming from learning by doing while labor externalities are associated with thick market effects. The production function of a representative firm is thus $AF(K,L)e(\bar{K},\bar{L})$, with $F(K,L)$ homogeneous of degree one, $e(\bar{K},\bar{L})$ increasing in each argument and $A>0$ a scaling parameter. Denoting, for any $L\neq 0$, $x=K/L$ the capital stock per labor unit, we may define the production function in intensive form as $Af(x)e(\bar{K},\bar{L})$.

Assumption 1. *$f(x)$ is $\mathbf{C}^r$ over $\mathbb{R}_+$ for r large enough, increasing ($f'(x)>0$) and concave ($f''(x)<0$) over $\mathbb{R}_{++}$.*

The interest factor R_t and the wage rate w_t then satisfy:

$$R_t = Af'(x_t)e(\bar{K}_t,\bar{L}_t)+1-\mu, \quad w_t = A[f(x_t)-x_tf'(x_t)]e(\bar{K}_t,\bar{L}_t) \tag{1}$$

with $\mu\in[0,1]$ the depreciation rate of capital.

We may also compute the share of capital in total income:

$$s(x)=\frac{xf'(x)}{f(x)}\in(0,1) \tag{2}$$

the elasticity of capital-labor substitution:

$$\sigma(x)=-\frac{(1-s(x))f'(x)}{xf''(x)}>0 \tag{3}$$

and the elasticities of $e(\bar{K}_t,\bar{L}_t)$ with respect to capital and labor:

$$\varepsilon_{eK}(\bar{K},\bar{L})=\frac{e_1(\bar{K},\bar{L})\bar{K}}{e(\bar{K},\bar{L})}, \quad \varepsilon_{eL}(\bar{K},\bar{L})=\frac{e_2(\bar{K},\bar{L})\bar{L}}{e(\bar{K},\bar{L})} \tag{4}$$

We consider positive externalities:

Assumption 2. *For any given $\bar{K},\bar{L}>0$, $\varepsilon_{eK}(\bar{K},\bar{L})\geq 0$ and $\varepsilon_{eL}(\bar{K},\bar{L})\geq 0$.*

Considering the aggregate consumption C_t, the capital accumulation equation is then

$$K_{t+1}=L_tAf(x_t)e(\bar{K}_t,\bar{L}_t)+(1-\mu)K_t-C_t \tag{5}$$

with K_0 given.

2.2 Preferences and intertemporal equilibrium

We consider an economy populated by a large number of identical infinitely-lived agents. We assume without loss of generality that the total population is constant and normalized to one, i.e. $N = 1$. At each period a representative agent supplies elastically an amount of labor $l \in [0, \ell]$, with $\ell > 0$ his endowment of labor. He then derives utility from consumption c and leisure $\mathcal{L} = \ell - l$ according to a non-separable function $u(c, \mathcal{L})$ which satisfies:

Assumption 3. *$u(c, \mathcal{L})$ is $\mathbf{C}^r$ over $\mathbb{R}_+ \times [0, \ell]$ for r large enough, increasing with respect to each argument, concave, homogeneous of degree one over $\mathbb{R}_{++} \times (0, \ell)$ and such that, for all $c, \mathcal{L} > 0$, $\lim_{c/\mathcal{L} \to 0} u_2/u_1 = 0$ and $\lim_{c/\mathcal{L} \to +\infty} u_2/u_1 = +\infty$.*

Homogeneity is introduced to completely characterize preferences in terms of the share of consumption within total utility $\alpha \in (0, 1)$ and the elasticity of intertemporal substitution in consumption $\epsilon_{cc} \in (0, +\infty)$ defined as follows:

$$\alpha(c, \mathcal{L}) = \frac{u_1(c,\mathcal{L})c}{u(c,\mathcal{L})}, \quad \epsilon_{cc}(c, \mathcal{L}) = -\frac{u_1(c,\mathcal{L})}{u_{11}(c,\mathcal{L})c} \tag{6}$$

Notice that the share of leisure within total utility is given by $1 - \alpha(c, \mathcal{L})$.

Since $N_t = 1$ for all $t \geq 0$, we get $L_t = l_t$ and $C_t = c_t$. The intertemporal maximization program of the representative agent is thus given as follows:

$$\begin{aligned} \max_{c_t, l_t, K_t} \quad & \sum_{t=0}^{+\infty} \delta^t u(c_t, \ell - l_t) \\ s.t. \quad & K_{t+1} = l_t A f(x_t) e(\bar{K}_t, \bar{l}_t) + (1 - \mu) K_t - c_t \\ & K_0 = k_0, \{\bar{K}_t, \bar{l}_t\}_{t=0}^{+\infty} \text{ given} \end{aligned} \tag{7}$$

where $\delta \in (0, 1)$ denotes the discount factor. Following Michel (1990), we introduce the generalized Lagrangian at time $t \geq 0$:

$$\mathcal{L}_t \ = u(c_t, \ell - l_t) + \delta \lambda_{t+1} \Big[l_t A f(x_t) e(\bar{K}_t, \bar{l}_t) + (1 - \mu) K_t - c_t \Big] - \lambda_t K_t$$

with λ_t the shadow price of capital K_t. Considering the prices (1), we derive the following first order conditions together with the transversality condition

$$u_1(c_t, \ell - l_t) = \delta\lambda_{t+1}, \quad u_2(c_t, \ell - l_t) = \delta\lambda_{t+1} w_t, \quad \delta\lambda_{t+1} R_t = \lambda_t \tag{8}$$

$$\lim_{t\to+\infty} \delta^t u_1(c_t, \ell - l_t) K_{t+1} \;=\; 0 \tag{9}$$

All firms being identical, the competitive equilibrium conditions imply that $\bar{K} = K$ and $\bar{l} = l$. By manipulating equations (8), we easily obtain the following system of Euler equations

$$-u_2(c_t, \ell - l_t) + w_t u_1(c_t, \ell - l_t) \;=\; 0 \tag{10}$$

$$\delta R_{t+1} u_1(c_{t+1}, \ell - l_{t+1}) - u_1(c_t, \ell - l_t) \;=\; 0 \tag{11}$$

Under Assumption 3, solving equation (10) with respect to c_t gives a consumption demand function $c(K_t, l_t)$. From the capital accumulation equation (5) and (11), we finally derive the following system of difference equations in K and l:

$$\begin{aligned} l_t A f(x_t) e(K_t, l_t) + (1-\mu)K_t - c(K_t, l_t) - K_{t+1} &= 0 \\ \delta R_{t+1} u_1(c(K_{t+1}, l_{t+1}), \ell - l_{t+1}) - u_1(c(K_t, l_t), \ell - l_t) &= 0 \end{aligned} \tag{12}$$

An intertemporal equilibrium is then a path $\{K_t, l_t\}_{t\geq 0}$, with $(K_t, l_t) \in \mathbb{R}_{++} \times (0, \ell)$ and $K_0 = k_0 > 0$, that satisfies equations (12) and the transversality condition (9).

3 Steady state

A steady state is a 4-uple (K^*, l^*, x^*, c^*) such that $x^* = K^*/l^*$ and:

$$\begin{aligned} &Af'(x^*)e(K^*, l^*) = \tfrac{1-\delta(1-\mu)}{\delta} \equiv \tfrac{\theta}{\delta}, \quad c^* = l^* A f(x^*) e(K^*, l^*) - \mu K^* \\ &u_2(c^*, \ell - l^*) = A[f(x^*) - x^* f'(x^*)] e(K^*, l^*) u_1(c^*, \ell - l^*) \end{aligned} \tag{13}$$

We use the scaling parameter A in order to give conditions for the existence of a normalized steady state (NSS in the sequel) such that $x^* = 1$.

Proposition 1. *Under Assumptions 1-3, there exist* $A^* > 0$ *such that when* $A = A^*$*, a NSS satisfying* $(K^*, l^*, x^*, c^*) = (\bar{l}, \bar{l}, 1, \bar{l}(\theta - s\delta\mu)/s\delta)$*, with* $\bar{l} \in (0, \ell)$*, is the unique solution of (13).*

Proof: See Appendix 7.1. □

Remark 1: Using a continuity argument we derive from Proposition 1 that there exists an intertemporal equilibrium for any initial capital stock k_0 in the neighborhood of K^*. Notice also that Proposition 1 ensures the existence and uniqueness of the NSS. However, the presence of externalities implies that one or two other steady states may co-exist. This point will be discussed later through a bifurcation analysis.

In the rest of the paper, we evaluate all the shares and elasticities previously defined at the NSS. From (2), (3), (4) and (6), we consider indeed $s(1) = s$, $\sigma(1) = \sigma$, $\varepsilon_{eK}(\bar{l}, \bar{l}) = \varepsilon_{eK}$, $\varepsilon_{eL}(\bar{l}, \bar{l}) = \varepsilon_{eL}$, $\alpha(\bar{c}, \ell - \bar{l}) = \alpha$ and $\epsilon_{cc}(\bar{c}, \ell - \bar{l}) = \epsilon_{cc}$.

Remark 2: Considering (1) and the shares defined by (2) and (6), the first order condition (10) evaluated along a NSS can be written as follows

$$\frac{\bar{l}}{\ell - \bar{l}} = \frac{\alpha}{1-\alpha} \frac{\theta(1-s)}{\theta - s\delta\mu}$$

Hence, choosing a particular value for the share of consumption into total utility $\alpha \in (0, 1)$ implies to consider a particular value for $\bar{l} \in (0, \ell)$.

