Notes for the Lecture

Nonlinear dynamics and indeterminacy in multisector growth models

presented at the XIIIth Summer School of the European Economic Association on "Endogenous Fluctuations"
Universidade Catolica Portuguesa, September 9-14, 2002

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1 Multisector optimal growth models

1.1 Some basic facts on one sector models

Consider a standard discrete-time one-sector optimal growth model with inelastic labor supply. Consider the following variables at time t:
- L_t: the labor supply which is proportional to population;
- K_t: the capital stock;
- Y_t: the production which is given by a standard neo-classical technology $F(k_t, L_t)$ homogeneous of degree one and concave;
- C_t: the consumption;
- $I_t = K_{t+1} - K_t$: the net investment;
- μK_t: the replacement investment with $\mu \in [0,1]$ the rate of depreciation of capital.

In a closed economy the equilibrium on the good market implies

$$F(K_t, L_t) = C_t + I_t + \mu K_t \quad \Leftrightarrow \quad K_{t+1} = F(K_t, L_t) + (1-\mu)K_t - C_t$$

Assuming that population grows at a constant rate n, i.e. $L_{t+1} = (1+n)L_t$, we may rewrite the previous equation in intensive form with $k_t = K_t/L_t$ and $c_t = C_t/L_t$:

$$(1+n)k_{t+1} = f(k_t) + (1-\mu)k_t - c_t$$

with $f(k) = F(k, 1)$. To simplify the exposition we assume that investment is reversible.[1]

Assumption 1 . *$f(k)$ is C^2 and such that for any $k > 0$, $f'(k) > 0$, $f''(k) < 0$, $f(0) = 0$, $f'(0) = +\infty$ and $f'(+\infty) = 0$.*

We consider the existence of a representative consumer who maximises the discounted sum of utility derived from consumption. His utility function $u(c)$ is given by:

Assumption 2 . *$u(c)$ is C^2 and such that for any $c > 0$, $u'(c) > 0$, $u''(c) < 0$, $u(0) = 0$, $u'(0) = +\infty$ and $u'(+\infty) = 0$.*

The maximisation program is

$$\begin{array}{ll} \max\limits_{\{c_t\}_{t=0}^{+\infty}} & \sum\limits_{t=0}^{+\infty} \delta^t u(c_t) \\ s.t. & (1+n)k_{t+1} = f(k_t) + (1-\mu)k_t - c_t \\ & k_0 \text{ given} \end{array} \tag{1}$$

[1]This assumption does not fundamentaly modify the results.

with $\delta \in (0,1]$ the discount factor which characterizes the rate of time preferences. We may then define the indirect utility function

$$V(k_t, k_{t+1}) = u(f(k_t) - (1+n)k_{t+1} + (1-\mu)k_t)$$

Under Assumptions 1-2, $V(x,y)$ is increasing with respect to x, decreasing with respect to y and strictly concave. The maximisation program (1) may be written as follows

$$\begin{array}{ll} \max_{\{k_t\}_{t=0}^{+\infty}} & \sum_{t=0}^{+\infty} \delta^t V(k_t, k_{t+1}) \\ s.t. & (k_t, k_{t+1}) \in \mathcal{D} \\ & k_0 \text{ given} \end{array}$$

with

$$\mathcal{D} = \left\{ (k_t, k_{t+1}) \in \mathbb{R}_+^2 / \frac{(1-\mu)k_t}{1+n} \leq k_{t+1} \leq \frac{f(k_t) + (1-\mu)k_t}{1+n} \right\}$$

the set of admissible paths. The first order condition for an interior maximum is given by the Euler equation

$$V_2(k_t, k_{t+1}) + \delta V_1(k_{t+1}, k_{t+2}) = 0$$

which is a second-order non-linear implicit difference equation. We also need to satisfy the transversality condition

$$\lim_{t \to +\infty} \delta^t k_t V_1(k_t, k_{t+1}) = 0$$

It is therefore easy to derive

$$\begin{array}{rcl} V_1(k_t, k_{t+1}) & = & u'(c_t)[f'(k_t) + 1 - \mu] \\ V_2(k_t, k_{t+1}) & = & -(1+n)u'(c_t) \end{array}$$

The steady state is obtained considering $k_{t+1} = k_t = k_\delta^*$ in the Euler equation. The modified golden rule k_δ^* is obtained as the unique solution of:

$$f'(k_\delta^*) = \frac{1+n}{\delta} - (1-\mu)$$

Notice that if $\delta = 1$ we get the well-known golden rule $f'(k_\delta^*) = \mu + n$. We also get $c_\delta^* = f(k_\delta^*) - (\mu+n)k_\delta^* > 0$.

Linearizing the Euler equation around the modified golden rule gives the following characteristic polynomial

$$P(\lambda) = \lambda^2 + \lambda \frac{\delta V_{11}^* + V_{22}^*}{\delta V_{12}^*} + \delta^{-1} = 0$$

with $V_{ij}^* = V_{ij}(k_\delta^*, k_\delta^*)$. We then derive that the two roots λ_1 and λ_2 satisfy

$$\lambda_1\lambda_2 = \delta^{-1} \geq 1$$
$$\lambda_1 + \lambda_2 = -\frac{\delta V_{11}^* + V_{22}^*}{\delta V_{12}^*}$$

From the definition of the modified golden rule we get

$$V_{12}^* = -(1+n)^2 u''(c_\delta^*)\delta^{-1} > 0$$

It follows that both roots are positive and at least one is greater than 1. Considering that

$$V_{11}^* = (1+n)^2 u''(c_\delta^*)\delta^{-2} + u'(c_\delta^*)f''(k_\delta^*)$$
$$V_{22}^* = (1+n)^2 u''(c_\delta^*)$$

we get

$$P(1) = -\frac{\delta u'(c_\delta^*)f''(k_\delta^*)}{(1+n)^2 u''(c_\delta^*)} < 0$$

Since $P(0) = \delta^{-1} > 0$ and $\lim_{\lambda \to +\infty} P(\lambda) = +\infty$, we conclude that one root is necessarily less than 1. The steady state is thus saddle-point stable and uniqueness allows to state the following Turnpike Theorem:[2]

Theorem 1 . *Under Assumptions 1-2, for any given k_0 and $\delta \in (0,1]$, there exists one unique optimal path which monotonically converges to the modified golden rule k_δ^*.*

1.2 Two sector models

We consider now a two-sector optimal growth model with one pure consumption good y_0 and one capital good y. The labor supply is still assumed to be inelastic. Total labor is normalised to 1 and each good is produced with a standard constant returns to scale technology:

$$y_0 = f^0(k_0, l_0)$$
$$y = f^1(k_1, l_1)$$

with $k_0 + k_1 \leq k$, k being the total stock of capital, and $l_0 + l_1 \leq 1$.

Assumption 3 . *Each production function $f^i : \mathbb{R}_+^2 \to \mathbb{R}_+$, $i = 0, 1$, is C^2, increasing in each argument, concave, homogeneous of degree one and such that for any $x > 0$, $f_1^i(0,x) = f_2^i(x,0) = +\infty$, $f_1^i(+\infty, x) = f_2^i(x, +\infty) = 0$.*

[2]See Michel [23] for a presentation of the decentralized equilibrium and for a discussion of the influence of debt and public spending on the equilibrium.

For any given (k, y), we define a temporary equilibrium by solving the following problem of optimal allocation of productive factors between the two sectors:

$$\begin{array}{rll} T(k,y) = & \max\limits_{k_0,k_1,l_0,l_1} & f^0(k_0,l_0) \\ & s.t. & y \leq f^1(k_1,l_1) \\ & & k_0 + k_1 \leq k \\ & & l_0 + l_1 \leq 1 \\ & & k_0, k_1, l_0, l_1 \geq 0 \end{array} \tag{2}$$

The value function $T(k, y)$ is called the social production function and describes the frontier of the production possibility set. Since Benhabib and Nishimura [5], we know that because of the constant returns to scale of technologies, $T(k, y)$ is concave non-strictly so that its Hessian matrix

$$H_T = \begin{pmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{pmatrix}$$

is singular, which means $|H_T| = T_{11}T_{22} - T_{12}^2 = 0$. Moreover, in another contribution, Benhabib and Nishimura [4] have shown that $T(k, y)$ is C^1. We will assume in the following that $T(k, y)$ is at least C^2.

We introduce a slightly different assumption for the preferences of the representative agent in order to have the possibility to consider a linear utility function.

Assumption 4 . *$u(c)$ is C^2 and such that for any $c > 0$, $u'(c) > 0$, $u''(c) \leq 0$, $u(0) = 0$.*

$T(k, y)$ gives the maximum production level of the consumption good which will be entirely consumed by the representative agent. Under Assumption 4, we have indeed $c_t = T(k_t, y_t)$ at each time $t \geq 0$. The maximisation program of the representative agent is

$$\begin{array}{ll} \max\limits_{\{y_t\}_{t=0}^{+\infty}} & \sum\limits_{t=0}^{+\infty} \delta^t u(T(k_t, y_t)) \\ s.t. & (1+n)k_{t+1} = y_t + (1-\mu)k_t \\ & k_0 \text{ given} \end{array} \tag{3}$$

As in the previous section, we may define the indirect utility function:

$$V(k_t, k_{t+1}) = u(T(k_t, (1+n)k_{t+1} - (1-\mu)k_t))$$

and the maximisation program (3) may be written as follows

$$\begin{array}{ll} \max_{\{k_t\}_{t=0}^{+\infty}} & \sum_{t=0}^{+\infty} \delta^t V(k_t, k_{t+1}) \\ s.t. & (k_t, k_{t+1}) \in \mathcal{D} \\ & k_0 \text{ given} \end{array}$$

with

$$\mathcal{D} = \left\{ (k_t, k_{t+1}) \in \mathbb{R}_+^2 / \frac{(1-\mu)k_t}{1+n} \leq k_{t+1} \leq \frac{f^1(k_t, 1) + (1-\mu)k_t}{1+n} \right\}$$

the set of admissible paths. The first order condition for an interior maximum is given by the Euler equation

$$V_2(k_t, k_{t+1}) + \delta V_1(k_{t+1}, k_{t+2}) = 0$$

which is a second-order non-linear implicit difference equation. We also need to satisfy the transversality condition

$$\lim_{t \to +\infty} \delta^t k_t V_1(k_t, k_{t+1}) = 0$$

Before going into the analysis of the dynamical properties of the Euler equation, we need to characterise precisely the properties of the social production function $T(k, y)$.

1.2.1 A characterisation of $T(k, y)$

Consider the optimisation program (13). The associated Lagrangian is

$$\mathcal{L} = f^0(k_0, l_0) + q[f^1(k_1, l_1) - y] + w[k - k_0 - k_1] + w_0[1 - l_0 - l_1]$$

with q the price of the investment good, w the rental rate of capital and w_0 the wage rate, all in terms of the price of the consumption good. The first order conditions for a maximum are as follows:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial k_0} &= f_1^0(k_0, l_0) - w = 0 \\ \frac{\partial \mathcal{L}}{\partial l_0} &= f_2^0(k_0, l_0) - w_0 = 0 \\ \frac{\partial \mathcal{L}}{\partial k_1} &= q f_1^1(k_1, l_1) - w = 0 \\ \frac{\partial \mathcal{L}}{\partial l_1} &= q f_1^1(k_1, l_1) - w_0 = 0 \end{aligned} \tag{4}$$

We derive easily from all these conditions

$$\begin{aligned} w &= f_1^0(k_0, l_0) = q f_1^1(k_1, l_1) \\ w_0 &= f_2^0(k_0, l_0) = q f_1^2(k_1, l_1) \\ q &= \frac{f_1^0(k_0, l_0)}{f_1^1(k_1, l_1)} = \frac{f_2^0(k_0, l_0)}{f_2^1(k_1, l_1)} \end{aligned} \tag{5}$$

Moreover, solving the first order conditions give optimal demand functions for capital and labor, namely $k_0(k, y)$, $l_0(k, y)$, $k_1(k, y)$ and $l_1(k, y)$. We get:

$$\begin{aligned} T(k, y) &= f^0(k_0(k, y), l_0(k, y)) \\ y &= f^1(k_1(k, y), l_1(k, y)) \\ k &= k_0(k, y) + k_1(k, y) \\ 1 &= l_0(k, y) + l_1(k, y) \end{aligned} \tag{6}$$

We may thus compute the first partial derivatives of $T(k, y)$.

$$\begin{aligned} T_1(k, y) &= f_1^0(k_0(k, y), l_0(k, y)) \frac{\partial k_0(k, y)}{\partial k} + f_2^0(k_0(k, y), l_0(k, y)) \frac{\partial l_0(k, y)}{\partial k} \\ T_2(k, y) &= f_1^0(k_0(k, y), l_0(k, y)) \frac{\partial k_0(k, y)}{\partial y} + f_2^0(k_0(k, y), l_0(k, y)) \frac{\partial l_0(k, y)}{\partial y} \end{aligned}$$

From (6) we obtain:

$$\frac{\partial k_0(k, y)}{\partial k} = 1 - \frac{\partial k_1(k, y)}{\partial k}, \quad \frac{\partial k_0(k, y)}{\partial y} = -\frac{\partial k_1(k, y)}{\partial y}$$

$$\frac{\partial l_0(k, y)}{\partial k} = -\frac{\partial l_1(k, y)}{\partial k}, \quad \frac{\partial l_0(k, y)}{\partial y} = -\frac{\partial l_1(k, y)}{\partial y}$$

$$1 = f_1^1(k_1(k, y), l_1(k, y)) \frac{\partial k_1(k, y)}{\partial y} + f_2^1(k_1(k, y), l_1(k, y)) \frac{\partial l_1(k, y)}{\partial y}$$

$$0 = f_1^1(k_1(k, y), l_1(k, y)) \frac{\partial k_1(k, y)}{\partial k} + f_2^1(k_1(k, y), l_1(k, y)) \frac{\partial l_1(k, y)}{\partial k}$$

Substituting this into $T_1(k, y)$ and $T_2(k, y)$, and using (5), we get

$$\begin{aligned} T_1(k, y) &= f_1^0 - \left(f_1^0 \frac{\partial k_1(k, y)}{\partial k} + f_2^0 \frac{\partial l_1(k, y)}{\partial k} \right) \\ &= f_1^0 - q \left(f_1^1 \frac{\partial k_1(k, y)}{\partial k} + f_2^1 \frac{\partial l_1(k, y)}{\partial k} \right) \\ &= f_1^0(k_0(k, y), l_0(k, y)) \\ &= q f_1^1(k_1(k, y), l_1(k, y)) \\ &= w(k, y) \end{aligned}$$

$$
\begin{aligned}
T_2(k,y) &= -f_1^0\frac{\partial k_1(k,y)}{\partial y} - f_2^0\frac{\partial l_1(k,y)}{\partial y} \\
&= -q\left(f_1^1\frac{\partial k_1(k,y)}{\partial y} + f_2^1\frac{\partial l_1(k,y)}{\partial y}\right) \\
&= -\frac{f_1^0(k_0(k,y),l_0(k,y))}{f_1^1(k_1(k,y),l_1(k,y))} = -\frac{f_2^0(k_0(k,y),l_0(k,y))}{f_2^1(k_1(k,y),l_1(k,y))} \\
&= -q(k,y)
\end{aligned}
$$