4 Characteristic polynomial and geometrical method

Let us linearize the dynamical system (12) around the NSS. We get the following Proposition:

Proposition 2. *Under Assumptions 1-3, the characteristic polynomial is*

$$\mathcal{P}(\lambda) = \lambda^2 - \lambda\mathcal{T} + \mathcal{D} \tag{14}$$

with

$$\begin{aligned}
\mathcal{D} &= \tfrac{1}{\delta}\left\{1 + \tfrac{\theta}{s}\tfrac{\varepsilon_{eK}[\sigma(1-s)+(1-\alpha)s]+s\varepsilon_{e,L}[1-\alpha-\sigma]}{\sigma\varepsilon_{e,L}(\theta-1+\alpha)+(1-s)\theta+(1-\alpha)s}\right\} \\
\mathcal{T} &= 1 + \mathcal{D} + \varepsilon_{eK}\theta\tfrac{(1-s)\frac{\theta}{s\delta}-\frac{\theta-s\delta\mu}{s\delta}(1-\alpha)\epsilon_{cc}+\sigma\frac{1-\delta}{\delta}-\sigma\frac{\theta-s\delta\mu}{s\delta}\left(1+\frac{\alpha\theta(1-s)}{(1-\alpha)(\theta-s\delta\mu)}\right)}{\sigma\varepsilon_{e,L}(\theta-1+\alpha)+(1-s)\theta+(1-\alpha)s} \\
&+ \theta\tfrac{(1-s)\frac{\theta}{s\delta}\varepsilon_{eL}+\frac{\theta-s\delta\mu}{s\delta}\left[(1-s)\left(1+\frac{\alpha\theta(1-s)}{(1-\alpha)(\theta-s\delta\mu)}\right)-\varepsilon_{eL}(1-\alpha)\epsilon_{cc}\right]+\sigma\varepsilon_{e,L}\frac{1-\delta}{\delta}}{\sigma\varepsilon_{e,L}(\theta-1+\alpha)+(1-s)\theta+(1-\alpha)s}
\end{aligned}$$

Proof: See Appendix 7.2. □

Our aim is to discuss the local indeterminacy properties of equilibria, i.e. the existence of a continuum of equilibrium paths starting from the same initial capital stock and converging to the NSS. Our model consists in one predetermined variable, the capital stock, and one forward variable, the labor supply. Therefore, the NSS is locally indeterminate if and only if the local stable manifold is two-dimensional.

A necessary condition for the occurrence of local indeterminacy is $\mathcal{D} \in (-1, 1)$. Notice from Proposition 2 that if there is no externality coming from labor, i.e. $\varepsilon_{eL} = 0$, then $\mathcal{D} > 1$ and we get the following result:

Proposition 3. *Under Assumptions 1-3, if $\varepsilon_{eL} = 0$ the NSS is locally determinate.*[6]

As already shown in formulations with additively separable preferences,[7] we find again that labor externalities are a fundamental ingredient while capital

[6]In an OLG model, we also show in Lloyd-Braga *et al.* (2006) that when only capital externalities enter the technology and the homogeneous utility function is characterized by a large share of young agents' consumption over the wage income, the steady state is locally determinate.

[7]See Pintus (2006).

externalities are not. However, with non-separable preferences, such a conclusion is not obvious. Hintermaier (2001) (see Theorem 4, p. 14) shows indeed in a one-sector model with Cobb-Douglas technology and no externalities in labor that there are non-separable preferences consistent with indeterminacy if capital externalities are high enough. Notice though that this existence result is obtained through numerical simulations which do not allow to determine the precise formulation of the utility function. Proposition 3 then implies that such a result is not compatible with linear homogenous utility functions. Building on this conclusion, we will consider in the rest of the paper that only labor externalities enter the technology:

Assumption 4. $\varepsilon_{eK} = 0$ *and* $\varepsilon_{eL} > 0$

It follows that

$$\begin{aligned} \mathcal{D} &= \tfrac{1}{\delta}\left\{1 + \theta\varepsilon_{e,L}\tfrac{1-\alpha-\sigma}{\sigma\varepsilon_{e,L}(\theta-1+\alpha)+(1-s)\theta+(1-\alpha)s}\right\} \\ \mathcal{T} &= 1 + \mathcal{D} + \theta\tfrac{(1-s)\frac{\theta}{s\delta}\varepsilon_{eL}+\frac{\theta-s\delta\mu}{s\delta}\left[(1-s)\left(1+\frac{\alpha\theta(1-s)}{(1-\alpha)(\theta-s\delta\mu)}\right)-\varepsilon_{eL}(1-\alpha)\epsilon_{cc}\right]+\sigma\varepsilon_{e,L}\frac{1-\delta}{\delta}}{\sigma\varepsilon_{e,L}(\theta-1+\alpha)+(1-s)\theta+(1-\alpha)s} \end{aligned}$$

4.1 The Δ-segment

As in Grandmont *et al.* (1998), we study the variations of the trace $\mathcal{T}$ and the determinant $\mathcal{D}$ in the $(\mathcal{T}, \mathcal{D})$ plane as one of the parameters of interest is made to vary continuously in its admissible range. This methodology allows to fully characterize the local stability of the NSS, as well as the occurrence of local bifurcations. Let us then start by considering the locus of points $(\mathcal{T}(\sigma), \mathcal{D}(\sigma))$ as the elasticity of capital-labor substitution σ continuously changes in $(0, +\infty)$. From Proposition 2, solving $\mathcal{T}$ and $\mathcal{D}$ with respect to σ allows to get the following linear relationship $\Delta(\mathcal{T})$:

$$\mathcal{D} = \Delta(\mathcal{T}) \equiv \mathcal{S}(\mathcal{T} - 1) - \tfrac{\mathcal{A}_1\mathcal{A}_7+\mathcal{A}_2(\mathcal{A}_5-\epsilon_{cc}\mathcal{A}_6)}{\mathcal{A}_1\mathcal{A}_4+\mathcal{A}_2\mathcal{A}_3+\delta[\mathcal{A}_4(\mathcal{A}_5-\epsilon_{cc}\mathcal{A}_6)-\mathcal{A}_3\mathcal{A}_7]} \qquad (15)$$

with

$$\begin{aligned}
&\mathcal{A}_1 = (1-s)\theta + (1-\alpha)s + (1-\alpha)\theta\varepsilon_{e,L} > 0, \quad \mathcal{A}_2 = \varepsilon_{eL}(1-\alpha) > 0 \\
&\mathcal{A}_3 = (1-s)\theta + (1-\alpha)s > 0, \quad \mathcal{A}_4 = \varepsilon_{eL}(\theta - 1 + \alpha) \\
&\mathcal{A}_5 = \tfrac{\theta(1-s)}{s\delta}\left[\theta\varepsilon_{e,L} + (\theta - s\delta\mu)\left(1 + \tfrac{\alpha\theta(1-s)}{(1-\alpha)(\theta-s\delta\mu)}\right)\right] > 0 \\
&\mathcal{A}_6 = \tfrac{\theta - s\delta\mu}{s\delta}\theta\varepsilon_{e,L}(1-\alpha) > 0, \quad \mathcal{A}_7 = \theta\varepsilon_{e,L}\tfrac{1-\delta}{\delta} > 0
\end{aligned} \tag{16}$$

and where the slope $\mathcal{S}$ of $\Delta(\mathcal{T})$ is

$$\mathcal{S} = \tfrac{\mathcal{A}_1\mathcal{A}_4 + \mathcal{A}_2\mathcal{A}_3}{\mathcal{A}_1\mathcal{A}_4 + \mathcal{A}_2\mathcal{A}_3 + \delta[\mathcal{A}_4(\mathcal{A}_5 - \epsilon_{cc}\mathcal{A}_6) - \mathcal{A}_3\mathcal{A}_7]}, \tag{17}$$

Figure 1 provides an illustration of $\Delta(\mathcal{T})$. We also introduce three other relevant lines: line AC ($\mathcal{D} = \mathcal{T} - 1$) along which one characteristic root is equal to 1, line AB ($\mathcal{D} = -\mathcal{T} - 1$) along which one characteristic root is equal to -1 and segment BC ($\mathcal{D} = 1$, $|\mathcal{T}| < 2$) along which the characteristic roots are complex conjugate with modulus equal to 1. These lines divide the space $(\mathcal{T}, \mathcal{D})$ into three different types of regions according to the number of characteristic roots with modulus less than 1. When $(\mathcal{T}, \mathcal{D})$ belongs to the interior of triangle ABC, the NSS is locally indeterminate (a sink). Let σ^F, σ^T and σ^H in $(0, +\infty)$ be the values of σ at which $\Delta(\mathcal{T})$ respectively crosses the lines AB, AC and the segment BC. Then as σ respectively goes through σ^F, σ^T or σ^H, a flip, transcritical or Hopf bifurcation generically occurs. Indeed the existence of the NSS being always ensured under the conditions of Proposition 1, a saddle-node bifurcation cannot occur. Moreover, pitchfork bifurcations require some non-generic condition.[8] In order to simplify the exposition we then concentrate on the generic case and we associate in the rest of the paper the critical value σ^T to a transcritical bifurcation, i.e. an exchange of stability between the NSS and another steady state.

[8] Some second derivative of the map which defines the dynamical system (12) needs to be equal to zero. As shown in Ruelle (1989), this is a non-generic configuration.

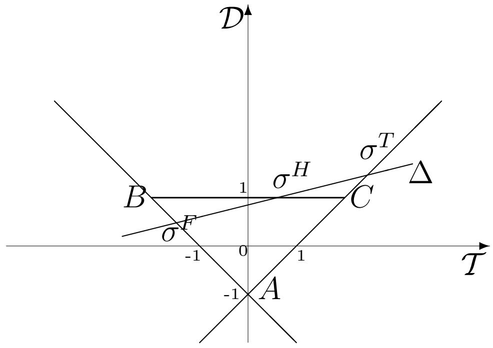

Figure 1: Stability triangle and $\Delta(\mathcal{T})$ line.

As $\sigma \in (0, +\infty)$, only a part of $\Delta(\mathcal{T})$ is relevant. We need to compute the starting and end points of the pair $(\mathcal{T}(\sigma), \mathcal{D}(\sigma))$. We easily get

$$\mathcal{D}(0) \equiv \mathcal{D}_0 = \tfrac{1}{\delta}\left\{1 + \theta\varepsilon_{e,L}\tfrac{1-\alpha}{(1-s)\theta+(1-\alpha)s}\right\}, \quad \mathcal{T}(0) \equiv \mathcal{T}_0 = 1 + \mathcal{D}_0 + \tfrac{\mathcal{A}_5 - \epsilon_{cc}\mathcal{A}_6}{\mathcal{A}_3}$$

$$\mathcal{D}(+\infty) \equiv \mathcal{D}_\infty = -\tfrac{1-\alpha}{\delta(\theta-1+\alpha)}, \quad \mathcal{T}(+\infty) \equiv \mathcal{T}_\infty = \tfrac{\theta-(1-\alpha)(1+\delta)}{\delta(\theta-1+\alpha)}$$

From now on, $\Delta(\mathcal{T})$ will be a segment from $(\mathcal{T}_0, \mathcal{D}_0)$ to $(\mathcal{T}_\infty, \mathcal{D}_\infty)$.

4.2 The Δ_0-half-line

Assuming fixed values for δ, μ, α and $\varepsilon_{e,L}$, we study now how the segment Δ evolves in the $(\mathcal{T}, \mathcal{D})$ plane as the elasticity of intertemporal substitution in consumption ϵ_{cc} varies continuously in $(0, +\infty)$. This amounts to study how the starting point $(\mathcal{T}_0, \mathcal{D}_0)$ and the end point $(\mathcal{T}_\infty, \mathcal{D}_\infty)$ move (and therefore how the slope $\mathcal{S}$ of Δ changes) with ϵ_{cc}. Notice that for given values of δ, μ, α and $\varepsilon_{e,L}$, $(\mathcal{T}_\infty, \mathcal{D}_\infty)$ and $\mathcal{D}_0$ are fixed while $\mathcal{T}_0$ is a linear function of ϵ_{cc}. Let us then analyze how $\mathcal{T}_0(\epsilon_{cc})$ changes with ϵ_{cc}. We easily derive

$$\lim_{\epsilon_{cc}\to 0} \mathcal{T}_0 \equiv \mathcal{T}_0^0 = 1 + \mathcal{D}_0 + \tfrac{\mathcal{A}_5}{\mathcal{A}_3} > 2, \quad \lim_{\epsilon_{cc}\to +\infty} \mathcal{T}_0 \equiv \mathcal{T}_0^\infty = -\infty \qquad (18)$$

We may then define a half-line Δ_0 of starting points $(\mathcal{T}_0, \mathcal{D}_0)$ which describes the possible values of $\mathcal{T}_0$ when ϵ_{cc} covers $(0, +\infty)$: Δ_0 is a horizontal half-line located at the fixed value $\mathcal{D}_0 > 1$. In graphical terms, Δ_0 is characterized by a "reversed" orientation as ϵ_{cc} increases from 0 to $+\infty$: it starts in $(\mathcal{T}_0^0, \mathcal{D}_0)$, with $\mathcal{T}_0^0 > 2$, and ends in $(-\infty, \mathcal{D}_0)$. Hence the horizontal half-line Δ_0 lies above line BC and crosses line AB for some value of $\epsilon_{cc} \in (0, +\infty)$. Moreover the slope $\mathcal{S}$ tends to zero while ϵ_{cc} increases to $+\infty$. In this extreme case, Δ tends to Δ_0.

Our main objective is to give conditions for local indeterminacy of equilibria under small labor externalities. A critical issue is to locate precisely Δ_0 with respect to line AC and the end point $(\mathcal{T}_\infty, \mathcal{D}_\infty)$ with respect to lines AB, BC and AC. As shown in Lemma 1 below, according to the fixed value of the share of consumption into total utility α, these intersections may be actually classified in simple basic cases depending on the sign of $\mathcal{D}_\infty$ and on whether the end point is above or below the line AB.

Lemma 1. *Under Assumptions 1-4, the half-line Δ_0 always crosses the line AC for some value of $\epsilon_{cc} \in (0, +\infty)$ and the following results hold:*

a) When $\alpha > 1 - \theta \equiv \alpha_1$:

i) $\mathcal{D}_\infty < 0$,

ii) $\partial \mathcal{D}/\partial \sigma < 0$,

iii) $1 - \mathcal{T}_\infty + \mathcal{D}_\infty < 0$,

iv) $1 + \mathcal{T}_\infty + \mathcal{D}_\infty > 0$ if and only if $\alpha > 1 - \theta/2 \equiv \alpha_2 (> \alpha_1)$.

b) When $\alpha \in (0, \alpha_1)$ and $\varepsilon_{eL} < [(1-s)\theta + (1-\alpha)s]/[(1-\alpha)(1-\alpha-\theta)] \equiv \bar{\varepsilon}$:

i) $\mathcal{D}_\infty > \mathcal{D}_0 > 1$,

ii) $\partial \mathcal{D}/\partial \sigma < 0$ and there exists $\sigma^* > 0$ such that $\lim_{\sigma \to \sigma^*_-} \mathcal{D}(\sigma) = -\infty$ and $\lim_{\sigma \to \sigma^*_+} \mathcal{D}(\sigma) = +\infty$,

iii) $1 - \mathcal{T}_\infty + \mathcal{D}_\infty > 0$,

iv) $1 + \mathcal{T}_\infty + \mathcal{D}_\infty > 0$.

Proof: See Appendix 7.3.

□

Remark 3: Notice that when $\alpha \in (\alpha_1, 1)$, $\mathcal{D}_\infty < -1$ if and only if $\alpha \in (\alpha_1, \alpha_3)$ with $\alpha_3 = 1 - \delta\theta/(1+\delta)(> \alpha_2)$.

As referred previously, local indeterminacy arises when Δ crosses the triangle ABC. In configuration a) with $\alpha > \alpha_1$, this may be the case since $\mathcal{D}_0 > 1$, $\mathcal{D}_\infty < 0$ and $\mathcal{D}(\sigma)$ is a decreasing function of σ. Of course, such a property requires some restrictions on the values of the elasticity of intertemporal substitution in consumption ϵ_{cc}. These restrictions then depend on the precise localization of the end point $(\mathcal{T}_\infty, \mathcal{D}_\infty)$, i.e. on whether the expressions $1 - \mathcal{T}_\infty + \mathcal{D}_\infty$ and $1 + \mathcal{T}_\infty + \mathcal{D}_\infty$ have or not the same sign. Two cases need therefore to be considered: when $\alpha \in (\alpha_2, 1)$, $(\mathcal{T}_\infty, \mathcal{D}_\infty)$ is located above line AB $(1 + \mathcal{T}_\infty + \mathcal{D}_\infty > 0)$ but below line AC $(1 - \mathcal{T}_\infty + \mathcal{D}_\infty < 0)$ as illustrated with E_1 in Figure 2, and when $\alpha \in (\alpha_1, \alpha_2)$, $(\mathcal{T}_\infty, \mathcal{D}_\infty)$ is located below lines AB and AC as illustrated with E_2.

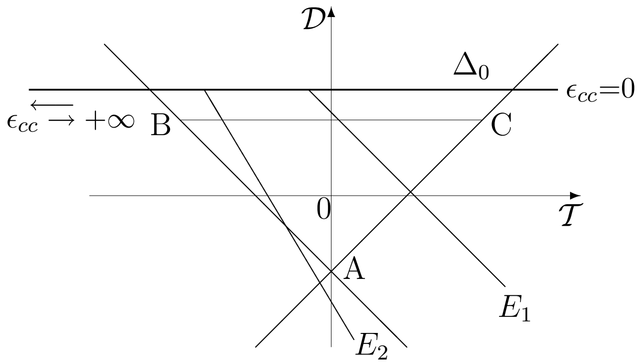

Figure 2: $\alpha \in (\alpha_1, 1)$.

In configuration b) with $\alpha \in (0, \alpha_1)$, we still have $\mathcal{D}_0 > 1$ but now, provided the amount of labor externalities is small enough, i.e. $\varepsilon_{eL} \in (0, \bar{\varepsilon})$, we get $\mathcal{D}_\infty > \mathcal{D}_0$ and $\mathcal{D}(\sigma)$ is a decreasing function of σ. Intersections between Δ and the triangle ABC may then still occur. In such a case, as shown in Figure 3, starting from one point on Δ_0, when σ increases, the point $(\mathcal{T}(\sigma), \mathcal{D}(\sigma))$ moves downwards along a segment Δ as $\sigma \in (0, \sigma^*)$, with $\mathcal{D}(\sigma)$ going through $-\infty$ when $\sigma = \sigma^*$ and finally decreasing from $+\infty$ as $\sigma > \sigma^*$ until it reaches the end point $(\mathcal{T}_\infty, \mathcal{D}_\infty)$ which is located above lines AB and AC. The occurrence of local indeterminacy is again obtained under particular restrictions on the elasticity of intertemporal substitution in consumption ϵ_{cc}.

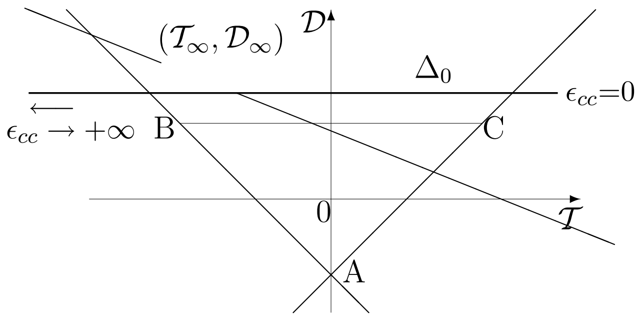

Figure 3: $\alpha \in (0, \alpha_1)$ and $\varepsilon_{eL} \in (0, \bar{\varepsilon})$.

5 Indeterminacy with small externalities

As shown in Lemma 1, we have to consider two cases depending on whether the end point $(\mathcal{T}_\infty, \mathcal{D}_\infty)$ is located in a region in which the NSS is a saddle-point (as E_1 in Figure 2), or a source (as E_2 in Figure 2 and in Figure 3).

5.1 The case $\alpha \in (\alpha_2, 1)$

When $\alpha \in (\alpha_2, 1)$, consider the localization of Δ_0 and $(\mathcal{T}_\infty, \mathcal{D}_\infty)$ derived from Lemma 1. We have to define some critical values of ϵ_{cc} which correspond to the crossings of the segment Δ with particular points: ϵ_{cc}^1, ϵ_{cc}^2, ϵ_{cc}^3 are respectively associated with a segment Δ that crosses the points B, A, C, while ϵ_{cc}^4, ϵ_{cc}^5 are respectively associated with a segment Δ that crosses the points of intersection between Δ_0 and $\mathcal{P}(1) = 0$, $\mathcal{P}(-1) = 0$.