Therefore $T_1(k,y)$ gives the rental rate of capital and $T_2(k,y)$ minus the price of the investment good. We have now to compute the second derivatives of $T(k,y)$. As already mentioned above, we know that $T(k,y)$ is a concave function. It follows that:

$$
\begin{aligned}
T_{11}(k,y) &= \frac{\partial w}{\partial k}(k,y) \leq 0 \\
T_{22}(k,y) &= -\frac{\partial q}{\partial y}(k,y) \leq 0
\end{aligned}
$$

However the sign of the cross derivative is not obvious:

$$
T_{12}(k,y) = \frac{\partial w}{\partial y}(k,y) = T_{21}(k,y) = -\frac{\partial q}{\partial k}(k,y)
$$

To study this derivative we start from the homogeneity property of the production functions. We have:

$$
\begin{array}{c} y_0 = k_0 f_1^0 + l_0 f_2^0 \\ y_1 = k_1 f_1^1 + l_1 f_2^1 \end{array} \quad \Leftrightarrow \quad \begin{array}{c} 1 = \frac{k_0}{y_0}w + \frac{l_0}{y_0}w_0 \\ 1 = \frac{k_1}{y_1}\frac{w}{q} + \frac{l_1}{y_1}\frac{w_0}{q} \end{array}
$$

We finally obtain:

$$
\begin{pmatrix} w_0 & w \end{pmatrix} \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = \begin{pmatrix} 1 & q \end{pmatrix} \tag{7}
$$

with

$$
a_{00} = l_0/y_0, \quad a_{10} = k_0/y_0, \quad a_{01} = l_1/y_1, \quad a_{11} = k_1/y_1
$$

the capital and labor coefficients in each sector. Equation (7) gives the factor-price frontier and corresponds to the equality between price and cost. Differentiating this equation gives:

$$
\begin{pmatrix} dw_0 & dw \end{pmatrix} \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} + \begin{pmatrix} w_0 & w \end{pmatrix} \begin{pmatrix} da_{00} & da_{01} \\ da_{10} & da_{11} \end{pmatrix} = \begin{pmatrix} 1 & dq \end{pmatrix} \tag{8}
$$

By definition we have

$$da_{00} = \frac{y_0 dl_0 - l_0 dy_0}{(y_0)^2}, \quad da_{10} = \frac{y_0 dk_0 - k_0 dy_0}{(y_0)^2}$$

$$da_{01} = \frac{y_1 dl_1 - l_1 dy_1}{(y_1)^2}, \quad da_{11} = \frac{y_1 dk_1 - k_1 dy_1}{(y_1)^2}$$

Substituting this into the second term in the l.h.s. of equation (8) gives:

$$\begin{pmatrix} w_0 & w \end{pmatrix} \begin{pmatrix} da_{00} & da_{01} \\ da_{10} & da_{11} \end{pmatrix} = \begin{pmatrix} \dfrac{y_0(w_0 dl_0 + w dk_0) - dy_0(w_0 l_0 + w k_0)}{(y_0)^2} \\ \dfrac{y_1(w_0 dl_1 + w dk_1) - dy_1(w_0 l_1 + w k_1)}{(y_1)^2} \end{pmatrix}^t$$

Using again the homogeneity property together with the price equations (5), we obtain

$$y_0 = w k_0 + w_0 l_0$$

$$q y_1 = w k_1 + w_0 l_1$$

Then we get after substitution

$$\begin{pmatrix} w_0 & w \end{pmatrix} \begin{pmatrix} da_{00} & da_{01} \\ da_{10} & da_{11} \end{pmatrix} = \begin{pmatrix} \dfrac{w_0 dl_0 + w dk_0 - dy_0}{y_0} \\ \dfrac{w_0 dl_1 + w dk_1 - q dy_1}{y_1} \end{pmatrix}^t$$

Considering finally the technologies $y_0 = f^0(k_0, l_0)$ and $y = f^1(k_1, l_1)$, total differenciation taking into account the first order conditions (27) gives

$$dy_0 = w dk_0 + w_0 dl_0$$

$$q dy_1 = w dk_1 + w_0 dl_1$$

Then[3]

$$\begin{pmatrix} w_0 & w \end{pmatrix} \begin{pmatrix} da_{00} & da_{01} \\ da_{10} & da_{11} \end{pmatrix} = 0$$

so that

$$\begin{pmatrix} dw_0 & dw \end{pmatrix} \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = \begin{pmatrix} 1 & dq \end{pmatrix}$$

Eliminating dw_0 we can solve this system to get

$$\frac{dq}{dw} = a_{01} \left(\frac{a_{11}}{a_{01}} - \frac{a_{10}}{a_{00}} \right) \equiv b$$

b is a relative capital intensity difference. The sign of b is thus positive if and only if the investment good is capital intensive. Now consider the cross derivative $T_{12}(k, y)$. We can write:

[3] This result is a direct consequence of the envelope theorem.

$$T_{12} = -\frac{\partial q}{\partial w}\frac{\partial w}{\partial k} = -T_{11}b$$

We conclude therefore that the sign of $T_{12}(k,y)$ is given by the sign of b. Notice also that $T_{22}(k,y)$ may be written as

$$\begin{aligned} T_{22} &= -\frac{\partial q}{\partial w}\frac{\partial w}{\partial y} \\ &= \left(\frac{\partial q}{\partial w}\right)^2 \frac{\partial w}{\partial k} \\ &= T_{11}b^2 \end{aligned}$$

Remark: The derivative dw/dq is well-known in trade theory as the Stolper-Samuelson effect. It follows from the above computations that

$$\frac{dw}{dq} = (b)^{-1}$$

Considering also the full employment equation derived from the stock constraints in program (13), we get the factor market clearing equation

$$\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \begin{pmatrix} y_0 \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ k \end{pmatrix}$$

At constant prices, the input coefficients remain fixed and we obtain the associated Rybczinski effect

$$\frac{dy}{dk} = (b)^{-1}$$

We therefore find the well-known duality between the Rybczinski and Stolper-Samuelson effects.

1.2.2 Dynamical analysis

We may now consider the Euler equation. It is easy to derive

$$\begin{aligned} V_1(k_t, k_{t+1}) &= u'(c_t)[T_1(k_t, y_t) - (1-\mu)T_2(k_t, y_t)] \\ V_2(k_t, k_{t+1}) &= -(1+n)u'(c_t)T_2(k_t, y_t) \end{aligned}$$

The steady state is obtained considering $k_{t+1} = k_t = k^*_\delta$, $y_{t+1} = y_t = y^*_\delta = \mu k^*_\delta$ in the Euler equation. At the steady state the Euler equation becomes

$$(1+n)T_2(k^*_\delta, \mu k^*_\delta) = \delta[T_1(k^*_\delta, \mu k^*_\delta) - (1-\mu)T_2(k^*_\delta, \mu k^*_\delta)] \quad (9)$$

Then we get:

$$
\begin{aligned}
-\frac{T_1(k_\delta^*, \mu k_\delta^*)}{T_2(k_\delta^*, \mu k_\delta^*)} &= \frac{1+n}{\delta} - (1-\mu) \\
\Leftrightarrow \quad f_1^1(k_1(k_\delta^*, \mu k_\delta^*), l_1(k_\delta^*, \mu k_\delta^*)) &= \frac{1+n}{\delta} - (1-\mu)
\end{aligned}
\tag{10}
$$

This formula provides a generalization of the modified golden rule defined in the one-sector model. The l.h.s. of the previous equation is indeed the marginal productivity of capital in the investment good sector. Applying the proof of Theorem 3.1 in Becker and Tsyganov [1] to the case of one homogeneous agent applies, we get the same result as in the standard one-sector Ramsey model:

Proposition 1 . *Under Assumptions 3-4, there exists a unique steady state k^* solution of equation* (10)*.*

We have now to linearize the Euler equation around the modified golden rule. As in the previous section we derive the following characteristic polynomial

$$P(\lambda) = \lambda^2 \delta V_{12}^* + \lambda(\delta V_{11}^* + V_{22}^*) + V_{12}^* = 0$$

with $V_{ij}^* = V_{ij}(k_\delta^*, k_\delta^*)$. Provided $V_{12}^* \neq 0$, the two roots λ_1 and λ_2 satisfy

$$
\begin{aligned}
\lambda_1 \lambda_2 &= \delta^{-1} \geq 1 \\
\lambda_1 + \lambda_2 &= -\frac{\delta V_{11}^* + V_{22}^*}{\delta V_{12}^*}
\end{aligned}
$$

Notice first that contrary to the one-sector model, the sign of the cross derivative is not obvious. Using equation (9), we have indeed

$$V_{12}^* = -\frac{(1+n)^2}{\delta} u''(c_\delta^*)(T_2^*)^2 + (1+n)u'(c_\delta^*)[T_{12}^* - (1-\mu)T_{22}^*] \tag{11}$$

It follows easily that if at the steady state the investement good is capital intensive, i.e. if $T_{12}^* \geq 0$, then the cross derivative V_{12}^* is positive. In this case, as in the one-sector model, the two roots are positive and one is necessarily greater than 1. Moreover, the optimal path is monotone. However, as we will show later on local stability depends on the value of the discount factor δ. If at the steady the consumption good is capital intensive, i.e. $T_{12}^* < 0$, the cross derivative may be negative when the utility function is not too concave. If this case appears, the optimal path oscillates. Using again equation (9), we also have:

$$V_{11}^* = \frac{(1+n)^2}{\delta} u''(c_\delta^*)(T_2^*)^2 + u'(c_\delta^*)[T_{11}^* - 2(1-\mu)T_{12}^* + (1-\mu)^2 T_{22}^*] \leq 0$$

$$V_{22}^* = (1+n)^2 u''(c_\delta^*)(T_2^*)^2 + (1+n)^2 u'(c_\delta^*) T_{22}^* \leq 0$$

We know that $T(k_t, y_t)$ is concave non-strictly. By construction, the indirect utility function $V(k_t, k_{t+1})$ is thus concave. However strict concavity is not a priori ensured. This will be however enough to characterise the discriminant of the characteristic polynomial:

$$\begin{aligned}\Delta &= (\delta V_{11}^* + V_{22}^*)^2 - 4\delta(V_{12}^*)^2 \\ &= (\delta V_{11}^* + V_{22}^* + 2\sqrt{\delta}V_{12}^*)(\delta V_{11}^* + V_{22}^* - 2\sqrt{\delta}V_{12}^*) \\ &= \begin{pmatrix} 1 & \sqrt{\delta} \end{pmatrix} \begin{pmatrix} V_{11}^* & V_{12}^* \\ V_{12}^* & V_{22}^* \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{\delta} \end{pmatrix} \begin{pmatrix} 1 & -\sqrt{\delta} \end{pmatrix} \begin{pmatrix} V_{11}^* & V_{12}^* \\ V_{12}^* & V_{22}^* \end{pmatrix} \begin{pmatrix} 1 \\ -\sqrt{\delta} \end{pmatrix}\end{aligned}$$

Since the Hessian matrix of V is negative semi-definite, we conclude that $\Delta \geq 0$, and the roots are real.

Consider now the characteristic polynomial when $\delta = 1$. We have:

$$\begin{aligned}\lambda_1\lambda_2 &= 1 \\ P(1) &= V_{11}^* + V_{22}^* + 2V_{12}^* \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} V_{11}^* & V_{12}^* \\ V_{12}^* & V_{22}^* \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \leq 0 \\ P(-1) &= -(V_{11}^* + V_{22}^* - 2V_{12}^*) \\ &= -\begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} V_{11}^* & V_{12}^* \\ V_{12}^* & V_{22}^* \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \geq 0\end{aligned}$$

Generically, even if a function is not strictly concave, a quadratic form derived from its Hessian matrix evaluated at some point, here the steady state, will be different from zero so that there does not exist any unit root. This property may depend on wether the utility function $u(c)$ is strictly concave or not. It follows that when $\delta = 1$ there is one root inside the unit circle and one root outside the unit circle. The steady state is thus saddle-point stable and we find the same turnpike property as in the one-sector model.

Consider now the case $\delta \in (0, 1)$. We have

$$\begin{aligned}\lambda_1\lambda_2 &= \delta^{-1} \\ P(1) &= \delta V_{11}^* + V_{22}^* + (1+\delta)V_{12}^* \\ P(-1) &= -(\delta V_{11}^* + V_{22}^* - (1+\delta)V_{12}^*)\end{aligned}$$

As was originally proved by Kurz [18] and Levhari and Liviatan [19], if λ is a characteristic root, then $(\delta\lambda)^{-1}$ is also a characteristic root. To get more precise results, we have to discuss two cases depending on the sign of V_{12}^*.

- Assume first that $V_{12}^* > 0$. Then $P(-1) > 0$ and a necessary and sufficient condition for the saddle-point property is $P(1) < 0$.

- Assume now that $V_{12}^* < 0$. Then $P(1) < 0$ and a necessary and sufficient condition for the saddle-point property is $P(-1) > 0$.
Despite these general properties, it may be proved for two-sector optimal growth models that if the indirect utility function satisfies $V_{12}^* > 0$, uniqueness and monotone convergence of the optimal path are obtained for any value of the discount factor.[4]

Theorem 2 . *Under Assumptions 3-4, let k_0 be in a neighbourhood of the steady state k_δ^* and $V_{12}^* > 0$. Then the optimal path is unique and converges toward k_δ^* for any $\delta \in (0, 1]$.*

Notice that this result will hold when the investment good is capital intensive at the steady state, i.e. $b > 0$. We derive indeed from equation (11) that $b > 0$ implies $V_{12}^* > 0$.

We have also the following local turnpike theorem when $V_{12}^* < 0$:

Theorem 3 . *Under Asumptions 3-4, let k_0 be in a neighbourhood of the steady state k_δ^* and $V_{12}^* < 0$. There exists $\delta^* \in (0, 1)$ such that for any $\delta \in (\delta^*, 1]$, the optimal path is unique and converges toward k_δ^*. This convergence is oscillatory. Moreover, δ^* is obtained as a solution of*

$$\delta V_{11}^* + V_{22}^* - (1 + \delta)V_{12}^* = 0$$

and the turnpike property holds as soon as

$$\delta V_{11}^* + V_{22}^* - (1 + \delta)V_{12}^* < 0 \tag{12}$$

When $V_{12}^* < 0$, we will see later on that some period-two cycles may appear while the steady state becomes locally unstable. In this case, persistent endogenous fluctuations arise. Notice from equation (11) that a necessary condition for $V_{12}^* < 0$ is a consumption good capital intensive at the steady state, i.e. $b < 0$.

In order to understand more precisely the conditions of Theorem 3, assume that population is constant, $n = 0$, and that capital fully depreciates within one period of time, $\mu = 1$. Since the consumption good is capital intensive at the steady state we have $T_{12}^* < 0$. Notice first that in this case the Hessian matrix of V evaluated at the steady state is

$$H_V^* = \begin{pmatrix} V_{11}^* & V_{12}^* \\ V_{12}^* & V_{22}^* \end{pmatrix} = u'(c_\delta^*) \begin{pmatrix} T_{11}^* & T_{12}^* \\ T_{12}^* & T_{22}^* \end{pmatrix} + u''(c_\delta^*) \begin{pmatrix} T_1^* \\ T_2^* \end{pmatrix} \begin{pmatrix} T_1^* & T_2^* \end{pmatrix}$$

[4] See for instance Bosi, Magris and Venditti [12] in which a proof of this result is given for a more general formulation with endogenous labor.