As previously explained in Remark 1, $\mathcal{D}_\infty < -1$ if $\alpha \in (\alpha_2, \alpha_3)$ while $\mathcal{D}_\infty \in (-1, 0)$ if $\alpha \in (\alpha_3, 1)$. It follows that, since the slope $\mathcal{S}$ of Δ tends to 0 as ϵ_{cc} goes to $+\infty$, the existence of the critical value ϵ_{cc}^2 requires $\alpha < \alpha_3$ and cannot occur when $\alpha \in (\alpha_3, 1)$, as shown in the following Figures.

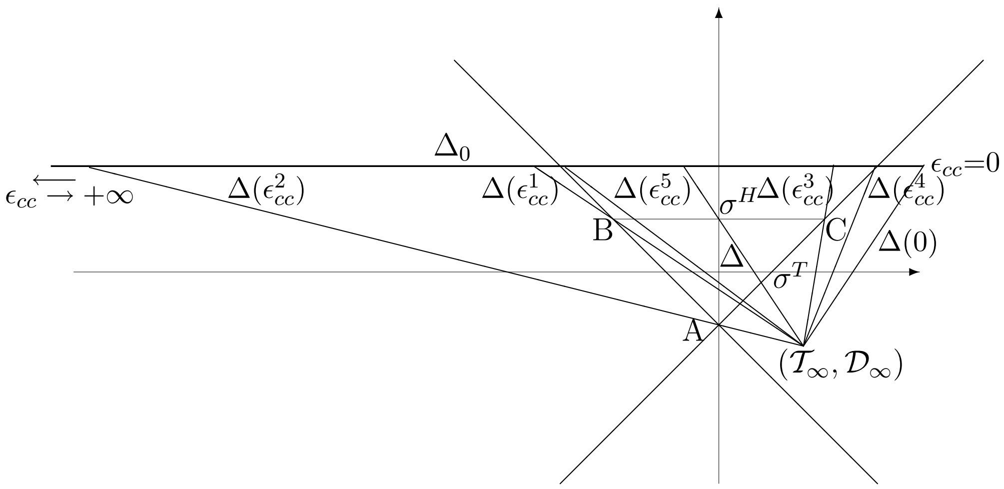

Figure 4: $\alpha \in (\alpha_2, \alpha_3)$.

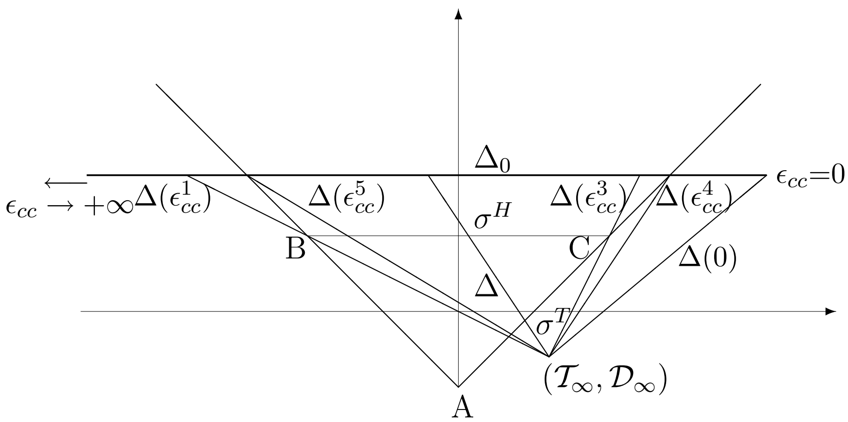

Figure 5: $\alpha \in (\alpha_3, 1)$.

All the detailed results on local stability of the NSS and bifurcations in the case $\alpha \in (\alpha_2, \alpha_3)$ corresponding to Figure 4 are now gathered into the following Proposition. Notice that the occurrence of local indeterminacy requires intermediary values for the elasticity of intertemporal substitution in consumption and the elasticity of capital-labor substitution. The configuration with $\alpha \in (\alpha_3, 1)$ is a particular case that will be discussed in a Corollary.

Proposition 4. *Under Assumptions 1-4, let $\alpha \in (\alpha_2, \alpha_3)$ be fixed as well as $\varepsilon_{eL} > 0$. There exist ϵ_{cc}^1, ϵ_{cc}^2, ϵ_{cc}^3, ϵ_{cc}^4 and ϵ_{cc}^5, with $\epsilon_{cc}^2 > \epsilon_{cc}^1 > \epsilon_{cc}^5 > \epsilon_{cc}^3 > \epsilon_{cc}^4 > 0$, such that the following results generically hold:*[9]

i) $\epsilon_{cc} \in (\epsilon_{cc}^2, +\infty)$. The NSS is a saddle-point for $\sigma \in (\sigma^F, +\infty)$, undergoes a flip bifurcation at $\sigma = \sigma^F$, becomes a source for $\sigma \in (\sigma^T, \sigma^F)$, undergoes a transcritical bifurcation at $\sigma = \sigma^T$ and becomes a saddle-point for $\sigma \in (0, \sigma^T)$.

ii) $\epsilon_{cc} \in (\epsilon_{cc}^1, \epsilon_{cc}^2)$. The NSS is a saddle-point for $\sigma \in (\sigma^T, +\infty)$, undergoes a transcritical bifurcation at $\sigma = \sigma^T$, becomes locally indeterminate for $\sigma \in (\sigma^F, \sigma^T)$, undergoes a flip bifurcation at $\sigma = \sigma^F$ and becomes a saddle-point for $\sigma \in (0, \sigma^F)$.

iii) $\epsilon_{cc} \in (\epsilon_{cc}^5, \epsilon_{cc}^1)$. The NSS is a saddle-point for $\sigma \in (\sigma^T, +\infty)$, undergoes a transcritical bifurcation at $\sigma = \sigma^T$, becomes locally indeterminate for $\sigma \in (\sigma^H, \sigma^T)$, undergoes a Hopf bifurcation at $\sigma = \sigma^H$, becomes a source for $\sigma \in (\sigma^F, \sigma^H)$, undergoes a flip bifurcation at $\sigma = \sigma^F$ and becomes a saddle-point for $\sigma \in (0, \sigma^F)$.

iv) $\epsilon_{cc} \in (\epsilon_{cc}^3, \epsilon_{cc}^5)$. The NSS is a saddle-point for $\sigma \in (\sigma^T, +\infty)$, undergoes a transcritical bifurcation at $\sigma = \sigma^T$, becomes locally indeterminate for $\sigma \in (\sigma^H, \sigma^T)$, undergoes a Hopf bifurcation at $\sigma = \sigma^H$ and becomes a source

[9] The expressions of the bifurcation values are given in Appendix 7.4.

for $\sigma \in (0, \sigma^H)$.

v) $\epsilon_{cc} \in (\epsilon_{cc}^4, \epsilon_{cc}^3)$. *The NSS is a saddle-point for* $\sigma \in (\sigma^T, +\infty)$, *undergoes a transcritical bifurcation at* $\sigma = \sigma^T$ *and becomes a source for* $\sigma \in (0, \sigma^T)$.

vi) $\epsilon_{cc} \in (0, \epsilon_{cc}^4)$. *The NSS is a saddle-point for any* $\sigma > 0$.

Proof: See Appendix 7.4. □

Proposition 4 shows that for any given amount of labor externalities $\varepsilon_{eL} > 0$, when the share of consumption into total utility is large, local indeterminacy of equilibria requires $\epsilon_{cc} \in (\epsilon_{cc}^3, \epsilon_{cc}^2)$ and $\sigma \in (\max\{\sigma^F, \sigma^H\}, \sigma^T)$. As explicitely stated in Proposition 4 and clearly apparent in Appendix 7.4, the critical values ϵ_{cc}^i, $i = 1, \ldots, 5$, for the elasticity of intertemporal substitution in consumption and the bifurcation values σ^j, $j = H, T, F$, for the elasticity of capital-labor substitution depend on ε_{eL}. In particular they all tend to infinity as the labor externality goes to zero. This property simply follows from the fact that in the limit case with $\varepsilon_{eL} = 0 = \varepsilon_{eK}$, the steady state is necessarily locally determinate. However, as shown later on, although we consider standard calibrations for the structural parameters with small labor externalities, local indeterminacy is obtained under realistic values for σ provided the elasticity of intertemporal substitution in consumption is large enough (but bounded away from $+\infty$). It is also worth noticing that, as shown in Appendix 7.2 (see (21)), a large elasticity of intertemporal substitution in consumption corresponds to a large elasticity of the labor supply.

Notice finally that in accordance with Remark 1, if $\alpha \in (\alpha_3, 1)$, the critical value ϵ_{cc}^2 cannot be defined, and we get:

Corollary 1. *Under Assumptions 1-4 if* $\alpha \in (\alpha_3, 1)$, *case i) in Proposition 4 does not occur and case ii) becomes valid for* $\epsilon_{cc} \in (\epsilon_{cc}^1, +\infty)$ *with* $\lim_{\epsilon_{cc} \to +\infty} \sigma^F = \lim_{\epsilon_{cc} \to +\infty} \sigma^T = +\infty$.

As shown in Proposition 4, when $\epsilon_{cc} > \epsilon_{cc}^1$ local indeterminacy requires the elasticity of capital-labor substitution to satisfy $\sigma > \sigma^F$. Corollary 1 then proves that a locally indeterminate NSS cannot co-exist with an infinite elasticity of intertemporal substitution in consumption ϵ_{cc}. This point is worthwhile to be stressed since homothetic preferences may be additively separable if and only if $\epsilon_{cc} = +\infty$. It follows that local indeterminacy with small externalities is fundamentally based on non-separability.

5.2 The case $\alpha \in (0, \alpha_2)$

We may now consider the cases with a lower share of consumption into total utility. As shown in Lemma 1, and Figures 2 and 3, the main consequence of this decrease is that the end point $(\mathcal{T}_\infty, \mathcal{D}_\infty)$ is now located in a region in which the NSS is a source. A direct implication of this is that the ranking of the critical values ϵ_{cc}^1, ϵ_{cc}^2 and ϵ_{cc}^5 is modified. Moreover, we have to distinguish two sub-cases depending on whether α is greater or lower than the bound α_1: if $\alpha \in (\alpha_1, \alpha_2)$, the end point is characterized by $\mathcal{D}_\infty < -1$, while if $\alpha \in (0, \alpha_1)$ and the amount of labor externalities is small enough, i.e. $\varepsilon_{eL} \in (0, \bar{\varepsilon})$, it is characterized by $\mathcal{D}_\infty > \mathcal{D}_0$. In both cases, $\mathcal{D}$ is a decreasing function of σ, and since $\mathcal{D}_0 > 1$, local indeterminacy occurs for some values $\epsilon_{cc} \in (0, +\infty)$. All the results are summarized in Figures 6-7 and detailed in the following Proposition:

Proposition 5. *Under Assumptions 1-4, let $\alpha > 0$ and $\varepsilon_{eL} > 0$ be fixed such that $\alpha \in (\alpha_1, \alpha_2)$ or $\alpha \in (0, \alpha_1)$ with $\varepsilon_{eL} \in (0, \bar{\varepsilon})$. Then there exist ϵ_{cc}^1, ϵ_{cc}^2, ϵ_{cc}^3, ϵ_{cc}^4 and ϵ_{cc}^5, with $\epsilon_{cc}^5 > \epsilon_{cc}^1 > \epsilon_{cc}^2 > \epsilon_{cc}^3 > \epsilon_{cc}^4 > 0$, such that the following results generically hold:*

i) $\epsilon_{cc} \in (\epsilon_{cc}^5, +\infty)$. The NSS is a source for $\sigma \in (\sigma^T, +\infty)$, undergoes a

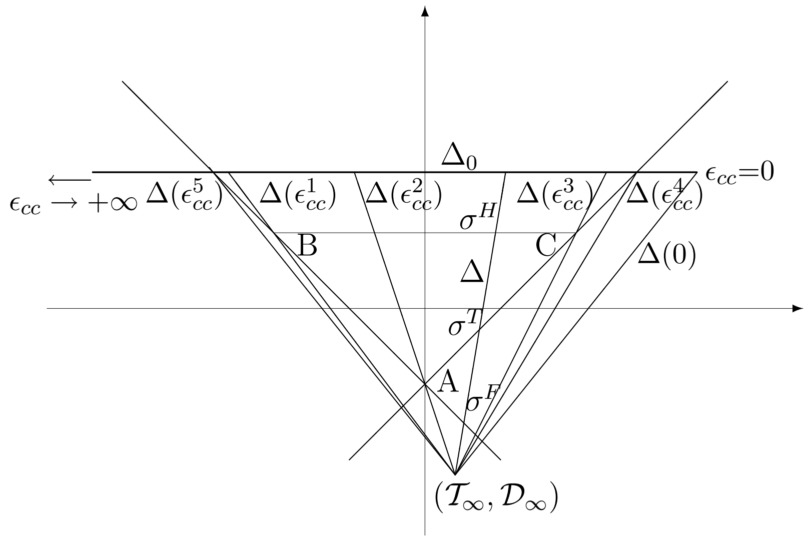

Figure 6: $\alpha \in (\alpha_1, \alpha_2)$

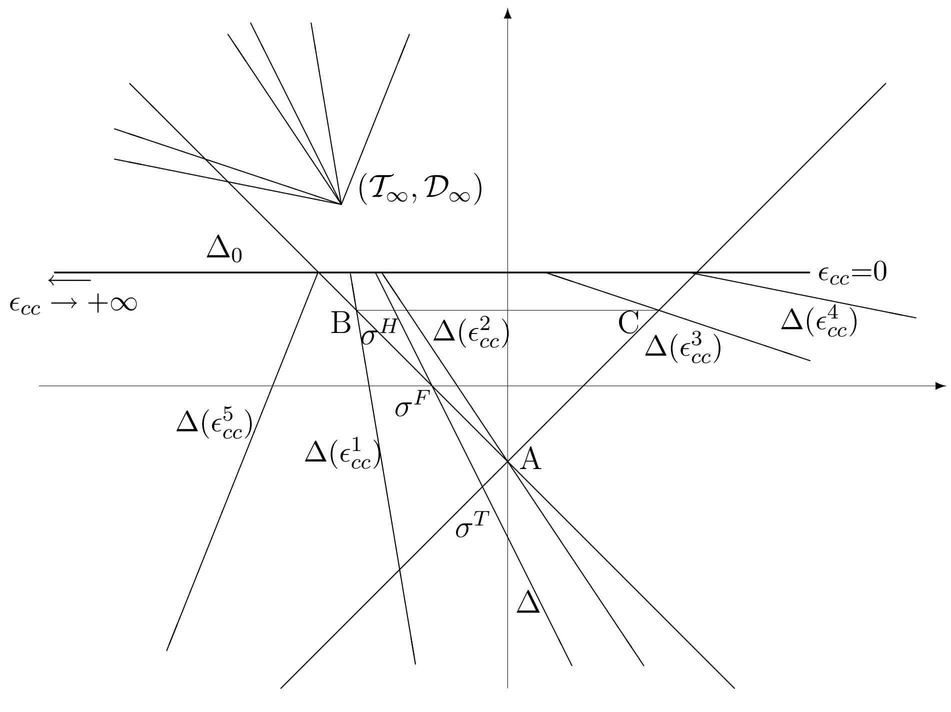

Figure 7: $\alpha \in (0, \alpha_1)$

transcritical bifurcation at $\sigma = \sigma^T$ *and becomes a saddle-point for* $\sigma \in (0, \sigma^T)$.

ii) $\epsilon_{cc} \in (\epsilon_{cc}^1, \epsilon_{cc}^5)$. *The NSS is a source for* $\sigma \in (\sigma^T, +\infty)$, *undergoes a transcritical bifurcation at* $\sigma = \sigma^T$, *becomes a saddle-point for* $\sigma \in (\sigma^F, \sigma^T)$, *undergoes a flip bifurcation* $\sigma = \sigma^F$ *and becomes a source for* $\sigma \in (0, \sigma^F)$.

iii) $\epsilon_{cc} \in (\epsilon_{cc}^2, \epsilon_{cc}^1)$. *The NSS is a source for* $\sigma \in (\sigma^T, +\infty)$, *undergoes a transcritical bifurcation at* $\sigma = \sigma^T$, *becomes a saddle-point for* $\sigma \in (\sigma^F, \sigma^T)$, *undergoes a flip bifurcation at* $\sigma = \sigma^F$, *becomes locally indeterminate for* $\sigma \in (\sigma^H, \sigma^F)$, *undergoes a Hopf bifurcation at* $\sigma = \sigma^H$ *and becomes a source for* $\sigma \in (0, \sigma^H)$.

iv) $\epsilon_{cc} \in (\epsilon_{cc}^3, \epsilon_{cc}^2)$. *The NSS is a source for* $\sigma \in (\sigma^F, +\infty)$, *undergoes a flip bifurcation at* $\sigma = \sigma^F$, *becomes a saddle-point for* $\sigma \in (\sigma^T, \sigma^F)$, *undergoes a transcritical bifurcation at* $\sigma = \sigma^T$, *becomes locally indeterminate for* $\sigma \in (\sigma^H, \sigma^T)$, *undergoes a Hopf bifurcation at* $\sigma = \sigma^H$ *and becomes a source for* $\sigma \in (0, \sigma^H)$.

v) $\epsilon_{cc} \in (\epsilon_{cc}^4, \epsilon_{cc}^3)$. *The NSS is a source for* $\sigma \in (\sigma^F, +\infty)$, *undergoes a flip bifurcation at* $\sigma = \sigma^F$, *becomes a saddle-point for* $\sigma \in (\sigma^T, \sigma^F)$, *undergoes a transcritical bifurcation at* $\sigma = \sigma^T$ *and becomes a source for* $\sigma \in (0, \sigma^T)$.

vi) $\epsilon_{cc} \in (0, \epsilon_{cc}^4)$. *The NSS is a source for* $\sigma \in (\sigma^F, +\infty)$, *undergoes a flip bifurcation at* $\sigma = \sigma^F$ *and becomes a saddle-point for* $\sigma \in (0, \sigma^F)$.

Proof: See Appendix 7.6. □

When the share of consumption into total utility is lower, the occurrence of local indeterminacy is based on similar conditions as in Proposition 4: intermediary values for the elasticity of intertemporal substitution in consumption ($\epsilon_{cc} \in (\epsilon_{cc}^3, \epsilon_{cc}^1)$) and the elasticity of capital-labor substitution ($\sigma \in (\sigma^H, \min\{\sigma^F, \sigma^T\})$), with these critical values tending to infinity as the labor externality goes to zero. The main difference between Proposition 5 and Proposition 4 concerns the fact that the values of the critical bounds ϵ_{cc}^i, $i = 1, \dots, 5$, and σ^j, $j = H, T, F$, are different since they all depend on α.

5.3 Local indeterminacy with Cobb-Douglas technology

Propositions 4 and 5 clearly show that the occurrence of local indeterminacy requires a minimal amount of capital-labor substitution, i.e. $\sigma > \sigma^H$. Based on this result, one question remains open: is it possible to get local indeterminacy with a Cobb-Douglas technology ? Since the critical value σ^H goes to $+\infty$ as $\epsilon_{e,L}$ goes to 0, this result will require a minimal amount of labor externalities.

Corollary 2. *A necessary condition for the occurrence of local indeterminacy with a Cobb-Douglas technology is* $\epsilon_{e,L} > \underline{\varepsilon} \equiv (1-\delta)[(1-s)\theta + (1-\alpha)s]/[(1-\delta)(1-\theta) + \alpha\delta\mu]$. *Notice that when* $\alpha \in (0, \alpha_1)$, $\underline{\varepsilon} < \bar{\varepsilon}$.

However, as we will show with some numerical simulations in the next section, the lower bound $\underline{\varepsilon}$ may remain low and local indeterminacy is compatible with small labor externalities. Of course, some additional restrictions on the values of the elasticity of intertemporal substitution in consumption are also necessary to get a locally indeterminate NSS. If the conditions of cases iii), iv) in Proposition 4 or cases iii), iv) in Proposition 5 are satisfied with $\epsilon_{e,L} > \underline{\varepsilon}$, the NSS of an economy with a Cobb-Douglas technology is locally indeterminate.