Using the Euler equation at the steady state, the determinant of H_V^* is

$$\begin{aligned} |H_V^*| &= (u'(c_\delta^*))^2|H_T^*| + (u''(c_\delta^*))^2 \left| \begin{pmatrix} T_1^* \\ -\delta T_1^* \end{pmatrix} \begin{pmatrix} T_1^* & -\delta T_1^* \end{pmatrix} \right| \\ &+ u'(c_\delta^*)u''(c_\delta^*) \begin{pmatrix} \delta T_1^* & T_1^* \end{pmatrix} H_T^* \begin{pmatrix} \delta T_1^* \\ T_1^* \end{pmatrix} \\ &= u'(c_\delta^*)u''(c_\delta^*)(T_1^*)^2[\delta^2 T_{11}^* + 2\delta T_{12}^* + T_{22}^*] \geq 0 \end{aligned}$$

As we suggested above, the Hessian matrix of V evaluated at the steady state will be non singular as soon as the utility function $u(c)$ is non linear. In particular, if $T_{12}^* < 0$ then $|H_V^*| > 0$.

Now consider equation (12):

$$\delta V_{11}^* + V_{22}^* - (1+\delta)V_{12}^* = u'(c_\delta^*)[\delta T_{11}^* + T_{22}^* - (1+\delta)T_{12}^*] + 2u''(c_\delta^*)\delta(1+\delta)(T_1^*)^2$$

It clearly appears that the curvature properties of the utility function and the social production function are very important. When $u(c)$ is very concave, the condition for the saddle-point property will be satisfied. On the contrary, when $u(c)$ is close to a linear function, condition (12) becomes

$$\delta T_{11}^* + T_{22}^* - (1+\delta)T_{12}^* < 0$$

In order to give economic intuition consider the results provided in the previous section which introduces the relative capital intensity difference b. We have indeed:

$$\delta T_{11}^* + T_{22}^* - (1+\delta)T_{12}^* = T_{11}^*(1+b)(\delta+b)$$

When the consumption good is capital intensive, we have $b < 0$ and the saddle-point property will hold as soon as $b < -1$ or $b > -\delta$. Notice however that the optimal path oscillates while converging to the steady state. On the contrary, when $\delta < 1$ and $b \in (-1, -\delta)$, the saddle-point property no longer holds and period-two cycles exist. An equivalent formulation in terms of the discount factor δ could be: $\delta < \delta^*$ with $\delta^* = -b < 1$. Benhabib and Nishimura [6] have indeed proved that under these conditions, δ^* is a Flip bifurcation value. Depending on some additional conditions based on non linear terms of the Euler equation, period-two cycles will appear either in a left or right neighbourhood of δ^* and will be either saddle-point stable or locally unstable.

Benhabib and Nishimura [6] provide an economic intuition for these results that can be summarised as follows. Assume that the consumption good is capital intensive, i.e. $b < 0$, and consider an instantaneous increase in the capital stock k_t. This results in two opposing forces:

- Since the consumption good is more capital intensive than the investment good, the trade-off in production becomes more favorable to the consumption good. Moreover, the Rybczinsky effect implies a decrease of the output of the capital good y_t. This tends to lower the investment and the capital stock in the next period k_{t+1}.

- In the next period the decrease of k_{t+1} implies again through the Rybczinsky effect an increase of the output of the capital good y_{t+1}. This mechanism is explained by the fact that the decrease of k_{t+1} improves the trade-off in production in favor of the investment good which is relatively less intensive in capital. Therefore this tends to increase the investment and the capital stock in period $t+2$, k_{t+2}. Notice also that the rise of y_{t+1} implies a decrease of the rental rate w_{t+1} and through the Stolper-Samuelson effect an increase of the price q_{t+1}.

So far the above discussion concerns the existence of oscillations but not that of persistent cycles. For cycles to be sustained, the oscillations in relative prices must not present intertemporal arbitrage opportunities. Thus possible gains from postponing consumption from periods when the marginal rate of transformation between consumption and investment is high to periods when it is low must not be worth it. In presence of a linear utility function, whether this is the case or not depends on the discount rate. A minimum level of myopia, i.e. a low enough value for the discount rate δ, is necessary.

1.3 Extensions to multisector models

Consider a multisector optimal growth model with one pure consumption good y_0 and n capital goods y_j, $j = 1, \cdots, n$. The labor supply is still assumed to be inelastic. Total labor is normalised to 1 and each good is produced with a standard constant returns to scale technology:

$$\begin{aligned} y_0 &= f^0(k_{10}, \cdots, k_{n0}, l_0) \\ y_j &= f^j(k_{1j}, \cdots, k_{nj}, l_j) \end{aligned}$$

with k_{ij} the amount of capital good i used in the production of good j, $\sum_{j=0}^n k_{ij} \leq k_i$, k_i being the total stock of capital i, and $\sum_{j=0}^n l_j \leq 1$.

Assumption 5 . *Each production function $f^i : \mathbb{R}_+^2 \to \mathbb{R}_+$, $i = 0, 1, \cdots, n$, is C^2, increasing in each argument, concave and homogeneous of degree one.*

For any given $(k, y) = (k_1, \cdots, k_n, y_1, \cdots, y_n)$, we solve the following problem of optimal allocation of productive factors between the $n+1$ sectors:

$$\begin{aligned} T(k, y) = \quad & \max_{k_{ij}, l_j} \quad f^0(k_{10}, \cdots, k_{n0}, l_0) \\ & s.t. \quad y_j \leq f^j(k_{1j}, \cdots, k_{nj}, l_j) \quad j = 1, \cdots, n \\ & \qquad \sum_{i=0}^{n} k_{ji} \leq k_j \quad j = 1, \cdots, n \\ & \qquad \sum_{i=0}^{n} l_i \leq 1 \\ & \qquad k_{ij}, l_i \geq 0 \quad i = 0, \cdots, n, \;\; j = 1, \cdots, n \end{aligned} \tag{13}$$

As shown in Benhabib and Nishimura [5], the social production function $T(k, y)$ is concave non-strictly. We will use again Assumption 4 to describe the preferences of the representative agent.

1.3.1 Discrete-time models

The maximisation program of the representative agent is:

$$\begin{aligned} \max_{\{y_{jt}\}_{t=0}^{+\infty}} \quad & \sum_{t=0}^{+\infty} \delta^t u(T(k_t, y_t)) \\ s.t. \quad & (1+n)k_{jt+1} = y_{jt} + (1-\mu)k_{jt} \quad j = 1, \cdots, n \\ & k_{j0} \text{ given} \quad j = 1, \cdots, n \end{aligned} \tag{14}$$

with $k_t = (k_{1t}, \cdots, k_{nt})$ and $y_t = (y_{1t}, \cdots, y_{nt})$. Without loss of generality, the rate of depreciation of capital $\mu \in [0, 1]$ is assumed to be identical accross sectors. As in the previous section, we define the indirect utility function:

$$V(k_t, k_{t+1}) = u(T(k_t, (1+n)k_{t+1} - (1-\mu)k_t))$$

and the maximisation program (14) may be written as follows

$$\begin{aligned} \max_{\{k_t\}_{t=0}^{+\infty}} \quad & \sum_{t=0}^{+\infty} \delta^t V(k_t, k_{t+1}) \\ s.t. \quad & (k_t, k_{t+1}) \in \mathcal{D} \\ & k_0 \text{ given} \end{aligned}$$

with the set of admissible paths

$$\mathcal{D} = \left\{ (k_t, k_{t+1}) \in \mathbb{R}_+^{2n} / \frac{(1-\mu)}{1+n} k_t \leq k_{t+1} \leq (1+n)^{-1}[g(k_t) + (1-\mu)k_t] \right\}$$

and $g(k_t)$ such that $T(k_t, g(k_t)) = 0$. The first order conditions for an interior maximum are given by the Euler equations

$$\frac{\partial V}{k_{jt+1}}(k_t, k_{t+1}) + \delta \frac{\partial V}{k_{jt+1}}(k_{t+1}, k_{t+2}) = 0, \quad j = 1, \cdots, n \tag{15}$$

which represent a set of n second-order non-linear implicit difference equations. We also need to satisfy the n transversality conditions

$$\lim_{t \to +\infty} \delta^t k_{jt} \frac{\partial V}{k_{jt}}(k_t, k_{t+1}) = 0, \quad j = 1, \cdots, n$$

Assumption 6 . *For any $\delta \in (0, 1]$, there exists one steady state solution $k^*_\delta = (k^*_{1\delta}, \cdots, k^*_{n\delta})$ of the system of Euler equations (15).*

It is now well-known since the contribution of Scheinkman [30] that under some non singularity conditions on the Hessian matrix of the indirect utility function $V(k_t, k_{t+1})$ at the steady state, there exists $\delta^* \in [0, 1)$ such that for any $\delta \in (\delta^*, 1]$, the steady state is saddle-point stable. When δ crosses δ^* from above, the turnpike property is lost and there may exist some endogenous fluctuations. Contrary to the two-sector case, some complex eigenvalues may exist so that we also need to discuss the possible existence of a Hopf bifurcation. Extensions to this multisector framework of some results available in two-sector models have been provided by Cartigny and Venditti [15] and Venditti [32]. The main conclusions are the following:

Theorem 4 . *Under Assumptions 4, 5 and 6, the following cases hold:*

i) If the matrix $V_{12}(x, x)$ is symmetric negative semi-definite for any $(x, x) \in int\mathcal{D}$ and the matrix $[V_{22}(x, x) - V_{12}(x, x)]$ is positive definite for any $(x, x) \in int\mathcal{D}$, then the bound $\delta^* \in (0, 1)$ is a Flip bifurcation value and period-two cycles exist in a right or left neighbourhood of δ^*.

ii) If $V_{12}(x, x)$ is non symmetric and $\bar{y}^t V_{22}(x, x)y + |\bar{y}^t V_{12}(x, x)y| > 0$ for all $(x, x) \in int\mathcal{D}$ and all $y \in \mathbb{C}^n$ such that $|y| \neq 0$ with some complex bifurcating eigenvalues, then the bound $\delta^* \in (0, 1)$ is a Hopf bifurcation value and quasi-periodic cycles exist in a right or left neighbourhood of δ^*.

This Theorem provides strong sufficient conditions for the existence of endogenous fluctuations. Notice that the symmetry assumption in *i)* is not necessary. It is introduced to rule out the existence of complex eigenvalues. We may try now to get more economic intuition for these results. In particular, we are looking for similar interpretations in terms if capital intensity differences as in two-sector models. Using the same technique as in Section 1.2.1, we can show that the first derivatives of the social production function give the rental rates of capital and the prices of the investment goods:

$$
\begin{aligned}
\frac{\partial T}{\partial k_j}(k,y) &= w_j(k,y), \quad j=1,\cdots,n \\
\frac{\partial T}{\partial y_j}(k,y) &= -q_j(k,y), \quad j=1,\cdots,n
\end{aligned}
\tag{16}
$$

Moreover, we have

$$
\begin{aligned}
T_{12}(k,y) &= -T_{11}(k,y)B \\
&= -T_{11}(k,y)\begin{pmatrix} b_{110} & \cdots & b_{1n0} \\ \vdots & \ddots & \vdots \\ b_{n10} & \cdots & b_{nn0} \end{pmatrix}\begin{pmatrix} a_{01} & & 0 \\ & \ddots & \\ 0 & & a_{0n} \end{pmatrix}
\end{aligned}
\tag{17}
$$

where

$$b_{ji0} = \frac{a_{ji}}{a_{0i}} - \frac{a_{j0}}{a_{00}}$$

is the relative intensity difference in capital j between the sector of capital i and the consumption good sector. Notice that when $n = 1$, we get $B = b$ with b as defined in Section 1.2.1. Similarly, we get

$$T_{22}(k,y) = B^t T_{11}(k,y)B$$

Assume for simplification that utility is linear $u(c) = c$ and population is constant $n = 0$. Then

$$
\begin{aligned}
V_{12}(k,k) &= -T_{11}(k,\mu k)B - (1-\mu)B^t T_{11}(k,\mu k)B \\
V_{22}(k,k) - V_{12}(k,k) &= T_{11}(k,\mu k)B + (2-\mu)B^t T_{11}(k,\mu k)B
\end{aligned}
$$

Conditions *i)* in Theorem 4 require that B is a symmetric negative definite matrix. Notice that this property is similar to the condition $b < 0$ in two-sector models which means that the consumption good is capital intensive. In a multisector framework, this condition is more difficult to interpret. It implies however that the consumption good is relatively capital intensive with respect to each investment good.

Concerning the existence of quasi-periodic cycles, assume also for simplification that capital fully depreciates within one period $\mu = 1$. We then have:

$$\bar{y}^t V_{22} y + |\bar{y}^t V_{12} y| = \bar{y}^t B^t T_{11} B y + \sqrt{\bar{y}^t T_{11} B y \bar{y}^t B^t T_{11} y}$$

Since T_{11} is negative semi-definite, conditions *ii)* require that B is non symmetric positive definite.

1.3.2 Continuous-time models

In continuous time, the optimisation program of the representative agent becomes

$$\begin{array}{lll} \max\limits_{\{y_i(t)\}_{t\geq 0}} & \displaystyle\int_{t=0}^{+\infty} e^{-\delta t} u(T(k(t), y(t))dt & \\ s.t. & \dot{k}_j(t) = y_j(t) - (\mu + n)k_j(t) & j = 1, \cdots, n \\ & k_j(0) \text{ given} & j = 1, \cdots, n \end{array} \tag{18}$$

with $\delta \geq 0$ the discount rate, $n \geq 0$ the rate of growth of population and $\mu \geq 0$ the rate of depreciation of capital which is assumed to be identical accross sectors. As in discrete-time models, we can define an indirect utility function

$$U(k(t), \dot{k}(t)) = u(T(k(t), \dot{k}(t) + (\mu + n)k(t)))$$

with $k(t) = (k_1(t), \cdots, k_n(t))$. The program (19) then becomes

$$\begin{array}{ll} \max\limits_{\{y_i(t)\}_{t\geq 0}} & \displaystyle\int_{t=0}^{+\infty} e^{-\delta t} U(k(t), \dot{k}(t))dt \\ s.t. & (k(t), \dot{k}(t) \in \mathcal{D} \\ & k(0) \text{ given} \end{array} \tag{19}$$

with the set of admissible paths

$$\mathcal{D} = \left\{ (k(t), \dot{k}(t)) \in \mathbb{R}^n_+ \times \mathbb{R}^n / - (\mu + n)k(t) \leq \dot{k}(t) \leq g(k(t)) - (\mu + n)k(t) \right\}$$

and $g(k(t))$ such that $T(k(t), g(k(t))) = 0$. We now introduce the following Hamiltonian function

$$H(k(t), p(t)) = \max_{\dot{k}(t)} U(k(t), \dot{k}(t)) + p(t)\dot{k}(t)$$

with $p(t) = (p_1(t), \cdots, p_n(t))$ the vector of shadow prices of the n capital goods. It can be easily shown that since the indirect utility function is concave, the Hamiltonian $H(k(t), p(t))$ is concave in k and convex in p. The first order conditions for an interior solution give

$$p_j(t) = -\frac{\partial U}{\partial \dot{k}_j(t)}(k(t), \dot{k}(t)), \quad j = 1, \cdots, n \tag{20}$$

Applying the Pontryagin Maximum Principle leads to the following system of $2n$ nonlinear differential equations

$$\begin{array}{lll} \dot{k}_j(t) = \dfrac{\partial H}{\partial p_j(t)}(k(t), p(t)) & & j = 1, \cdots, n \\ \dot{p}_j(t) = -\dfrac{\partial H}{\partial k_j(t)}(k(t), p(t)) + \delta p_j(t) & & j = 1, \cdots, n \end{array} \tag{21}$$

The n transversality conditions may be formulated as

$$\lim_{t \to +\infty} e^{-\delta t} p_j(t) k_j(t) = 0, \quad j = 1, \cdots, n$$

Notice that it follows from equations (16) and (20) that

$$p_j(t) = u'(c(t)) q_j(t)$$

and $p_j(t)$ may be interpreted as the utility price of the investment good j.