This conclusion drastically differs from most of the contributions of the literature. When additively separable preferences are considered with small externalities, the existence of local indeterminacy requires a technology with an elasticity of capital-labor substitution sufficiently larger than one. The Cobb-Douglas specification is thus ruled-out.[10]

Even worse conclusions have been reached with non-separable preferences characterized by the class of utility functions as specified in King *et al.* (1988). The restrictions for the concavity of the utility function indeed precludes the existence of local indeterminacy as soon as mild external effects (i.e. a decreasing labor demand function) are considered.[11] It is worth noticing however that Hintermaier (2001) also shows that there exist non-separable preferences for which local indeterminacy arises under a Cobb-Douglas technology and mild externalities. His proof is based on numerical simulations. The main problem with these conclusions, besides the fact that the form of utility function is not specified, concerns the amount of externalities necessary for local indeterminacy, i.e. at least 35%.

Our main results then prove that the local stability properties of the steady state are highly sensitive to the specification of preferences. Using a standard non-separable homothetic formulation allows to easily get a compatibility between local indeterminacy and very low externalities, even in the case of a Cobb-Douglas technology.

A last question remains however: Why our results cannot apply with KPR preferences ? One basic reason explains this fact: Consider $\psi = -lv'(l)/v(l) \geq 0$ and $\gamma = l[v''(l)v(l) - v'(l)^2]/[v(l)v'(l)]$. As shown in Pintus (2004), concavity of $U(c,l)$ requires $\gamma \geq \psi(1/\varsigma - 1)$. Since the elasticity of intertemporal substitution in consumption is $\epsilon_{cc} = 1/\varsigma$ and the elasticity of

[10]See Hintermaier (2001, 2003), Pintus (2006).

[11]See Hintermaier (2001) and Pintus (2004) in which the utility function is given by $U(c,l) = [cv(l)]^{1-\varsigma}/(1-\varsigma)$, with $v(l)$ a decreasing function characterizing the disutility of labor.

the labor supply with respect to real wage is $\epsilon_{lw} = 1/\gamma$, the above inequality becomes $1/\epsilon_{lw} \geq \psi(\epsilon_{cc} - 1)$, and large ϵ_{cc} imply low ϵ_{lw}. But as shown in Propositions 4-5 and in Appendix 7.2 (see equation (21)), local indeterminacy in our framework requires large enough values for both ϵ_{cc} and ϵ_{lw}.

5.4 Numerical illustrations

Considering standard values for the share of capital in total income, $s = 1/3$, the depreciation rate of capital, $\mu = 2.5\%$, and the discount factor, $\delta = 0.98$, we easily compute $\theta = 0.0455$ and the bounds $\alpha_1 = 0.9555$, $\alpha_2 = 0.97775$ and $\alpha_3 = 0.9779$.

We illustrate all the various cases exhibited in the previous section:

i) If $\alpha = 0.98 \in (\alpha_3, 1)$, the critical bound ϵ_{cc}^2 does not exist. Proposition 4 then shows that the occurrence of local indeterminacy requires $\epsilon_{cc} > \epsilon_{cc}^3 \approx 136753.4$ when $\varepsilon_{eL} = 1\%$. Assuming $\epsilon_{cc} = 136780$, case iv) in Proposition 4 implies that the steady state is locally indeterminate if $\sigma \in (\sigma^H, \sigma^T)$ with $\sigma^H = 1.67$ and $\sigma^T = 4.57$. In accordance with Corollary 2, the minimal amount of labor externalities implying a compatibility between local indeterminacy and a Cobb-Douglas technology is $\underline{\varepsilon} = 1.68\%$. Considering $\varepsilon_{eL} = 2\%$ we get $\epsilon_{cc}^3 \approx 68397.2$. Therefore, when $\epsilon_{cc} = 68420$, the steady state is locally indeterminate if $\sigma \in (\sigma^H, \sigma^T)$ with $\sigma^H = 0.845$ and $\sigma^T = 3.33$.

ii) If $\alpha = 0.9778 \in (\alpha_2, \alpha_3)$, Proposition 4 shows that the occurrence of local indeterminacy requires $\epsilon_{cc} \in (\epsilon_{cc}^3, \epsilon_{cc}^2)$ with $\epsilon_{cc}^2 \approx 286378.1$ and $\epsilon_{cc}^3 \approx 111052$ when $\varepsilon_{eL} = 1\%$. Assuming $\epsilon_{cc} = 111070$, the steady state is locally indeterminate if $\sigma \in (\sigma^H, \sigma^T)$ with $\sigma^H = 1.7$ and $\sigma^T = 3.87$. Moreover we have $\underline{\varepsilon} = 1.72\%$. Considering $\varepsilon_{eL} = 2\%$ we get $\epsilon_{cc}^2 \approx 143219.2$ and $\epsilon_{cc}^3 \approx 55544.5$. Therefore, when $\epsilon_{cc} = 55570$, the steady state is locally indeterminate if $\sigma \in (\sigma^H, \sigma^T)$ with $\sigma^H = 0.86$ and $\sigma^T = 3.95$.

iii) If $\alpha = 0.96 \in (\alpha_1, \alpha_2)$, Proposition 5 shows that the occurrence of local indeterminacy requires $\epsilon_{cc} \in (\epsilon_{cc}^3, \epsilon_{cc}^1)$ with $\epsilon_{cc}^1 \approx 121414$ and $\epsilon_{cc}^3 \approx 34356.7$ when $\varepsilon_{eL} = 1\%$. Assuming $\epsilon_{cc} = 34370$, the steady state is locally indeterminate if $\sigma \in (\sigma^H, \sigma^T)$ with $\sigma^H = 1.97$ and $\sigma^T = 4.87$. Moreover we have $\underline{\varepsilon} = 2.01\%$. Considering $\varepsilon_{eL} = 2.02\%$ we get $\epsilon_{cc}^1 \approx 60118.2$ and $\epsilon_{cc}^3 \approx 17018.6$. Therefore, when $\epsilon_{cc} = 17030$, the steady state is locally indeterminate if $\sigma \in (\sigma^H, \sigma^T)$ with $\sigma^H = 0.998$ and $\sigma^T = 3.47$.

iv) If $\alpha = 0.9 \in (0, \alpha_1)$, Proposition 5 shows that the occurrence of local indeterminacy requires $\epsilon_{cc} \in (\epsilon_{cc}^3, \epsilon_{cc}^1)$ and $\varepsilon_{eL} < \bar{\varepsilon} \approx 11.3$. When $\varepsilon_{eL} = 1\%$, we get $\epsilon_{cc}^1 \approx 55210.6$ and $\epsilon_{cc}^3 \approx 5579.1$. Assuming $\epsilon_{cc} = 5580$, the steady state is locally indeterminate if $\sigma \in (\sigma^H, \sigma^T)$ with $\sigma^H = 2.86$ and $\sigma^T = 3.32$. Moreover we have $\underline{\varepsilon} = 3.06\%$. Considering $\varepsilon_{eL} = 3.1\%$ we get $\epsilon_{cc}^1 \approx 17785.9$ and $\epsilon_{cc}^3 \approx 1805.4$. Therefore, when $\epsilon_{cc} = 1810$, the steady state is locally indeterminate if $\sigma \in (\sigma^H, \sigma^T)$ with $\sigma^H = 0.988$ and $\sigma^T = 3.5$.

While labor externalities are restricted to be small, these numerical illustrations show that, provided the elasticity of intertemporal substitution in consumption is large enough (but bounded away from $+\infty$), local indeterminacy of equilibria and endogenous fluctuations rely on plausible values for the elasticity of capital-labor substitution.[12] They also clearly show that there is a trade-off between the value of the share of consumption within total utility α and the critical bounds for the elasticity of intertemporal substitution in consumption ϵ_{cc} and the elasticity of capital-labor substitution

[12]Duffy and Papageorgiou (2000) consider a panel of 82 countries over a 28-year period to estimate a CES production function specification. They find that capital and labor have an elasticity of substitution significantly greater than unity (i.e. $\sigma \in [1.14, 3.24]$) in the richest group of countries.

σ: ϵ_{cc}^i, $i = 1, \ldots, 5$ are indeed increasing functions of α while σ^j, $j = H, T, F$, are decreasing functions of α. It follows that to get lower values for ϵ_{cc}^3, we need to consider low values for α but, in order to keep reasonable values for σ^H we have to increase the amount of labor externalities. For instance, if $\alpha = 0.2$ and $\varepsilon_{eL} = 4\%$ we get $\epsilon_{cc}^3 \approx 26.8$ so that choosing $\epsilon_{cc} = 27$ leads to the existence of local indeterminacy for any $\sigma \in (\sigma^H, \sigma^T)$ with $\sigma^H = 3.08$ and $\sigma^T = 3.89$.

These numerical illustrations finally show that, in accordance with Corollary 2, local indeterminacy is compatible with a Cobb-Douglas technology if the labor externality is large enough. However, the lower bound $\underline{\varepsilon}$ remains very low, and in any case much lower than the amount required in Hintermaier (2001). It is also worth mentioning that all our indeterminacy results have been obtained with a decreasing aggregate labor demand function, i.e. $\varepsilon_{eL} - s/\sigma < 0$.

6 Concluding comments

In this paper we have studied a Ramsey one-sector model with non-separable homothetic preferences, endogenous labor and productive external effects arising from average capital and labor. We have shown that indeterminacy cannot occur with capital externalities alone but that it can occur when there are only mild labor externalities provided that the elasticity of capital-labor substitution and the elasticity of intertemporal substitution in consumption to be large enough. However, when slightly larger labor externalities are considered, we have proved that local indeterminacy is fully compatible with a Cobb-Douglas technology and a negatively sloped aggregate labor demand function. We have thus shown that contrary to what the recent literature suggests, the existence of expectations-driven fluctuations under standard parameterizations for the fundamentals may be much more likely with non-separable preferences than with additively separable ones.