Assumption 7 . *For any $\delta \geq 0$, there exists one steady state (k_δ^*, p_δ^*), solution of $\dot{k}(t) = \dot{p}(t) = 0$.*

The linearisation of system (21) around (k_δ^*, p_δ^*) gives the following Jacobian matrix:

$$J = \begin{pmatrix} H_{21}^* - \frac{\delta}{2} I & H_{22}^* \\ -H_{11}^* & -H_{12}^* + \frac{\delta}{2} I \end{pmatrix} + \frac{\delta}{2} I = J_1 + \frac{\delta}{2} I \tag{22}$$

with $H_{ij}^* = H_{ij}(k_\delta^*, p_\delta^*)$. As originally proved by Kurz [18], if we denote by $\theta_i(\delta)$ one eigenvalue of matrix J_1 and $\lambda_i(\delta)$ one eigenvalue of matrix J, then $\lambda_i(\delta) = \theta_i(\delta) + \delta/2$. Moreover, if $\theta_i(\delta)$ is an eigenvalue of J_1 then $-\theta_i(\delta)$ and the complex conjugate $\bar{\theta}_i(\delta)$, $-\bar{\theta}_i(\delta)$ are also eigenvalues of J_1. Hence, if all the eigenvalues of J_1 satisfy

$$|Re(\theta_i(\delta))| > \delta/2$$

with $Re(\theta_i(\delta))$ the real part of $\theta_i(\delta)$, the steady state is saddle-point stable. It can be easily shown (see Kurz [18] and Levhari and Liviatan [19]) that when $\delta = 0$ and if the Hessian matrix of the indirect utility function $U(k, \dot{k})$ is non singular when evaluated at the steady state, the turnpike property holds and the optimal path locally converges to the steady state. This convergence is monotone or oscillating depending on whether or not there exist complex eigenvalues. From this point of view, if the optimal growth model is symmetric, i.e. if the matrix of cross derivatives $U_{12}(k_\delta^*, 0)$ is symmetric, all the roots are real and the optimal path is monotone. Notice that this symmetry property obviously holds in the case of one and two-sector models so that the existence of oscillations in continuous-time models requires at least $n = 2$ investment goods.

Many papers deal with the robustness of the turnpike property in continuous time models when $\delta > 0$, for example Cass and Shell [16], Brock and Scheinkman [13], Magill [20], Benhabib and Nishimura [5], Cartigny and Venditti [14]. The main results are the following:

Theorem 5 . *Under Assumptions 4, 5 and 7, the following cases hold:*

i) If the matrix $U_{12}(x, 0) + U_{21}(x, 0)$ is negative semi-definite for any $x > 0$, then for all positive δ the steady state is saddle-point stable.

ii) If the matrix $U_{12}(x, 0) + U_{21}(x, 0)$ is positive definite for any $x > 0$, then there exists $\delta^* > 0$ such that for all $\delta < \delta^*$ the steady state is saddle-point stable.

As in two-sector discrete-time models, the dynamical properties of the optimal path depend on the matrix of cross derivatives of the indirect utility function $U_{12}(k, 0)$ and thus the matrix of cross derivatives of the social production function $T_{12}(k, y)$. We will discuss later on some possible economic interpretations based on this fact.

Using some precise measures of concavity and convexity for the Hamiltonian $H(k(t), p(t))$, Rockafellar [28] has proved a very nice result which appears to be very useful to get intuition about the loss of stability of the steady state. Rockafellar uses the following definitions:

Definition 1 . *Let $\mathbb{R}^n$ be endowed with the Euclidean norm $\|.\|$, and $D \subset \mathbb{R}^n$ be a non-empty convex set. Let $f : D \to \mathbb{R}$ be a real-valued concave function. Let α be the least upper bound of the set of real numbers a such that the function $f(x) + (1/2)a\|x\|^2$ is concave over D. If $\alpha \geq 0$, f is called α-concave, and if $\alpha > 0$, f is said to be strongly concave.*

Definition 2 . *Let $\mathbb{R}^n$ be endowed with the Euclidean norm $\|.\|$, and $D \subset \mathbb{R}^n$ be a non-empty convex set. Let $f : D \to \mathbb{R}$ be a real-valued convex function. Let β be the least upper bound of the set of real numbers b such that the function $f(x) - (1/2)b\|x\|^2$ is convex over D. If $\beta \geq 0$, f is called β-convex, and if $\beta > 0$, f is said to be strongly convex.*

Using these definitions to characterise the Hamiltonian, he provides the following result:

Theorem 6 . *Under Assumptions 4, 5 and 7, let the Hamiltonian $H(k, p)$ be α-concave in k and β-convex in p in a neighbourhood of the steady state (k^*_δ, p^*_δ), with $\alpha, \beta > 0$. Then if $\delta^2 < 4\alpha\beta$, the steady state is saddle-point stable.*

Theorem 6 gives a nice intuition which does not seem to have an equivalent in the discrete-time formulation of optimal growth models. The strong concavity-convexity properties of the Hamiltonian $H(k, p)$ require some strong concavity properties of the indirect utility function $U(k, \dot{k})$. Rockafellar shows therefore that the robustness of the saddle-point property increases with the concavity of the indirect utility function.[5]

Most of the contributions on multisector optimal growth models are based on reduced from formulations using indirect utility functions. This is the case in particular with Rockafellar [28] in which the strong concavity-convexity assumptions are made on the Hamiltonian. However, nothing is said about the conditions on the fundamentals, i.e. the utility and production functions, that should be introduced in order for these assumptions to be satisfied. Venditti [34] provides sufficient conditions for strong concavity of the indirect utility function.

Theorem 7 . *Under Assumption 4, if the following conditions are satisfied:*

i) there exists $r > 0$ *such that* $\inf_{x \geq 0} u'(x) = r$,

ii) the production function of the consumption good f^0 *is* γ*-concave with* $\gamma > 0$,

iii) the production function of each investment good f^j, $j = 1, \cdots, n$, *is concave and* L_j*-Lipschitz,*[6]

then the indirect utility function $U(k, \dot{k})$ *is* (α, β)*-concave with*

$$\alpha \geq \gamma\theta^2 r \text{ and } \beta \geq \gamma\theta^2 r L^{-2}$$

where $L = \max_j L_j$ *and* θ *a constant characterising the constraints of stock and accumulation of capital which does not depend on* γ *and* r.

This result is not completely satisfactory since condition *ii)* is not compatible with the standard assumption of constant returns to scale. Strong concavity implies indeed that the technology of the consumption good has decreasing returns. Notice also that there is no assumption on strong concavity of preferences. It could be interesting to introduce such a restriction in order to weaken the assumptions on technologies.

Consider again Theorem 6. We may also derive from this result that if the indirect utility function is not too concave, the saddle-point property may be lost for small values of the discount rate δ. Before exploring the range of values of the discount rate for which the turnpike property does not hold, we need to wonder about the properties of the optimal path when the steady state is locally unstable. Benhabib and Nishimura [4] are the

[5] Montrucchio [24] provides a turnpike Theorem for discrete-time models based on strong concavity of the indirect utility function. However the intuition appears to be much more complicated than in the continuous-time framework.

[6] This property implies that the rate of growth of each technology f^j is bounded from above.

first to provide a clear answer to this question. They show in a three-sector continuous-time optimal growth model with linear utility and Cobb-Douglas technologies that a Hopf bifurcation, and thus periodic cycles, may occur when δ crosses the bound δ^* from below. Their results have been generalised to non-linear utilities and any standard technologies by Medio [22], Cartigny and Venditti [14] and Venditti [33].

Theorem 8 . *Consider the Jacobian matrix J_1 defined by equation (22) and its eigenvalues denoted $\theta_i(\delta)$. Under Assumptions 4, 5 and 7, if the following conditions are satisfied:*

i) the matrix $U_{12}(k^*_\delta, 0)$ is non-symmetric and for all positive δ, the matrix $[U_{12}(k^*_\delta, 0) + U_{21}(k^*_\delta, 0)]$ is positive definite,

ii) the matrix

$$\mathcal{M}(\delta) = 2U_{22}(k^*_\delta, 0) + \delta[U_{12}(k^*_\delta, 0) + U_{21}(k^*_\delta, 0)]$$

becomes positive definite when δ goes from 0 to $+\infty$,[7]
then there exists $0 < \delta^* < +\infty$ and an eigenvalue $\theta_j(\delta)$ such that $|Re(\theta_j(\delta^*))| = \delta^*/2$. If moreover $\theta_j(\delta^*)$ is complex then the bound δ^* is a Hopf bifurcation value and periodic cycles exist in a right or left neighbourhood of δ^*.

In order to get economic interpretations of the conditions provided by Theorems 5 and 8, consider again the matrix B of capital intensity differences defined by equation (17). Assume for simplification that utility is linear $u(c) = c$ and population is constant $n = 0$. Then

$$U_{12}(k^*_\delta, 0) = [T_{12}(k^*_\delta, \mu k^*_\delta) + \mu T_{22}(k^*_\delta, \mu k^*_\delta)]$$

We know that $T_{11}(k^*_\delta, \mu k^*_\delta)$ is a negative semi-definite matrix. If B is also a negative semi-definite matrix then the condition *i)* in Theorem 5 will be satisfied. On the contrary if we assume that $T_{11}(k^*_\delta, \mu k^*_\delta)$ is non singular and B is positive definite, then from *ii)* in Theorem 5 the saddle-point property is only preserved for small discount rates δ. This is also a necessary conditions for the occurrence of periodic cycles as shown in Theorem 8. Notice that we find the same restriction on matrix B as for the occurrence of Hopf bifurcation in the discrete-time framework.

We may now explore the intuition derived from Theorem 6 given by Rockafellar: when the indirect utility function is not too concave, is it possible to get periodic cycles in continuous-time optimal growth models with low discounting ? This question has been partially answered by Benhabib and

[7] Note that the concavity of the indirect utility function implies that the matrix $\mathcal{M}(\delta)$ is negative semi-definite when $\delta = 0$.

Rustichini [10]. They consider the same model as Benhabib and Nishimura [4] with Cobb-Douglas technologies and linear utility function. They show that for any given discount rate δ^* strictly positive but as close as zero as we want, there exist some Cobb-Douglas technologies such that the associated optimal growth model is characterised by periodic cycles for any δ in a left or right neighbourhood of δ^*. Benhabib and Rustichini thus provide a very nice result but they do not discuss the role of the concavity properties. Using another precise concept of concavity, Venditti [33] provides some results that may give a theoretical justification of the conclusions given by Benhabib and Rustichini.

Definition 3 . *Let $\mathbb{R}^n$ be endowed with the Euclidean norm $\|.\|$, and $D \subset \mathbb{R}^n$ be a non-empty convex set. Let $f : D \rightarrow \mathbb{R}$ be a real-valued concave function. Let γ be the greatest lower bound of the set of real numbers g such that the function $f(x) + (1/2)g\|x\|^2$ is convex over D. If $+\infty > \gamma > 0$, f is called concave-γ, or equivalently weakly concave.*

It clearly appears from this definition that if a function is characterised by a small γ, then this function is close to a linear function.

The next result will be obtained under the assumption that the indirect utility function is such that $U(., \dot{k})$ is concave-γ for any admissible $\dot{k}$. Since we will consider different values for γ we now parameterise the indirect utility function by γ using the following notations

$$U^\gamma(k, \dot{k})$$

We now introduce an assumption that will be discussed later on.

Assumption 8 . *For any admissible $\dot{k}$, $U^\gamma(k, \dot{k})$ is concave-γ in a neighbourhood of $(k^*_\delta, 0)$ with $U^\gamma_{12}(k^*_\delta, 0)$ non-symmetric and $[U^\gamma_{12}(k^*_\delta, 0)+U^\gamma_{21}(k^*_\delta, 0)]$ positive definite for all positive δ. Moreover, for all δ in a neighbourhood of 0, there exist $\sigma < 1$ and a constant $c > 0$, which does not depend on γ, such that*

$$\lambda_m(\gamma) \geq c\gamma^\sigma$$

with $\lambda_m(\gamma)$ the smaller eigenvalue of the matrix $[U^\gamma_{12}(k^*_\delta, 0) + U^\gamma_{21}(k^*_\delta, 0)]$.

We may then modify Theorem 8 as follows:

Theorem 9 . *Consider the Jacobian matrix J_1 defined by equation (22) and its eigenvalues denoted $\theta_i(\delta)$. Under Assumptions 4, 5, 7 and 8, if for a given $\gamma > 0$ the matrix*

$$\mathcal{M}(\gamma, \delta) = 2U^\gamma_{22}(k^*_\delta, 0) + \delta[U^\gamma_{12}(k^*_\delta, 0) + U^\gamma_{21}(k^*_\delta, 0)] \tag{23}$$

becomes positive definite when δ *goes from* 0 *to* $+\infty$*, then there exists* $0 < \delta^* < +\infty$ *and an eigenvalue* $\theta_j(\delta)$ *such that* $|Re(\theta_j(\delta^*))| = \delta^*/2$*. If moreover* $\theta_j(\delta^*)$ *is complex then the bound* δ^* *is a Hopf bifurcation value and periodic cycles exist in a right or left neighbourhood of* δ^**. In addition,* δ^* *satisfies*

$$\delta^* < (2/c)\gamma^{1-\sigma}$$

and if γ *is chosen arbitrarily close to* 0*, then* δ^* *is arbitrarily close to* 0*.*

The last part of Assumption 8, which seems to be particularly restrictive and to have no economic interpretation, is actually very similar to the condition (23) in Theorem 9. Indeed the matrix $\mathcal{M}(\gamma, \delta)$ is positive definite if and only if the cross effects between stocks and flows are strong enough to offset the direct effects of stocks. The last part of Assumption 8 is exactly a specialisation of this property: it ensures that if the degree of weak concavity γ gets closer to 0, the matrix $[U_{12}^{\gamma}(k_{\delta}^*, 0) + U_{21}^{\gamma}(k_{\delta}^*, 0)]$ does not tend toward a zero matrix too quickly.

This Theorem confirms the intuition derived from the result of Rockafellar [28]: endogenous fluctuations may arise in continuous-time multisector optimal growth models with low discounting when the indirect utility function is not too concave.

We need now to compare this result with the conclusions reached by Benhabib and Rustichini [10]. We will assume with them that the utility function is linear so that $U(k(t), \dot{k}(t)) = T(k(t), \dot{k}(t) + (\mu + n)k(t))$. A comparison is possible if we first provide some conditions on the fundamentals to get weak concavity for the indirect utility function. Some sufficient conditions have been provided by Venditti [31]:

Proposition 2 . *Under Assumption 5, if the production function of the consumption good* f^0 *is concave-*α*,* $\alpha > 0$*, then* $T(., y)$ *and* $U(., \dot{k})$ *are concave-*γ *respectively for all admissible* y *and* $\dot{k}$ *with* $\gamma \leq \alpha(1 + n)$*.*

From this result, it is possible to check whether the conditions on the Cobb-Douglas technologies given by Benhabib and Rustichini imply lower concavity for the consumption good technology when the Hopf bifurcation value δ^* is chosen closer to 0. Some numerical simulations available in Venditti [33] confirm this fact.

2 Indeterminacy in multisector models

We are dealing with growth models with pre-determined variables, the capital stocks, and forward variables, the prices or the investment levels. Each capital stock is characterised by a given initial condition. On the contrary, the initial conditions on prices or investment levels are free and need to be determined under some equilibrium and optimality conditions. In the previous section we have analysed the saddle-point property of steady states. This means that for some given initial conditions on capital stocks, there exists at most one set of initial conditions on prices or investment levels that satisfy the equilibrium and optimality conditions. We will be concerned in his section with *local indeterminacy* of equilibria. We will say in what follows that one equilibrium is locally indeterminate if for some given initial conditions on capital stocks, there exist many sets of initial conditions on prices or investment levels that satisfy the equilibrium and optimality conditions. If an equilibrium is not locally indeterminate we will say that it is locally determinate.

2.1 Some basic facts on one sector models

We will focus our presentation on continuous-time models. Using the same notations as in Section 1.1.1, we assume now that there is a Romer-type [29] positive externality in the aggregate production function which represents the effect of knowledge on production and productivity. We consider

$$Y(t) = F(K(t), L(t), A(t))$$

with $A(t)$ the externality at time t which will be equal at the equilibrium to $K(t)/L(t)$. For any given A, $F(.,.,A)$ is increasing, concave and homogeneous of degree 1, and labor is inelastic. Without loss of generality, we will assume that the population is constant and that capital does not depreciate. In intensive formulation, the capital accumulation equation becomes:

$$\dot{k}(t) = f(k(t), A(t)) - c(t)$$

with $f(k, A) = F(k, 1, A)$.

Assumption 9 . *$f(k, A)$ is C^2 and such that for any $k, A > 0$, $f_1(k, A) > 0$, $f_2(k, A) \geq 0$, $f_{11}(k, A) < 0$, $f_{12}(k, A) \geq 0$ and $f(0, A) = 0$.*

The parameterized maximisation program of a representative consumer is

$$\begin{array}{cl} \max\limits_{\{c(t)\}_{t\geq 0}} & \displaystyle\int_{t=0}^{+\infty} e^{-\delta t} u(c(t))dt \\ s.t. & \dot{k}(t) = f(k(t), A(t)) - c(t) \\ & k(0), \{A(t)\}_{t\geq 0} \text{ given} \end{array} \tag{24}$$

We may then define the indirect utility function

$$U(k(t), \dot{k}(t), A(t)) = u(f(k(t), A(t)) - \dot{k}(t))$$

Under Assumptions 2 and 9, for any given $A > 0$, $U(x, y, A)$ is increasing with respect to x, decreasing with respect to y and strictly concave. The maximisation program (24) may be written as follows

$$\begin{array}{cl} \max_{\{c(t)\}_{t\geq 0}} & \int_{t=0}^{+\infty} e^{-\delta t} U(k(t), \dot{k}(t), A(t)) dt \\ s.t. & (k(t), \dot{k}(t)) \in \mathcal{D}(A(t)) \\ & k_0, \{A(t)\}_{t\geq 0} \text{ given} \end{array}$$

with

$$\mathcal{D}(A(t)) = \Big\{ (k(t), \dot{k}(t)) \in \mathbb{R}_+^2 / 0 \leq \dot{k}(t) \leq f(k(t), A(t)) \Big\}$$

the set of admissible paths for any given externality $A(t)$. The first order condition for an interior maximum is given by the Euler equation

$$U_1(k(t), \dot{k}(t), A(t)) + \delta U_2(k(t), \dot{k}(t), A(t)) - \frac{d}{dt} U_2(k(t), \dot{k}(t), A(t)) = 0$$

which is a second-order non-linear implicit differential equation. An equilibrium path is a parameterised solution of the Euler equation, denoted $k(t, \{A(t)\}_{t\geq 0})$, which satisfies the fixed point relationship between $k(t)$ and $A(t)$:

$$k(t, \{A(t)\}_{t\geq 0}) = A(t) \text{ for all } t \geq 0 \text{ with } k(0) = A(0)$$

We will assume that an equilibrium path solution to this fixed point problem exists. It also has to satisfy the transversality condition

$$\lim_{t \to +\infty} e^{-\delta t} k(t) U_1(k(t), \dot{k}(t), k(t)) = 0$$

Along an equilibrium path, the Euler equation easily rewrites as

$$u'(c)[f_1(k, k) - \delta] + u''(c)[(f_1(k, k) + f_2(k, k))\dot{k} - \ddot{k}] = 0$$

A steady state k^*_δ is obtained considering $\dot{k} = \ddot{k} = 0$ in the Euler equation, i.e. k^*_δ is a solution of

$$f_1(k, k) = \delta$$

Contrary to the optimal growth framework, existence and uniqueness are no longer ensured under Assumption 9. We will assume that there exists one locally unique steady state k^*_δ which gives a corresponding consumption level $c^*_\delta = f(k^*_\delta, k^*_\delta)$. Linearizing the Euler equation around this steady state gives the following characteristic polynomial

$$P(\lambda) = \lambda^2 - \lambda[\delta + f_2(k_\delta^*, k_\delta^*)] - \frac{u'(c_\delta^*)}{u''(c_\delta^*)}[f_{11}(k_\delta^*, k_\delta^*) + f_{12}(k_\delta^*, k_\delta^*)] = 0$$

We then derive that the two roots λ_1 and λ_2 satisfy

$$\begin{aligned} \lambda_1\lambda_2 &= -\frac{u'(c_\delta^*)}{u''(c_\delta^*)}[f_{11}(k_\delta^*, k_\delta^*) + f_{12}(k_\delta^*, k_\delta^*)] \\ \lambda_1 + \lambda_2 &= \delta + f_2(k_\delta^*, k_\delta^*) \geq 0 \end{aligned}$$

We obtain a generalisation of the Kurz [18] and Levhari-Liviatan [19] result: if λ is a characteristic root, then $-\lambda + \delta + f_2(k_\delta^*, k_\delta^*)$ is also a characteristic root. Local stability of the steady state depends on the sign of $\lambda_1\lambda_2$:

i) If the external effect is not too strong, i.e. $f_{11}(k_\delta^*, k_\delta^*) + f_{12}(k_\delta^*, k_\delta^*) \leq 0$, then the characteristic roots have opposite signs and the steady state is saddle-point stable.

ii) If the external effect is strong, i.e. $f_{11}(k_\delta^*, k_\delta^*) + f_{12}(k_\delta^*, k_\delta^*) > 0$, k_δ^* is locally unstable and either there exists another steady state which is saddle-point stable, or there does not exist any equilibrium path. As already proved by Boldrin and Rustichini [11] in a discrete-time framework, we get:

Theorem 10 . *Under Assumptions 2 and 9, any steady state k_δ^* is locally determinate.*

Benhabib and Farmer [2] have shown that local indeterminacy in one-sector models requires the consideration of elastic labor supply. However, their conditions for local indeterminacy imply the existence of very strong external effects which give rise to an equilibrium labor demand that increases with the wage rate.

2.2 Multisector models with sector specific externalities

In order to weaken their conditions for local indeterminacy, Benhabib and Farmer [3] consider a two-sector model with Cobb-Douglas technologies and sector-specific rather than aggregate externalities. They provide conditions which are compatible with mild externalities and downward sloping labor demand curves. However they assume that each sector is characterised by the same private technology. Benhabib and Nishimura [7] have extended their results to distinct private Cobb-Douglas technologies and provide some nice conditions in terms of capital intensity differences. Even if they still consider an elastic labor supply in order to provide a version of a standard real business cycles model, similar conditions for local indeterminacy may be obtained with inelastic labor. We will still consider this assumption in what follows.

2.2.1 A basic model of production

We consider a multisector economy with one pure consumption good y_0 and n capital goods y_j, $j = 1, \cdots, n$, in which each good is produced with a CES technology. This formulation encompasses the Cobb-Douglas case considered by Benhabib and Nishimura [7] and is taken from Nishimura and Venditti [25]. A representative firm in each industry faces the function:

$$y_j = \left(\sum_{i=0}^{n} \beta_{ij} x_{ij}^{-\rho} + e_j \right)^{-1/\rho}, \quad j = 0, \cdots, n \tag{25}$$

with $\beta_{ij} > 0$ and $\rho > -1$. We assume that the elasticity of substitution $\sigma = 1/(1+\rho) \geq 0$ is identical accross sectors. x_{0j} is the amount of labor used to produce good $j = 0, \cdots, n$ and x_{ij} is the amount of capital good $i = 1, \cdots, n$ used to produce good $j = 0, \cdots, n$. The externality, e_j, is assumed to be sector-specific, i.e.

$$e_j = \sum_{i=0}^{n} b_{ij} x_{ij}^{-\rho}$$

with $b_{ij} \geq 0$. We then define the *social production function* as

$$y_j = \left(\sum_{i=0}^{n} (\beta_{ij} + b_{ij}) x_{ij}^{-\rho} \right)^{-1/\rho} \tag{26}$$

The returns to scale are therefore constant at the social level, and decreasing at the private level. Note however that our formulation is compatible with constant returns at the private level if we assume that there exists a factor in fixed supply such as land in the technologies.[8] In this case, the income of the representative consumer will be increased by the rental of land.

A firm in each industry maximizes its profit given output price q_j and input prices $w_0, \cdots, w_n$. Its profit is:

$$\pi_j = q_j y_j - \sum_{i=0}^{n} w_i x_{ij}$$

The first order conditions subject to the private technologies (25) are the following

$$q_j \beta_{ij} \left(y_j / x_{ij} \right)^{1+\rho} = w_i, \quad i, j = 0, \cdots, n \tag{27}$$

[8] The technology of sector j may be formulated as follows

$$y_j = \left(\sum_{i=0}^{n} \beta_{ij} x_{ij}^{-\rho} + e_j T_j^{-\rho} \right)^{-1/\rho}$$

with T_j the amount of land used in the production of good j, which is normalized to 1 in our formulation.

From (27) we have

$$x_{ij}/y_j = (q_j \beta_{ij}/w_i)^{\frac{1}{1+\rho}} \equiv a_{ij}(w_i, q_j), \quad i, j = 0, \cdots, n \tag{28}$$

We call a_{ij} the input coefficients from the *private* viewpoint. If the agents take account of externalities as endogenous variables in profit maximization, the first order conditions subject to the social technologies (26) are

$$q_j(\beta_{ij} + b_{ij})\,(y_j/x_{ij})^{1+\rho} = w_i, \quad i, j = 0, \cdots, n$$

and the input coefficients become

$$\bar{a}_{ij}(w_i, q_j) = \left(q_j \hat{\beta}_{ij}/w_i\right)^{1/(1+\rho)}, \quad i, j = 0, \cdots, n$$

with $\hat{\beta}_{ij} = \beta_{ij} + b_{ij}$. We call $\bar{a}_{ij}$ the input coefficients from the *social* viewpoint. However, as we will show below, the factor-price frontier, which gives a relationship between input prices and output prices, is not exactly expressed with the input coefficients from the social viewpoint. We define

$$\hat{a}_{ij}(w_i, q_j) \equiv (\hat{\beta}_{ij}/\beta_{ij}) a_{ij}(w_i, q_j)$$

which we will call the *quasi* input coefficients from the *social* viewpoint, and it is easy to derive that

$$\hat{a}_{ij}(w_i, q_j) = \bar{a}_{ij}(w_i, q_j) \left(\hat{\beta}_{ij}/\beta_{ij}\right)^{\rho/(1+\rho)}$$

Notice that $\hat{a}_{ij} = \bar{a}_{ij}$ if the production function is Cobb-Douglas i.e. $\rho = 0$.

Based on these input coefficients we will now establish various Lemmas. We first show that the factor-price frontier is determined by the quasi input coefficients from the social viewpoint.

Lemma 1 . *Denote* $q = (q_0, \cdots, q_n)'$, $w = (w_0, \cdots, w_n)'$ *and* $\hat{A}(w, q) = [\hat{a}_{ij}(w_i, q_j)]$. *Then* $q = \hat{A}(w, q)'w$.

The total stock of factors is a vector $x = (x_0, \cdots, x_n)'$ with $x_i = \sum_{j=0}^{n} x_{ij}$. From the fhe full employment conditions, we derive the factor market clearing equation which depends on the input coefficients from the private perspective.

Lemma 2 . *Denote* $x = (x_0, \cdots, x_n)'$, $y = (y_0, \cdots, y_n)'$ *and* $A(w, q) = [a_{ij}(w_i, q_j)]$. *Then* $A(w, q)y = x$.

We now examine some comparative statics. Since the function $\hat{A}(w, q)$ is homogeneous of degree zero in w and q, the factor-price frontier satisfies:

Lemma 3 . $dq = \hat{A}(w, q)'dw$.

The factor market clearing equation finally satisfies:

Lemma 4 . $A(w,q)dy + \sum_{j=0}^{n} y_j \frac{\partial a^j}{\partial w} dw = dx$ *with* $\frac{\partial a^j}{\partial w} = \left[\frac{\partial a_{ij}}{\partial w_s}\right]_{s=0,\cdots,n}$.