7 Appendix

7.1 Proof of Proposition 1

Consider equations (13): $(x^*, l^*, c^*) = (1, \bar{l}, \bar{c})$ is a steady state if and only if there exists a value for A such that:

$$\bar{c} = \bar{l}Af(1)e(\bar{l},\bar{l}) - \mu\bar{l}, \quad \frac{u_2(\bar{c},\ell-\bar{l})}{u_1(\bar{c},\ell-\bar{l})} = A[f(1) - f'(1)]e(\bar{l},\bar{l}), \quad Af'(1)e(\bar{l},\bar{l}) = \frac{\theta}{\delta}$$

Solving the third equation gives

$$A = \frac{\theta}{\delta f'(1)e(\bar{l},\bar{l})} \equiv A^*$$

and considering $A = A^*$ into the first and second equations implies

$$\bar{c} = \bar{l}\frac{\theta - s\delta\mu}{s\delta} \equiv \bar{l}\mathcal{C}, \quad \frac{u_2(\bar{l}\mathcal{C},\ell-\bar{l})}{u_1(\bar{l}\mathcal{C},\ell-\bar{l})} \equiv g(\bar{l}) = \frac{\theta}{\delta}\frac{1-s}{s}$$

with $s = s(1)$. Under Assumption 3 we get $\lim_{\bar{l}\to 0} g(\bar{l}) = 0$ and $\lim_{\bar{l}\to \ell} g(\bar{l}) = +\infty$ with $g'(\bar{l}) > 0$. It follows that there exists a unique NSS with $x^* = 1$ and $l^* = \bar{l} \in (0, \ell)$.

□

7.2 Proof of Proposition 2

Consider the system of difference equations of order one in K and l:

$$\begin{aligned} l_tAf(x_t)e(K_t,l_t) + (1-\mu)K_t - c(K_t,l_t) - K_{t+1} &= 0 \\ \delta R_{t+1}u_1(c(K_{t+1},l_{t+1}),\ell - l_{t+1}) - u_1(c(K_t,l_t),\ell - l_t) &= 0 \end{aligned} \quad (19)$$

with $c(K_t, l_t)$ the solution of

$$-u_2(c_t, \ell - l_t) + w_tu_1(c_t, \ell - l_t) = 0 \quad (20)$$

From the prices (1) we derive

$$\frac{dw}{dK} = \frac{w}{K}\left(\varepsilon_{eK} + \frac{s}{\sigma}\right), \qquad \frac{dw}{dl} = \frac{w}{l}\left(\varepsilon_{eL} - \frac{s}{\sigma}\right)$$

$$\frac{dR}{dK} = \frac{R-(1-\mu)}{K}\left(\varepsilon_{eK} - \frac{1-s}{\sigma}\right), \quad \frac{dR}{dl} = \frac{R-(1-\mu)}{l}\left(\varepsilon_{eL} + \frac{1-s}{\sigma}\right)$$

Since the utility function is defined over consumption and leisure, we can define the following elasticities:

$$\epsilon_{cc} = -\frac{u_1}{u_{11}c},\ \epsilon_{\mathcal{L}c} = -\frac{u_2}{u_{21}c},\ \epsilon_{c\mathcal{L}} = -\frac{u_1}{u_{12}\mathcal{L}},\ \epsilon_{\mathcal{L}\mathcal{L}} = -\frac{u_2}{u_{22}\mathcal{L}}$$

However, it is more convenient to write the linearized dynamical system in terms of elasticities with respect to labor. Let $v(c,l) \equiv u(c, \ell - l)$. We easily get $v_1(c,l) = u_1(c, \ell - l)$, $v_2(c,l) = -u_2(c, \ell - l)$, $v_{12}(c,l) = -u_{12}(c, \ell - l)$ and $v_{22}(c,l) = u_{22}(c, \ell - l)$ and thus:

$$\epsilon_{lc} = -\frac{v_2}{v_{21}c} = \epsilon_{\mathcal{L}c}, \quad \epsilon_{cl} = -\frac{v_1}{v_{12}l} = -\epsilon_{c\mathcal{L}}\frac{\ell-l}{l}, \quad \epsilon_{ll} = -\frac{v_2}{v_{22}l} = -\epsilon_{\mathcal{L}\mathcal{L}}\frac{\ell-l}{l}$$

Since $v(c,l)$ is decreasing and convex with respect to l, $\epsilon_{ll} < 0$.

Considering the shares (2) and (6), equation (20) at the NSS becomes

$$\frac{l}{\ell-l} = \frac{\alpha}{1-\alpha}\frac{\theta}{\delta c}\frac{1-s}{s}$$

Using then equations (19) evaluated at the NSS we get

$$\frac{l}{\ell-l} = \frac{\alpha}{1-\alpha}\frac{\theta(1-s)}{\theta-s\delta\mu}$$

From the homogeneity property of the utility function we know that $u_{12} = -(c/\mathcal{L})u_{11}$ and $u_{22} = (c/\mathcal{L})^2 u_{11}$. We then derive:

$$\epsilon_{lc} = -\epsilon_{cc}\frac{1-\alpha}{\alpha} < 0,\ \epsilon_{cl} = \epsilon_{cc}\frac{1-\alpha}{\alpha}\frac{\theta-s\delta\mu}{\theta(1-s)} > 0,\ \epsilon_{ll} = -\epsilon_{cc}\left(\frac{1-\alpha}{\alpha}\right)^2\frac{\theta-s\delta\mu}{\theta(1-s)} < 0$$

and we get $\epsilon_{cc}\epsilon_{ll} - \epsilon_{cl}\epsilon_{lc} = 0$. Notice also that from a total differenciation of equation (20) evaluated at the NSS, we can define the elasticity of the labor supply with respect to the real wage as follows:

$$\frac{dl}{dw}\frac{w}{l} \equiv \epsilon_{lw} = -\alpha\epsilon_{ll} > 0 \tag{21}$$

Thus, for any $\alpha \in (0,1)$, ϵ_{lw} may be equivalently appraised through ϵ_{ll}.

We may now compute the derivatives of $c(K_t, l_t)$:

$$\frac{dc}{dK} = \frac{c}{K}\left(\varepsilon_{eK} + \frac{s}{\sigma}\right)(1-\alpha)\epsilon_{cc}, \quad \frac{dc}{dl} = \frac{c}{l}\left[\left(\varepsilon_{eL} - \frac{s}{\sigma}\right)(1-\alpha)\epsilon_{cc} - \frac{\alpha}{1-\alpha}\frac{\theta(1-s)}{\theta - s\delta\mu}\right]$$

Tedious computations based on these results allow to get from (19):

$$\begin{pmatrix} \frac{dK_{t+1}}{K} \\ \frac{dl_{t+1}}{l} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{J_{21}}{J_{22}} & \frac{1}{J_{22}} \end{pmatrix} \begin{pmatrix} J_{11} & J_{12} \\ (1-\alpha)\left(\varepsilon_{eK} + \frac{s}{\sigma}\right) & (1-\alpha)\left(\varepsilon_{eL} - \frac{s}{\sigma}\right) \end{pmatrix} \begin{pmatrix} \frac{dK_t}{K} \\ \frac{dl_t}{l} \end{pmatrix}$$

with

$$J_{11} = \frac{\theta}{s\delta}\left(\varepsilon_{eK} + s\right) + 1 - \mu + \epsilon_{cc}(1-\alpha)\left(\varepsilon_{eK} + \frac{s}{\sigma}\right)\frac{\theta - s\delta\mu}{s\delta}$$

$$J_{12} = \frac{\theta}{s\delta}\left(\varepsilon_{eL} + 1 - s\right) + \epsilon_{cc}(1-\alpha)\left(\varepsilon_{eL} - \frac{s}{\sigma}\right)\frac{\theta - s\delta\mu}{s\delta} - \frac{\alpha}{1-\alpha}\frac{\theta(1-s)}{s\delta}$$

$$J_{21} = \theta\left(\varepsilon_{eK} - \frac{1-s}{\sigma}\right) + (1-\alpha)\left(\varepsilon_{eK} + \frac{s}{\sigma}\right), \quad J_{22} = \theta\left(\varepsilon_{eL} + \frac{1-s}{\sigma}\right) + (1-\alpha)\left(\varepsilon_{eL} - \frac{s}{\sigma}\right)$$

The characteristic polynomial follows after straightforward simplifications.

□

7.3 Proof of Lemma 1

Consider the starting point $(\mathcal{T}_0, \mathcal{D}_0)$ with

$$\mathcal{D}_0 = \frac{\mathcal{A}_1}{\delta \mathcal{A}_3} > 1, \quad \mathcal{T}_0 = 1 + \mathcal{D}_0 + \frac{\mathcal{A}_5 - \epsilon_{cc}\mathcal{A}_6}{\mathcal{A}_3} \tag{22}$$

Since Δ_0 is a horizontal half-line starting at $(\mathcal{T}_0, \mathcal{D}_0)$ when $\epsilon_{cc} = 0$ and ending at $(-\infty, \mathcal{D}_0)$ when $\epsilon_{cc} = +\infty$, it crosses line AC if and only if $1 - \mathcal{T}_0 + \mathcal{D}_0 < 0$, i.e. if the starting point lies on the right of line AC when $\epsilon_{cc} = 0$. This is indeed the case since, from (22), we have $1 - \mathcal{T}_0 + \mathcal{D}_0 = -\mathcal{A}_5/\mathcal{A}_3 < 0$ when $\epsilon_{cc} = 0$. Therefore Δ_0 always crosses the line AC for some value of $\epsilon_{cc} \in (0, +\infty)$. Consider now the end point $(\mathcal{T}_\infty, \mathcal{D}_\infty)$ with

$$\mathcal{D}_\infty = -\frac{1-\alpha}{\delta(\theta - 1 + \alpha)}, \quad \mathcal{T}_\infty = \frac{\theta - (1-\alpha)(1+\delta)}{\delta(\theta - 1 + \alpha)}$$

and the following derivative:

$$\frac{\partial \mathcal{D}}{\partial \sigma} = -\frac{\theta \varepsilon_{e,L}[(1-\alpha)\varepsilon_{eL}(\theta-1+\alpha)+(1-s)\theta+(1-\alpha)s]}{\delta(\mathcal{A}_3+\sigma\mathcal{A}_4)^2}$$

We easily derive the following results:

a) Let $\alpha > 1-\theta \equiv \alpha_1$. Then $\mathcal{D}_\infty < 0$, $\partial\mathcal{D}/\partial\sigma < 0$ and $1-\mathcal{T}_\infty+\mathcal{D}_\infty < 0$. Moreover $1+\mathcal{T}_\infty+\mathcal{D}_\infty > 0$ if and only if $\alpha > 1-\theta/2 \equiv \alpha_2 \in (\alpha_1, 1)$. We also get $\mathcal{D}_\infty < -1$ if and only if $\alpha < \alpha_3 = 1-\delta\theta/(1+\delta)$, with $\alpha_3 \in (\alpha_2, 1)$.