We now define some $(n+1)\times(n+1)$ positive matrices

$$D = \left[\beta_{ij}^{1/(1+\rho)}\right] \quad \text{and} \quad \hat{D} = \left[\hat{\beta}_{ij}/\beta_{ij}^{\rho/(1+\rho)}\right]$$

which we assume to be non-singular. This matrices are based on the coefficients that characterise the CES functions. We also define the two following diagonal matrices of rental rates and prices

$$W = \begin{pmatrix} w_0^{\frac{1}{1+\rho}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & w_n^{\frac{1}{1+\rho}} \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} q_0^{\frac{1}{1+\rho}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & q_n^{\frac{1}{1+\rho}} \end{pmatrix}$$

Using the first order conditions we may derive a useful relationship between the matrices of input coefficients and the matrices of CES coefficients. From (28) we get $A = W^{-1}DQ$, $\hat{A} = W^{-1}\hat{D}Q$ and thus $A^{-1} = Q^{-1}D^{-1}W$, $\hat{A}^{-1} = Q^{-1}\hat{D}^{-1}W$. Note also from Lemmas 1-2 and the above diagonal matrices that

$$W^{-1}w = \hat{D}'^{-1}Q^{-1}q \quad \Leftrightarrow \quad \begin{pmatrix} w_0^{\frac{\rho}{1+\rho}} \\ \vdots \\ w_n^{\frac{\rho}{1+\rho}} \end{pmatrix} = \hat{D}'^{-1} \begin{pmatrix} q_0^{\frac{\rho}{1+\rho}} \\ \vdots \\ q_n^{\frac{\rho}{1+\rho}} \end{pmatrix} \tag{29}$$

and

$$y = Q^{-1}D^{-1}Wx \tag{30}$$

Factor rentals are functions of output prices only, $w_i = w_i(q)$, while outputs are functions of factor stocks and output prices, $y_i = y_i(x,q)$, $i = 0,\cdots,n$. We obtain finally

$$\begin{pmatrix} dy/dx & dy/dq \\ dw/dx & dw/dq \end{pmatrix} = \begin{pmatrix} A^{-1} & * \\ 0 & \hat{A}'^{-1} \end{pmatrix} \tag{31}$$

As we have shown in section 1.2.1, without external effects, i.e. $b_{ij} = 0$, the matrix $[\partial y/\partial x]$ reflects the Rybczynski theorem while the matrix $[\partial w/\partial q]$ reflects the Stolper-Samuelson theorem. From the duality between these two effects well known in trade theory we get $[\partial y/\partial x] = [\partial w/\partial q]'$. However, in presence of externalities, the Rybczynski effects depend on the input coefficients from the private perspective while the Stolper-Samuelson effects depend on the quasi input coefficients from the social perspective. The duality between these two effects is thus destroyed. This follows from the fact that with market distortions, true costs are not being minimized. Local indeterminacy of equilibria will be a consequence of this property.

2.2.2 Two-sector models

As in Benhabib and Nishimura [7], we consider the case of a linear utility function with labor normalised to 1 and $n = 1$. The maximisation program of the representative agent is:

$$\begin{aligned}
\max_{\{x_{ij}(t)\}} \quad & \int_0^{+\infty} e^{-\delta t} \left(\beta_{00}x_{00}(t)^{-\rho} + \beta_{10}x_{10}(t)^{-\rho} + e_0(t)\right)^{-1/\rho} dt \\
s.t. \quad & y_1(t) = \left(\beta_{01}x_{01}(t)^{-\rho} + \beta_{11}x_{11}(t)^{-\rho} + e_1(t)\right)^{-1/\rho} \\
& \dot{x}_1(t) = y_1(t) - \mu x_1(t) \\
& x_1(t) = x_{10}(t) + x_{11}(t) \\
& 1 = x_{00}(t) + x_{01}(t) \\
& x_1(0),\ \{e_j(t)\}_{t\geq 0},\ j = 0, 1 \text{ given}
\end{aligned} \tag{32}$$

where $\mu \geq 0$ is the depreciation rate of capital. The modified Hamiltonian in current value is:

$$\begin{aligned}
\mathcal{H} &= \left(\beta_{00}x_{00}^{-\rho} + \beta_{10}x_{10}^{-\rho} + e_0\right)^{-1/\rho} \\
&+ q_1\left(\left(\beta_{01}x_{01}^{-\rho} + \beta_{11}x_{11}^{-\rho} + e_1\right)^{-1/\rho} - \mu x_1\right) \\
&+ w_1\left(x_1 - x_{10} - x_{11}\right) + w_0\left(1 - x_{00} - x_{01}\right)
\end{aligned}$$

with $q_1(t)$ the price of the investment good and $w_1(t)$ the rental rate of capital and $w_0(t)$ the wage rate, all in terms of the price of the consumption good. The static first order conditions are given by:

$$\begin{aligned}
w_0 &= q_1\beta_{01}(y_1/x_{01})^{1+\rho} = \beta_{00}(y_0/x_{00})^{1+\rho} \\
w_1 &= q_1\beta_{11}(y_1/x_{11})^{1+\rho} = \beta_{10}(y_0/x_{10})^{1+\rho}
\end{aligned}$$

and they are equivalent to (27). It follows from (29) and (30) that the necessary conditions which describe the solution to problem (32) are given by the equations of motion:

$$\begin{aligned}
\dot{x}_1(t) &= y_1(x_1(t), q_1(t)) - \mu x_1(t) \\
\dot{q}_1(t) &= (\delta + \mu)q_1(t) - w_1(q_1(t))
\end{aligned} \tag{33}$$

We will assume that there exists a steady state (x_1^*, q_1^*) which solves $\dot{x}_1(t) = \dot{q}_1(t) = 0$. Linearizing (33) around (x_1^*, q_1^*) gives the 2×2 Jacobian matrix:

$$J = \begin{pmatrix} \partial y^1(x_1^*, q_1^*)/\partial x_1 - \mu & \partial y^1(x_1^*, q_1^*)/\partial q_1 \\ 0 & -\partial w_1(q_1^*)/\partial q_1 + \delta + \mu \end{pmatrix}$$

Since we have one pure consumption good, we need to eliminate from equality (31) the column and row which are associated with x_0, y_0, q_0 and w_0. To do so we introduce the following scalars:

$$B = \left[a_{11} - \frac{a_{10}a_{01}}{a_{00}}\right] \quad \text{and} \quad \hat{B} = \left[\hat{a}_{11} - \frac{\hat{a}_{10}\hat{a}_{01}}{\hat{a}_{00}}\right]$$

B is exactely the same capital intensity difference as the one, denoted b, in two-sector optimal growth models. In a framework with externalities, B gives the capital intensity difference at the private level. As shown by Lemma 4, B is associated with the Rybczynski effects. We may also define a similar concept at the social level. The existence of external effects breaks the duality between Rybczynski and Stolper-Samuelson effects. However, Lemma 3 shows that the Stolper-Samuelson effects are based on quasi input coefficients from the social viewpoint. $\hat{B}$ is thus an analog to B and is defined as the capital intensity difference at the quasi social level. The Jacobian matrix becomes:

$$J = \begin{pmatrix} B^{-1} - \mu & * \\ 0 & \delta + \mu - \hat{B}^{-1} \end{pmatrix}$$

Since this matrix is triangular, the characteristic roots are given by the diagonal terms.

We easily derive from the matrices D and $\hat{D}$ defined in section 2.2.1 that

$$\begin{aligned} B &= D_1 q_1/w_1 \text{ with } D_1 = \left[\beta_{11}^{\frac{1}{1+\rho}} - \frac{\beta_{10}^{\frac{1}{1+\rho}}\beta_{01}^{\frac{1}{1+\rho}}}{\beta_{00}^{\frac{1}{1+\rho}}}\right] \\ \hat{B} &= \hat{D}_1 q_1/w_1 \text{ with } \hat{D}_1 = \left[\frac{\hat{\beta}_{11}}{\beta_{11}^{\frac{\rho}{1+\rho}}} - \frac{\hat{\beta}_{10}\hat{\beta}_{01}}{\hat{\beta}_{00}}\frac{\beta_{00}^{\frac{\rho}{1+\rho}}}{\beta_{10}^{\frac{\rho}{1+\rho}}\beta_{01}^{\frac{\rho}{1+\rho}}}\right] \end{aligned}$$

Considering that at the steady state we have $w_1/q_1 = \delta + \mu$, the characteristic roots are:

$$\begin{aligned} \lambda_1 &= B^{-1} - \mu = D_1^{-1}(\delta + \mu) - \mu \\ \lambda_2 &= \delta + \mu - \hat{B}^{-1} = (\delta + \mu)(1 - \hat{D}_1^{-1}) \end{aligned} \tag{34}$$

We will now specialise the argument depending on the value of ρ.

Consider first the case of Cobb-Douglas technologies with $\rho = 0$. In this case we have $\beta_{ij} \in (0,1)$ and $\hat{\beta}_{00} + \hat{\beta}_{10} = \hat{\beta}_{01} + \hat{\beta}_{11} = 1$. As we have shown in section 2.2.1, with Cobb-Douglas technologies, quasi input coefficients at the social level and input coefficients at the social level are equal, $\hat{a}_{ij} = \bar{a}_{ij}$. It follows that $\hat{B}$ gives a capital intensity difference at the social level. Moreover the definitions of B, $\hat{B}$, D_1 and $\hat{D}_1$ implies the following at the steady state:

- the consumption good is capital intensive at the private level if and only if $a_{11}a_{00} - a_{10}a_{01} < 0$, i.e. $\beta_{11}\beta_{00} - \beta_{10}\beta_{01} < 0$;

- the consumption good is capital intensive at the social level if and only if $\hat{a}_{11}\hat{a}_{00} - \hat{a}_{10}\hat{a}_{01} < 0$, i.e. $\hat{\beta}_{11}\hat{\beta}_{00} - \hat{\beta}_{10}\hat{\beta}_{01} < 0$.

The characteristic roots become:

$$\begin{aligned}
\lambda_1 &= \frac{\beta_{00}}{\beta_{11}\beta_{00} - \beta_{10}\beta_{01}}(\delta + \mu) - \mu = \frac{\beta_{00}\delta + \mu[\beta_{00}(1 - \beta_{11}) + \beta_{10}\beta_{01}]}{\beta_{11}\beta_{00} - \beta_{10}\beta_{01}} \\
\lambda_2 &= -(\delta + \mu)\frac{\hat{\beta}_{01}}{\hat{\beta}_{11}\hat{\beta}_{00} - \hat{\beta}_{10}\hat{\beta}_{01}}
\end{aligned}$$

In a two-sector model with one predetermined variable, local indeterminacy is obtained if and only if the stable manifold has dimension 2, i.e. if both roots are negative. We obtain therefore:

Proposition 3 . *In a two-sector economy with Cobb-Douglas technologies, the steady state is locally indeterminate if and only if the consumption good is capital intensive from the private perspective* ($\beta_{11}\beta_{00} - \beta_{10}\beta_{01} < 0$)*, but labor intensive from the social perspective* ($\hat{\beta}_{11}\hat{\beta}_{00} - \hat{\beta}_{10}\hat{\beta}_{01} > 0$)*.*

As shown in Benhabib and Nishimura [7], local indeterminacy comes from the fact that external effects break the duality between Rybczynski and Stolper-Samuelson effects and requires a capital intensity reversal between the private and social levels. To get an economic intuition for this result, assume that the consumption good is capital intensive at the private level, or equivalently that the investment good is labor intensive at the private level. Starting from an equilibrium, consider an increase in the rate of investment above the level of its initial equilibrium, induced by an instantaneous increase in the relative price of the investment good. An increase in the stock of capital decreases its output at constant prices through the Rybczynski effect. The Stolper-Samuelson theorem however operates through the social factor intensities, and the consumption good is labor intensive from the social perspective, or equivalently the investment good is capital intensive from the social perspective. The initial rise in its price causes an increase in its rate of return w_1, and requires a price decline to maintain the overall return to capital equal to the discount rate, i.e.

$$\frac{\dot{q}_1(t)}{q_1(t)} + \frac{w_1(p_1(t))}{q_1(t)} = \delta + \mu$$

This offsets the initial rise in the relative price of the investment good and prices also reverses direction toward the steady state. The transversality condition still holds and this new path is also an equilibrium.

We now study the case of CES technologies with $\rho \neq 0$. The main objective now is to discuss the influence of the elasticity of substitution $\sigma = 1/(1+\rho)$ on the local determinacy properties of equilibria. All the results that will be presented have been established in Nishimura and Venditti [25]. We have now to deal with the quasi input coefficients at the social level which are different from the input coefficients at the social level, $\hat{a}_{ij} \neq \bar{a}_{ij}$.

Consider the characteristic roots (34). From Lemma 2, $x_1 = a_{10}y_0 + a_{11}y_1$. Moreover, at the steady state, $y_1 = \mu x_1$, and it follows

$$a_{10}y_0 + \mu a_{11}x_1 = x_1 \quad \Leftrightarrow \quad a_{10}y_0 = [1 - \mu a_{11}]x_1 > 0$$

Therefore

$$\lambda_1 = \frac{a_{00}}{a_{11}a_{00} - a_{10}a_{01}} - \mu = \frac{a_{00}[1 - \mu a_{11}] + a_{10}a_{01}\mu}{a_{11}a_{00} - a_{10}a_{01}}$$

From Lemma 1, $q_1 = \hat{a}_{01}w_0 + \hat{a}_{11}w_1$. Moreover, at the steady state, $(\delta + g)q_1 = w_1$, and it follows

$$(\delta+g)\hat{a}_{01}w_0 + (\delta+g)\hat{a}_{11}w_1 = w_1 \quad \Leftrightarrow \quad (\delta+g)\hat{a}_{01}w_0 = [1-(\delta+g)\hat{a}_{11}]w_1 > 0$$

Therefore

$$\lambda_2 = (\delta + g) - \frac{\hat{a}_{00}}{\hat{a}_{11}\hat{a}_{00} - \hat{a}_{10}\hat{a}_{01}} = -\frac{[1 - (\delta + g)\hat{a}_{11}]\hat{a}_{00} + (\delta + g)\hat{a}_{10}\hat{a}_{01}}{\hat{a}_{11}\hat{a}_{00} - \hat{a}_{10}\hat{a}_{01}}$$

We obtain finally:

Proposition 4 . *In a two-sector economy with CES technologies, the steady state is locally indeterminate if and only if the consumption good is capital intensive from the private perspective* ($a_{11}a_{00} - a_{10}a_{01} < 0$)*, but quasi labor intensive from the social perspective* ($\hat{a}_{11}\hat{a}_{00} - \hat{a}_{10}\hat{a}_{01} > 0$)*.*

The capital intensity reversal between the private and social levels exhibited with Cobb-Douglas technologies is modified here into a capital intensity reversal between the private and quasi social levels. The economic intuition provided by Benhabib and Nishimura is still valid with CES technologies but if we consider the true capital intensity difference from the social perspective we find a strong influence of the elasticity of substitution.

Using the definition of B, $\hat{B}$, D_1 and $\hat{D}_1$, we easily derive:

Lemma 5 . *At the steady state, the consumption good is:*

i) capital intensive from the private perspective if and only if $\beta_{11}\beta_{00} - \beta_{10}\beta_{01} < 0$*;*

ii) quasi capital intensive from the social perspective if and only if

$$\left(\frac{\hat{\beta}_{11}\hat{\beta}_{00}}{\hat{\beta}_{10}\hat{\beta}_{01}}\right) < \left(\frac{\beta_{11}\beta_{00}}{\beta_{10}\beta_{01}}\right)^{\frac{\rho}{1+\rho}}$$

iii) capital intensive from the social perspective if and only if $\hat{\beta}_{11}\hat{\beta}_{00} - \hat{\beta}_{10}\hat{\beta}_{01} < 0$.

Consider *ii*) of Lemma 5. When $\rho \geq 0$, if $\hat{\beta}_{11}\hat{\beta}_{00} - \hat{\beta}_{10}\hat{\beta}_{01} > 0$ and $\beta_{11}\beta_{00} - \beta_{10}\beta_{01} \leq 0$, then the consumption good sector is always quasi labor intensive from the social perspective. When $-1 < \rho < 0$, even if $\beta_{11}\beta_{00} - \beta_{10}\beta_{01} < 0$, the consumption good sector may be quasi labor intensive from the social perspective when $\hat{\beta}_{11}\hat{\beta}_{00} - \hat{\beta}_{10}\hat{\beta}_{01} < 0$ and dominates $\beta_{11}\beta_{00} - \beta_{10}\beta_{01}$. We may therefore derive the following result:

Proposition 5 . *In a two-sector economy with CES technologies, let the consumption good be capital intensive from the private perspective. Then:*

i) if the consumption good is labor intensive from the social perspective, there exists $\rho_1^* \in (-1, 0)$ *such that the steady state is locally indeterminate for any* $\rho > \rho_1^*$ *and saddle-point stable for any* $\rho \in (-1, \rho_1^*)$*;*

ii) if the consumption good is also capital intensive from the social perspective and

$$1 > \frac{\hat{\beta}_{11}\hat{\beta}_{00}}{\hat{\beta}_{10}\hat{\beta}_{01}} \geq \frac{\beta_{11}\beta_{00}}{\beta_{10}\beta_{01}},$$

there exists $\rho_2^* > 0$ *such that the steady state is locally indeterminate for any* $\rho > \rho_2^*$ *and saddle-point stable for any* $\rho \in (-1, \rho_2^*)$.

This Proposition shows that when the productive factors are sufficiently substitutable, local indeterminacy occurs if the consumption good is capital intensive at the private level. Note that local indeterminacy is still possible even if the consumption good is capital intensive at the social level. Condition *i*) only coincides with the result obtained by Benhabib and Nishimura

[7] in the particular case of Cobb-Douglas technologies. Therefore when CES production functions are considered, a capital intensity reversal is not always necessary for local indeterminacy.[9]

2.2.3 Extensions to multisector models

In a general multisector model with n capital goods, all the previous scalars become $n \times n$ matrices, and local conditions for stability or instability require strong properties on matrices of factor intensity differences D_1 or $\hat{D}_1$ which are now defined as follows. Notice that we parameterize these matrices by ρ since we will discuss the influence of the elasticity of substitution:

$$
\begin{aligned}
D_1(\rho) &= \left[\beta_{ij}^{\frac{1}{1+\rho}} - \frac{\beta_{i0}^{\frac{1}{1+\rho}} \beta_{0j}^{\frac{1}{1+\rho}}}{\beta_{00}^{\frac{1}{1+\rho}}} \right] \\
\hat{D}_1(\rho) &= \left[\frac{\hat{\beta}_{ij}}{\beta_{ij}^{\frac{\rho}{1+\rho}}} - \frac{\hat{\beta}_{i0}\hat{\beta}_{0j}}{\hat{\beta}_{00}} \frac{\beta_{00}^{\frac{\rho}{1+\rho}}}{\beta_{i0}^{\frac{\rho}{1+\rho}} \beta_{0j}^{\frac{\rho}{1+\rho}}} \right]
\end{aligned}
$$

Very few papers deal with the analysis of local indeterminacy in multisector models. Benhabib and Nishimura [8] provide some extensions of their 1998 results in a Cobb-Douglas framework. We will present in this section some results provided by Nishimura and Venditti [25] which are based on CES technologies. This framework covers the Cobb-Douglas formulation and the conditions for local indeterminacy are less demanding in terms of capital intensity differences than those exhibited by Benhabib and Nishimura.

In this section we provide a generalization of the results obtained for the two-sector model and propose new conditions for local indeterminacy. As in the two-sector case, we have to consider the coefficients β_{ij} and $\hat{\beta}_{ij}$ of the CES technologies. We will impose restrictions only on the sign of the diagonal terms of $D_1(\rho)$ and $\hat{D}_1(\rho)$ but not on the sign of the off-diagonal coefficients. We introduce the following sets of indices which characterize the sign of the diagonal terms when $\rho = 0$ in $D_1(0)$ and $\hat{D}_1(0)$:

$$\mathcal{D} = \{j \in \{1, \cdots, n\} \text{ such that } \beta_{jj}\beta_{00} - \beta_{j0}\beta_{0j} < 0\} \tag{35}$$

$$\widehat{\mathcal{D}} = \{k \in \{1, \cdots, n\} \text{ such that } \hat{\beta}_{kk}\hat{\beta}_{00} - \hat{\beta}_{k0}\hat{\beta}_{0k} < 0\} \tag{36}$$

When $j \in \mathcal{D}$ the consumption good is more intensive in capital good j than the capital good j itself at the private level. Similarly, when $k \in \widehat{\mathcal{D}}$ the

[9] See also Nishimura and Venditti [27] for some conditions for local indeterminacy in a two-sector discrete-time model with CES technologies and asymmetric elasticities of capital-labor substitution.

consumption good is more intensive in capital good k than the capital good k itself at the social level. Let us denote the number of elements in $\mathcal{D}$ and $\widehat{\mathcal{D}}$ respectively by $\#\mathcal{D}$ and $\#\widehat{\mathcal{D}}$. If $\#\mathcal{D} = n$ then $\beta_{ii}\beta_{00} - \beta_{i0}\beta_{0i} < 0$ for any $i = 1, \cdots, n$, and if $\#\widehat{\mathcal{D}} = 0$ then $\hat{\beta}_{ii}\hat{\beta}_{00} - \hat{\beta}_{i0}\hat{\beta}_{0i} \geq 0$ for any $i = 1, \cdots, n$. These restrictions are similar to the Metzler and Minkowsky conditions of Benhabib and Nishimura [8] which imply the existence of $2n$ eigenvalues with negative real parts.[10] Our main objective is to give conditions for the existence of at least $n+1$ eigenvalues with negative real parts from the sign patterns of the diagonal elements of matrices $D_1(\rho)$ and $\hat{D}_1(\rho)$.

In order to relate the sign of diagonal elements to the sign of the real parts of eigenvalues, we introduce the following dominant diagonal properties of matrices:

Definition 4 . *An $n \times n$ matrix $C = [c_{ij}]$ has a dominant diagonal if $|c_{ii}| > \sum_{j \neq i} |c_{ij}|$ for each $i = 1, \cdots, n$ or $|c_{ii}| > \sum_{i \neq j} |c_{ij}|$ for each $j = 1, \cdots, n$.*

This definition is stronger than the quasi-dominant diagonal introduced by McKenzie [21] since the weighting parameters are here equal to one. We also introduce strong dominant diagonal which requires both row dominance and column dominance.

Definition 5 . *An $n \times n$ matrix $C = [c_{ij}]$ has a strong dominant diagonal if $|c_{ii}| > \sum_{j \neq i} |c_{ij}|$ for each $i = 1, \cdots, n$ and $|c_{ii}| > \sum_{i \neq j} |c_{ij}|$ for each $j = 1, \cdots, n$.*

Assumption 10 . *There exists $\bar{\rho} > -1$ such that for any $\rho > \bar{\rho}$, $D_1(\rho)$ has a strong dominant diagonal, and $\hat{D}_1(\rho)$ has only real eigenvalues with dominant diagonal.*

Theorem 11 . *Let Assumption 10 hold, $\#\mathcal{D} \geq 1$ and*

$$\frac{\hat{\beta}_{ii}\hat{\beta}_{00}}{\hat{\beta}_{i0}\hat{\beta}_{0i}} \geq \frac{\beta_{ii}\beta_{00}}{\beta_{i0}\beta_{0i}} \tag{37}$$

for all $i = 1, \cdots, n$. Then there exists $-1 < \rho^* \leq \bar{\rho}$ such that the steady state is locally indeterminate for any $\rho > \rho^*$.

Theorem 11 only introduces a condition on the set $\mathcal{D}$ which, coupled with dominant diagonal properties on $D_1(\rho)$ and $\hat{D}_1(\rho)$, ensure the existence of

[10] A Metzler matrix has negative diagonal elements and positive off-diagonal elements, while a Minkowski matrix has positive diagonal elements and negative off-diagonal elements.

at least $n+1$ roots with negative real parts when ρ exceeds a bound ρ^*. However, we do not know a priori if this bound is positive or negative, i.e. if the range of values of ρ compatible with local indeterminacy includes the Cobb-Douglas case $\rho = 0$. To get this information we need to discuss the set $\widehat{D}$. In Nishimura and Venditti [25], some examples show that:

i) If $\#\widehat{\mathcal{D}} = 0$ and $\bar{\rho} \leq 0$, then $\rho^* < 0$;

ii) If $\#\widehat{\mathcal{D}} \geq 1$ and $\#\mathcal{D} - \#\widehat{\mathcal{D}} < 1$, then $\rho^* > 0$;

iii) Let $n \geq 2$. If $1 \leq \#\widehat{\mathcal{D}} < n$ and $\#\mathcal{D} - \#\widehat{\mathcal{D}} \geq 1$, then $\rho^* > 0$ but there exists $\underline{\rho}^* < \rho^*$, which may be negative, such that local indeterminacy still holds when $\rho \in]\underline{\rho}^*, \rho^*[$.

Theorem 11 suggests that local indeterminacy cannot arise with high substitutability. We may thus provide conditions for saddle-point stability. We first introduce an alternative restriction to Assumption 10.

Assumption 11 . *There exists $\bar{\rho} > -1$ such that for any $\rho \in (-1, \bar{\rho}]$, $D_1(\rho)$ has a strong dominant diagonal, and $\hat{D}_1(\rho)$ has only real eigenvalues with dominant diagonal.*

Proposition 6 . *Let Assumption 11 hold with $\#\mathcal{D} = n$. Then there exists $\hat{\rho} \geq \bar{\rho}$ such that the steady state is saddle-point stable for any $\rho \in (-1, \hat{\rho})$.*

Condition $\#\mathcal{D} = n$ with dominant diagonal garantees that the Jacobian matrix J has at least n negative eigenvalues. However, strong factor substitutability leads to the existence of a unique equilibrium path.

2.3 Two sector models with intersectoral externalities

2.3.1 A general formulation

In the previous section we have considered sector-specific externalities as initially formulated by Benhabib and Farmer [3]. Such a formulation does not introduce any direct additional intersectoral relationships into the model. A kind of symmetry similar to that obtained in standard optimal growth models is still preserved. An alternative way of introducing external effects into a multisector model is to consider aggregate externalities as in Romer [29]. One possible formulation has been provided by Boldrin and Rustichini [11]. They deal with a two-sector economy in which each technology is affected by some intersectoral external effects coming from the average stock of capital. It is assumed that this provides an approximation of the aggregate technolgical knowledge of the economy which has positive spillovers on the production activities.

Let us consider now the Boldrin and Rustichini formulation. We have a two-sector discrete-time growth model with one pure consumption good y_0 and one capital good y. The labor supply is inelastic and the population is constant. Total labor is normalised to 1 and each good is produced with a constant private returns to scale technology which also depends on an intersectoral externality A:

$$\begin{aligned} y_0 &= f^0(k_0, l_0, A) \\ y &= f^1(k_1, l_1, A) \end{aligned}$$

with $k_0 + k_1 \leq k$, k being the total stock of capital, and $l_0 + l_1 \leq 1$. At the equilibrium, the externality A will equal the aggregate capital stock k.

Assumption 12 . *Each production function $f^i(k^i, l^i, A)$, $i = 0, 1$, is C^2, increasing in each argument and, for any $A > 0$, concave, homogeneous of degree one and such that for any $l_i > 0$, $f^i_{11}(., l_i, A) < 0$.*

For any given (k, y, A), we define a temporary equilibrium by solving the following problem of optimal allocation of productive factors between the two sectors:

$$\begin{aligned} T(k, y, A) = \max_{k_0, k_1, l_0, l_1} \quad & f^0(k_0, l_0, A) \\ s.t. \quad & y \leq f^1(k_1, l_1, A) \\ & k_0 + k_1 \leq k \\ & l_0 + l_1 \leq 1 \\ & k_0, k_1, l_0, l_1 \geq 0 \end{aligned} \tag{38}$$

The value function $T(k, y, A)$ describes the frontier of the production possibility set from the private perspective. Under Assumption 12, for any given $A > 0$, $T(k, y, A)$ is concave. To simplify the formulation we will assume that the utility function of the representative agent is linear and that capital fully depreciates within one period of time. The maximisation program is

$$\begin{aligned} \max_{\{y_t\}_{t=0}^{+\infty}} \quad & \sum_{t=0}^{+\infty} \delta^t T(k_t, k_{t+1}, A_t) \\ s.t. \quad & (k_t, k_{t+1}) \in \mathcal{D}(A_t) \\ & k_0, \{A_t\}_{t=0}^{+\infty} \text{ given} \end{aligned} \tag{39}$$

with

$$\mathcal{D}(A_t) = \left\{ (k_t, k_{t+1}) \in \mathbb{R}^2_+ / 0 \leq k_{t+1} \leq f^1(k_t, 1, A_t) \right\}$$

the set of admissble paths for any given externalitie A_t. The first order condition for an interior maximum is given by the Euler equation

$$T_2(k_t, k_{t+1}, A_t) + \delta T_1(k_{t+1}, k_{t+2}, A_{t+1}) = 0$$

which is a second-order non-linear implicit difference equation. An equilibrium path is a parameterized solution of the Euler equation, denoted $k_t(\{A(t)\}_{t=0}^{+\infty})$, which satisfies the fixed point relationship between k_t and A_t:

$$k_t(\{A(t)\}_{t=0}^{+\infty}) = A_t \text{ for all } t \geq 0 \text{ with } k_0 = A_0$$

We will assume that an equilibrium path solution to this fixed point problem exists. It also has to satisfy the transversality condition

$$\lim_{t \to +\infty} \delta^t k_t T_1(k_t, k_{t+1}, k_t) = 0$$

Along an equilibrium path the Euler equation becomes

$$T_2(k_t, k_{t+1}, k_t) + \delta T_1(k_{t+1}, k_{t+2}, k_{t+1}) = 0$$

As in optimal growth models it is easy to show that

$$\begin{aligned} T_1(k, y, A) = &\ f_1^0(k_0(k, y, A), l_0(k, y, A), A) = q f_1^1(k_1(k, y, A), l_1(k, y, A), A) \\ = &\ w(k, y, A) \\ T_2(k, y, A) = &\ -\frac{f_1^0(k_0(k, y, A), l_0(k, y, A), A)}{f_1^1(k_1(k, y, A), l_1(k, y, A), A)} = -\frac{f_2^0(k_0(k, y, A), l_0(k, y, A), A)}{f_2^1(k_1(k, y, A), l_1(k, y, A), A)} \\ = &\ -q(k, y, A) \end{aligned}$$

A steady state $k_{t+1} = k_t = k_\delta^*$ is a solution of

$$f_1^1(k_1(k_\delta^*, k_\delta^*, k_\delta^*), l_1(k_\delta^*, k_\delta^*, k_\delta^*), k_\delta^*) = \delta^{-1} \quad (40)$$

Assumption 13 . *For any* $\delta \in (0, 1]$*, there exists a steady state solution of equation (40).*

Linearizing the Euler equation around the steady state gives the following characteristic polynomial

$$P(\lambda) = \lambda^2 \delta T_{12}^* + \lambda[\delta(T_{11}^* + T_{13}^*) + T_{22}^*] + T_{21}^* + T_{23}^* = 0$$

with $T_{ij}^* = T_{ij}(k_\delta^*, k_\delta^*, k_\delta^*)$. Provided $T_{12}^* \neq 0$, the two roots λ_1 and λ_2 satisfy

$$\begin{aligned} \lambda_1 \lambda_2 &= \delta^{-1} + T_{23}^* / \delta T_{12}^* \\ \lambda_1 + \lambda_2 &= -\frac{\delta(T_{11}^* + T_{13}^*) + T_{22}^*}{\delta T_{12}^*} \end{aligned}$$

We obtain a generalisation of the Kurz [18] and Levhari-Liviatan [19] result: if λ is a characteristic root, then $1/\delta\lambda + T_{23}^*/\delta\lambda T_{12}^*$ is also a characteristic root. We derive from this that if $T_{23}^*/T_{12}^* \geq 0$, then $\lambda_1 \lambda_2 > 1$ and the steady state is necessarily locally determinate. On the contrary, if $T_{23}^*/T_{12}^* < 0$,

$\lambda_1\lambda_2$ may be less than 1 and local indeterminacy of equilibria may occur. As in two-sector optimal growth models, the sign of T_{12}^* is ruled by the capital intensity difference at the private level. However, the sign of T_{23}^* is difficult to establish. We only know that

$$T_{23}^* = -\partial q/\partial A$$

Boldrin and Rustichini [11] provide formal conditions for local indeterminacy but it remains difficult to interpret these conditions in terms of the fundamentals.[11] In particular, there is no clear picture concerning the requirements on the capital intensity difference.