b) Let $\alpha < 1-\theta \equiv \alpha_1$. Then $1-\mathcal{T}_\infty+\mathcal{D}_\infty > 0$ and $1+\mathcal{T}_\infty+\mathcal{D}_\infty > 0$. Moreover we have $\mathcal{D}_\infty > \mathcal{D}_0 > 1$ and $\partial\mathcal{D}/\partial\sigma < 0$ if and only if $\varepsilon_{eL} < [(1-s)\theta+(1-\alpha)s]/[(1-\alpha)(1-\alpha-\theta)] \equiv \bar{\varepsilon}$. Notice that the denominator of $\mathcal{D}$ is equal to zero when

$$\sigma = \sigma^* \equiv \frac{(1-s)\theta+(1-\alpha)s}{\varepsilon_{eL}(1-\theta-\alpha)} > 0$$

Since under $\varepsilon_{eL} < \bar{\varepsilon}$, $\mathcal{D}(\sigma)$ is a decreasing function, we get $\lim_{\sigma\to\sigma^*_-}\mathcal{D}(\sigma) = -\infty$ while $\lim_{\sigma\to\sigma^*_+}\mathcal{D}(\sigma) = +\infty$

□

7.4 Proof of Proposition 4

Before proving Proposition 4 we have to examine the intersections of $\Delta(\mathcal{T})$ with points A, B and C, and the intersections of Δ_0 with lines AC and AB.

lemma 7.1. *Under Assumptions 1-4, there exist:*

i) $\epsilon^1_{cc} > 0$ *such that* $\Delta(\mathcal{T})$ *crosses* $(\mathcal{T},\mathcal{D}) = (-2,1)$,

ii) $\epsilon^2_{cc} > 0$ *such that* $\Delta(\mathcal{T})$ *crosses* $(\mathcal{T},\mathcal{D}) = (0,-1)$ *if and only if* $\alpha \in (0,\alpha_3)$ *with* $\alpha_3 = 1-\delta\theta/(1+\delta)$,

iii) $\epsilon^3_{cc} > 0$ *such that* $\Delta(\mathcal{T})$ *crosses* $(\mathcal{T},\mathcal{D}) = (2,1)$,

iv) $\epsilon^4_{cc} > 0$ *such that* Δ_0 *crosses the line* AC, *i.e.* $1-\mathcal{T}_0+\mathcal{D}_0 = 0$,

v) $\epsilon^5_{cc} > 0$ *such that* Δ_0 *crosses the line* AB, *i.e.* $1+\mathcal{T}_0+\mathcal{D}_0 = 0$.

Moreover the following rankings hold:

a) when $\alpha \in (\alpha_3, 1)$, $\epsilon^1_{cc} > \epsilon^5_{cc} > \epsilon^3_{cc} > \epsilon^4_{cc} > 0$,

b) when $\alpha \in (\alpha_2, \alpha_3)$, $\epsilon^2_{cc} > \epsilon^1_{cc} > \epsilon^5_{cc} > \epsilon^3_{cc} > \epsilon^4_{cc} > 0$, *c) when* $\alpha \in (\alpha_1, \alpha_2)$, $\epsilon_{cc}^5 > \epsilon_{cc}^1 > \epsilon_{cc}^2 > \epsilon_{cc}^3 > \epsilon_{cc}^4 > 0$,

d) when $\alpha \in (0, \alpha_1)$ *and* $\varepsilon_{eL} \in (0, \bar{\varepsilon})$, $\epsilon_{cc}^5 > \epsilon_{cc}^1 > \epsilon_{cc}^2 > \epsilon_{cc}^3 > \epsilon_{cc}^4 > 0$.

Proof: Under Assumptions 1-4, solving $\Delta(\mathcal{T})$ as defined in (15) with respect to ϵ_{cc} gives

$$\epsilon_{cc} = \frac{(1-\mathcal{T}+\mathcal{D})(\mathcal{A}_1\mathcal{A}_4+\mathcal{A}_2\mathcal{A}_3)+\mathcal{A}_5(\mathcal{A}_2+\delta\mathcal{D}\mathcal{A}_4)+\mathcal{A}_7(\mathcal{A}_1-\delta\mathcal{D}\mathcal{A}_3)}{\mathcal{A}_6(\mathcal{A}_2+\delta\mathcal{D}\mathcal{A}_4)}$$

We then easily derive

$$\epsilon_{cc}^1 = \frac{4(\mathcal{A}_1\mathcal{A}_4+\mathcal{A}_2\mathcal{A}_3)+\mathcal{A}_5(\mathcal{A}_2+\delta\mathcal{A}_4)+\mathcal{A}_7(\mathcal{A}_1-\delta\mathcal{A}_3)}{\mathcal{A}_6(\mathcal{A}_2+\delta\mathcal{A}_4)}$$

$$\epsilon_{cc}^2 = \frac{\mathcal{A}_5(\mathcal{A}_2-\delta\mathcal{A}_4)+\mathcal{A}_7(\mathcal{A}_1+\delta\mathcal{A}_3)}{\mathcal{A}_6(\mathcal{A}_2-\delta\mathcal{A}_4)}, \qquad \epsilon_{cc}^3 = \frac{\mathcal{A}_5(\mathcal{A}_2+\delta\mathcal{A}_4)+\mathcal{A}_7(\mathcal{A}_1-\delta\mathcal{A}_3)}{\mathcal{A}_6(\mathcal{A}_2+\delta\mathcal{A}_4)}$$

Notice that since $\lim_{\epsilon_{cc}\to+\infty} \mathcal{S} = 0$, the critical value ϵ_{cc}^2 cannot exist if the end point $(\mathcal{T}_\infty, \mathcal{D}_\infty)$ is such that $\mathcal{D}_\infty \in (-1, 0)$. As shown in Lemma 1, $\mathcal{D}_\infty \in (-\infty, -1) \cup (0, +\infty)$ if and only if $\alpha \in (0, \alpha_3)$.

Solving $1-\mathcal{T}_0+\mathcal{D}_0 = 0$ and $1+\mathcal{T}_0+\mathcal{D}_0 = 0$ with respect to ϵ_{cc} respectively gives

$$\epsilon_{cc}^4 = \frac{\mathcal{A}_5}{\mathcal{A}_6}, \quad \epsilon_{cc}^5 = \frac{\delta\mathcal{A}_5+2(\mathcal{A}_1+\delta\mathcal{A}_3)}{\delta\mathcal{A}_6}$$

A last step consists in ranking all the critical bounds ϵ_{cc}^i, $i = 1, \ldots, 5$. The segment Δ, having a fixed point $(\mathcal{T}_\infty, \mathcal{D}_\infty)$, continuously rotates counter-clockwise (clockwise) when $\alpha > \alpha_1$ ($\alpha < \alpha_1$) as ϵ_{cc} goes from 0 to $+\infty$. The rankings then directly follow from geometrical arguments based on Figures 4, 5, 6 and 7. Since the lowest bound ϵ_{cc}^4 is strictly positive, all the other critical values belong to $(0, +\infty)$. Recall however that ϵ_{cc}^2 cannot exist if $\alpha \in (\alpha_3, 1)$.

□

We may now prove Proposition 4. All the local stability results are derived from Lemmas 1, 7.1 and Figure 4. For a given value of ϵ_{cc}, it remains now to compute the bifurcation values of the elasticity of capital-labor substitution σ. The flip bifurcation value σ^F is such that $\Delta(\mathcal{T})$ crosses the line AB, i.e.

is a solution of $\mathcal{D} + \mathcal{T} + 1 = 0$. The Hopf bifurcation value σ^H is such that $\Delta(\mathcal{T})$ crosses the line BC, i.e. is a solution of $\mathcal{D} = 1$. The transcritical bifurcation value σ^T is such that $\Delta(\mathcal{T})$ crosses the line AC, i.e. is a solution of $\mathcal{D} - \mathcal{T} + 1 = 0$. Considering Proposition 2, straightforward computations give the expressions of these bifurcation values:

$$\sigma^F = -\frac{2(\mathcal{A}_1+\delta\mathcal{A}_3)+\delta(\mathcal{A}_5-\mathcal{A}_6\epsilon_{cc})}{2(\delta\mathcal{A}_4-\mathcal{A}_2)+\delta\mathcal{A}_7}, \quad \sigma^H = \frac{\mathcal{A}_1-\delta\mathcal{A}_3}{\mathcal{A}_2+\delta\mathcal{A}_4}, \quad \sigma^T = \frac{\mathcal{A}_6\epsilon_{cc}-\mathcal{A}_5}{\mathcal{A}_7} \tag{23}$$

□

7.5 Proof of Corollary 1

Consider σ^F and σ^T as defined in (23). From (16) we have $\mathcal{A}_i > 0$, $i = 1, \ldots, 7$, and we easily get $2(\delta\mathcal{A}_4 - \mathcal{A}_2) + \delta\mathcal{A}_7 > 0$ if and only if $\alpha > \alpha_1$. It follows that when $\alpha > \alpha_3$, $\lim_{\epsilon_{cc}\to+\infty} \sigma^F = \lim_{\epsilon_{cc}\to+\infty} \sigma^T = +\infty$. □

7.6 Proof of Proposition 5

All the local stability results are derived from Lemmas 1, 7.1 and Figure 5. As in Proposition 4, the bifurcation values of σ are given by (23). □

7.7 Proof of Corollary 2

Propositions 4 and 5 clearly show that $\sigma > \sigma^H$ is a necessary condition for the occurrence of local indeterminacy. Then for a Cobb-Douglas technology to be compatible with local indeterminacy, $\sigma^H < 1$ must be verified. Using (23) we get a condition on the amount of labor externalities which is valid for any $\alpha \in (0, 1)$:

$$\sigma^H < 1 \quad \Leftrightarrow \quad \epsilon_{e,L} > \underline{\varepsilon} \equiv \frac{(1-\delta)[(1-s)\theta+(1-\alpha)s]}{(1-\delta)(1-\theta)+\alpha\delta\mu}$$

Notice then that when $\alpha \in (0, \alpha_1)$, local indeterminacy also requires $\epsilon_{e,L} < \bar{\varepsilon}$. It is then easy to see that $\underline{\varepsilon} < \bar{\varepsilon}$ for any $\alpha \in (0, \alpha_1)$. □

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