2.3.2 A Cobb-Douglas formulation

The contribution of Boldrin and Rustichini [11] does not allow to compare the conditions for local indeterminacy in models with intersectoral externalities with those obtained in models with sector-specific externalities. In order to provide such a comparison, Nishimura and Venditti [26] provide a Cobb-Douglas formulation that encompasses the cases of sector-specific and intersectoral external effects.

Consider a discrete-time two-sector economy having an infinitely-lived representative agent with a linear single period utility function. Labor supply is inelastic and there are one pure consumption good, c, and one investment good, k. Each good is produced with a Cobb-Douglas technology. We assume that the consumption good production function contains positive intersectoral externalities given by a convex combination of the capital stocks of the two sectors. We denote by y and y_0 the outputs of sectors k and c:

$$y = k_1^{\beta_1} l_1^{\beta_2}, \quad y_0 = e k_0^{\alpha_1} l_0^{\alpha_2} \quad \text{with} \quad e = [\theta k_0 + (1-\theta)k_1]^a$$

where $\theta \in [0,1]$ and $a \geq 0$. Depending on the value of θ, our formulation encompasses the usual assumptions of sector specific externalities ($\theta = 1$), and global external effects ($\theta = 1/2$). We will also consider the case with purely intersectoral externalities ($\theta = 0$). Labor is normalized to one, $l_0 + l_1 = 1$, the total stock of capital is $k_0 + k_1 = k$ and total depreciation of capital occurs in one period.

Assumption 14 . *Returns to scale at the social level are non-increasing in the consumption good sector, i.e.* $\alpha_1 + \alpha_2 + a \leq 1$, *and constant in the investment good sector, i.e.* $\beta_1 + \beta_2 = 1$.

[11] See also Drugeon and Venditti [17] and Venditti [35] for more detailled conditions on local indeterminacy and local bifurcation of periodic cycles.

It can be easily shown that if $\beta_1/\beta_2 > (<)\alpha_1/\alpha_2$ the investment (consumption) good sector is capital intensive from the private perspective. Note that this definition is still valid with intersectoral external effects ($\theta < 1$). The condition $\beta_1/\beta_2 > (<)(\alpha_1 + a)/\alpha_2$ implies that the investment (consumption) good sector is capital intensive from the social perspective if the externalities are sector specific ($\theta = 1$).

The representative consumer's optimization program is given by

$$\begin{array}{ll} \max\limits_{\{k_{0t},l_{0t},k_{1t},l_{1t},k_t,y_t\}_{t=0}^{+\infty}} & \sum\limits_{t=0}^{\infty} \delta^t e_t k_{0t}^{\alpha_1} l_{0t}^{\alpha_2} \\ s.t. & y_t = k_{1t}^{\beta_1} l_{1t}^{\beta_2} \\ & 1 = l_{0t} + l_{1t} \\ & k_t = k_{0t} + k_{1t} \\ & k_{t+1} = y_t \\ & k_0 = \bar{k}_0, \{e_t\}_{t=0}^{+\infty} \quad given \end{array} \tag{41}$$

Denote by q_t, ω_{0t} and ω_t the price of the capital good, the wage rate of labor and the rental rate of the capital good at time $t \geq 0$, all in terms of the price of the consumption good. The Lagrangian at time $t \geq 0$ is:

$$\begin{aligned} \mathcal{L}_t &= \delta e_t k_{0t}^{\alpha_1} l_{0t}^{\alpha_2} + \delta q_t[k_{1t}^{\beta_1} l_{1t}^{\beta_2} - y_t] + \delta\omega_{0t}[1 - l_{0t} - l_{1t} + \delta\omega_t[k_t - k_{0t} - k_{1t}] \\ &+ \delta q_t y_t - q_{t-1} k_t \end{aligned}$$

For any (k_t, y_t), solving the first order conditions w.r.t. $(k_{0t}, l_{0t}, k_{1t}, l_{1t})$ gives inputs as functions $\hat{k}_0(k_t, y_t)$, $\hat{l}_0(k_t, y_t)$, $\hat{k}_1(k_t, y_t)$ and $\hat{l}_1(k_t, y_t)$.[12] We define the efficient production frontier as

$$T(k_t, y_t, e_t) = e_t \hat{k}_0(k_t, y_t)^{\alpha_1} \hat{l}_0(k_t, y_t)^{\alpha_2}$$

Using the envelope theorem we derive the equilibrium prices

$$q_t = -T_2(k_t, y_t, e_t) \tag{42}$$
$$\omega_t = T_1(k_t, y_t, e_t) \tag{43}$$

From the first order conditions w.r.t. (k_t, y_t) we obtain

$$-q_t + \delta\omega_{t+1} = 0 \tag{44}$$

Mixing equations (42-44) with $k_{t+1} = y_t$ gives the Euler equation:

$$T_2(k_t, k_{t+1}, e_t) + \delta T_1(k_{t+1}, k_{t+2}, e_{t+1}) = 0$$

Any sequence $\{k_t, e_t\}_{t=0}^{+\infty}$ needs also to satisfy the transversality condition

[12] Notice that the factors optimal demand functions do not depend on externalities e_t. This follows from the fact that the production functions are Cobb-Douglas.

$$\lim_{t\to+\infty} \delta^t k_t T_1(k_t, k_{t+1}, e_t) = 0$$

Let $\{k_t\}_{t=0}^{+\infty}$ denote a solution. This path depends on $\{e_t\}_{t=0}^{\infty}$. If expectations are realized, i.e. if $\{e_t\}_{t=0}^{+\infty}$ satisfies the following relationship:

$$e_t = \left[\theta \hat{k}_0(k_t, k_{t+1}) + (1-\theta)\hat{k}_1(k_t, k_{t+1})\right]^a \equiv \hat{e}(k_t, k_{t+1})$$

for $t = 0, 1, 2, ...$, then the sequence $\{k_t\}_{t=0}^{+\infty}$ is called an equilibrium path. Substituting $\hat{e}(k_t, k_{t+1})$ into equations (42-43) gives q_t and ω_t as functions of (k_t, k_{t+1}). Then the Euler equation along an equilibrium path becomes:

$$-q(k_t, k_{t+1}) + \delta\omega(k_{t+1}, k_{t+2}) = 0 \tag{45}$$

Lemma 6 . *There exists a unique steady state* $k^* = \frac{\alpha_1\beta_2(\rho\beta_1)^{1/\beta_2}}{\alpha_2\beta_1 + (\alpha_1\beta_2 - \alpha_2\beta_1)\delta\beta_1}$.

The linearization of (45) around k^* gives the characteristic polynomial:

$$\delta\frac{\partial\omega}{\partial y}(k^*, k^*)x^2 + x\Big[\delta\frac{\partial\omega}{\partial k}(k^*, k^*) - \frac{\partial q}{\partial y}(k^*, k^*)\Big] - \frac{\partial q}{\partial k}(k^*, k^*) = 0 \tag{46}$$

Consider the first order conditions w.r.t. $(k_{0t}, l_{0t}, k_{1t}, l_{1t})$ derived from the optimization program (41). Differenciating these conditions w.r.t. both $(k_{0t}, l_{0t}, k_{1t}, l_{1t})$ and (q, w_0, w), and using the implicit function theorem with the Cramer's rule allow to get the following result:

Lemma 7 . *The characteristic equation (46) is equivalent to the following*

$$P_{\theta,\delta}(x) = \delta\mathcal{A}(\theta,\delta)x^2 - \mathcal{B}(\theta,\delta)x + \mathcal{C}(\theta,\delta)$$

with

$$\begin{aligned}
\mathcal{A}(\theta,\delta) &= \alpha_2(\alpha_1-\beta_1) - \delta\alpha_1\beta_1\beta_2\tfrac{(1-\alpha_1-\alpha_2)}{1-\delta\beta_1} - \tfrac{a(1-2\theta)}{\phi}\beta_1[\alpha_2(1-\delta\beta_1)+\delta\alpha_1\beta_2] \\
\mathcal{B}(\theta,\delta) &= \tfrac{a\theta\alpha_2(1-\delta\beta_1)}{\phi} - \alpha_2(1-\alpha_1) - \delta(1+\delta)\alpha_1\beta_1\beta_2\tfrac{(1-\alpha_1-\alpha_2)}{1-\delta\beta_1} \\
&\quad + \delta\alpha_1\beta_2(\alpha_2-\beta_2) + \delta\alpha_2\beta_1(\alpha_1-\beta_1) + \delta a\tfrac{\beta_1\beta_2}{\phi}[\alpha_1\theta + \alpha_2(1-\theta)] \\
&\quad - \delta\tfrac{a(1-2\theta)}{\phi}\beta_1[\alpha_2(1-\delta\beta_1)+\delta\alpha_1\beta_2] \\
\mathcal{C}(\theta,\delta) &= \alpha_2(\alpha_1-\beta_1) + \tfrac{a}{\phi}\Big[\alpha_2\theta(1-\delta\beta_1) + \delta\alpha_1\beta_1\beta_2[\theta+\alpha_2(1-\theta)]\Big] \\
&\quad - \delta\alpha_1\beta_1\beta_2\tfrac{(1-\alpha_1-\alpha_2)}{1-\delta\beta_1} + \tfrac{a^2\theta(1-\theta)\delta\beta_1(1-\delta\beta_1)\alpha_2\beta_2}{\phi^2}
\end{aligned}$$

and $\phi = \theta(1-\delta\beta_1) + (1-\theta)\delta\beta_1$.

We assume in what follows that $\mathcal{A}(\theta, \delta) \neq 0$. We will now specialize the argument depending on the value of θ.

Assume first that there are only sector specific externalities in the consumption good sector. We show that the steady state is always locally determinate.

Proposition 7 . *If the externalities are sector specific ($\theta = 1$), the steady state k^* is locally determinate.*

A discrete-time extension of the two-sector model with sector-specific externalities considered by Benhabib and Nishimura [7] has been provided by Benhabib, Nishimura and Venditti [9]. They prove that if the consumption good is capital intensive from the private perspective, locally indeterminate equilibria may occur when the consumption good is either capital or labor intensive at the social level. In other words, a capital intensity reversal is not necessary. However, they assume that there are external effects on capital and labor in both sectors. Considering the same restriction as in the present paper, Proposition 7 shows that indeterminacy necessarily requires externalities coming from labor. Note that this result does not hold and indeterminacy is still possible if we assume that the investment good sector contains external effects on capital. The current formulation is motivated by its simplicity and by the fact that it will strongly enlighten the role of intersectoral external effects.

The intuition for these results is based on the Stolper-Samuelson theorem. Assume that the consumption good is capital intensive at the private level while the investment good is capital intensive at the social level.[13] Starting from an increase in the rate of investment induced by an instantaneous increase in the relative price of the capital good, the Rybczynski effect implies a decrease of its output at constant prices. Indeterminacy therefore requires a sufficient increase in one of the components of its return, w, and thus a price decline to maintain the overall return to capital equal to the discount rate. This indeed offsets the initial rise in the relative price of the investment good and prices also reverse direction toward the steady state. Benhabib, Nishimura and Venditti [9] show that this mechanism works when both sectors have capital and labor externalities. On the contrary, when there are only external effects coming from capital in the consumption good sector, the Stolper-Samuelson effect is not strong enough

[13] A similar intuition is obtained when the investment good is also labor intensive at the social level.

to generate a sufficient price decline which may offset the initial rise. It follows that prices become explosing and the transversality condition is violated.

Consider now the case in which the externality in the consumption good technology comes only from the capital stock of the investment sector ($\theta = 0$).

Assumption 15 . *min* $\left\{ \frac{\beta_1\alpha_2}{\alpha_1 + a}, 1, \frac{a\alpha_2(1-\alpha_1)}{\alpha_1+\alpha_2} \right\} > \beta_2$.

Given arbitrary $a > 0$, Assumption 15 may be satisfied if β_2 is chosen to be sufficiently small. Assumption 15 also implies that the investment good is capital intensive at the private level since $\beta_1\alpha_2/(\alpha_1 + a) > \beta_2$ implies $\beta_1/\beta_2 > (\alpha_1 + a)/\alpha_2 > \alpha_1/\alpha_2$.

Proposition 8 . *Let $\theta = 0$. Under Assumption 15, there exists $\delta_1 < 1$ such that the steady state is locally indeterminate for any $\delta \in]\delta_1, 1]$.*

Intersectoral externalities give rise to another mechanism from which indeterminacy arises. It can be shown when $\theta = 0$ and under constant social returns that the Stolper-Samuelson theorem now depends on the private factor intensities and more importantly on the way the capital stock is affected by a price variation. It follows that even if the investment good is capital intensive at the private level, an increase in the rate of investment induced by an instantaneous increase in the relative price of the capital good implies a complex reallocation of factors between sectors. This causes through Stolper-Samuelson effects a sufficient price decline which may offset the initial rise as soon as discounting is weak enough. For expectations-driven fluctuations to be sustained, the oscillations in relative prices must indeed present intertemporal arbitrage opportunities so that, starting from one initial equilibrium, the representative consumer may be able to select another equilibrium with, say, higher returns on capital and investment rates. When there is myopia associated with strong discounting, this mechanism no longer holds.

From Proposition 8 it is straightforward to extend the indeterminacy result to positive values of θ as in the following Theorem:

Theorem 12 . *Under Assumption 15, there exist $0 < \delta_1 < 1$ and a function $\theta^* :]\delta_1, 1] \to]0, 1[$ such that the steady state is locally indeterminate for each $\delta \in]\delta_1, 1]$ and $\theta \in [0, \theta^*(\delta)[$.*

Nishimura and Venditti [26] provide some example which shows that depending on the choice of δ, $\theta^*(\delta)$ in Theorem 12 can be close to one. They also give conditions for the existence of a Hopf bifurcation when the investment good is capital intensive at the private level.

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