

Capital depreciation, factor substitutability and indeterminacy*

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Dedicated to Professor Saber Elaydi on his 60th birthday

Abstract: *We consider a discrete-time two-sector CES economy with sector specific external effects and partial depreciation of capital. We show that the occurrence of local indeterminacy of equilibria depends on an interplay between factor substitutability and capital depreciation. When the elasticity of substitution is less than one, local indeterminacy may occur with low depreciation of capital. When the elasticity of substitution is greater than one, local indeterminacy is more likely if the rate of depreciation is high.*

Keywords: *Sector specific externalities, partial depreciation of capital, elasticity of capital-labor substitution, constant social returns, local indeterminacy.*

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1 Introduction

The aim of this paper is to study the interplay between the elasticity of capital-labor substitution and the rate of capital depreciation, and its influence on the determinacy properties of equilibria in a two-sector discrete-time economy with Constant Elasticity of Substitution (CES) technologies that are the most commonly used production functions by economists.

Benhabib and Farmer [1] and Boldrin and Rustichini [4] have studied local indeterminacy in multi-sector growth models with infinitely-lived agents and technological externalities.¹ Subsequently, Benhabib and Nishimura [2] and Benhabib, Nishimura and Venditti [3] have obtained factor intensity conditions in the framework of economies with Cobb-Douglas production functions, that are a special case of CES functions, and sector specific external effects. They have shown that local indeterminacy requires the consumption good to be capital intensive at the private level.² We extend these earlier analysis by explicitly considering factors substitutability and the role of partial depreciation of capital.

Unlike continuous-time models, introducing depreciation of capital creates additional difficulty in studying dynamical properties of equilibrium paths in discrete time models. In the current paper, we consider a discrete-time two-sector CES economy with sector specific externalities and an infinitely-lived representative agent. Our goal is to characterize the local stability properties of the steady state and to evaluate precisely the role of the elasticity of capital-labor substitution and the rate of depreciation of capital. We assume that the aggregate technology of each sector has constant social returns which implies that individual firms exhibit small decreasing returns. This divergence between private and social returns is explained by the existence of mild external effects.

We first prove that when the investment good is labor intensive at the

¹External effects are feedbacks from the other agents in the economy who also face identical maximizing problems.

²Bond, Wang and Yip [5] obtain similar results in an equivalent framework with taxation instead of external effects.

private level, if the elasticity of capital-labor substitution is less than 1, local indeterminacy may arise for low values of the rate of capital depreciation. Moreover, if the factors intensity difference is large enough, this property does not depend on this parameter. We also prove that when the elasticity of capital-labor substitution is greater than one, local indeterminacy is more likely if the rate of depreciation is high.

This paper is organized as follows. The next section sets up the basic model, proves the existence and uniqueness of the steady state and presents the characteristic roots. Section 3 presents our main results on local indeterminacy of equilibria. Section 4 contains some concluding comments.

2 The model

2.1 The basic structure

We consider a discrete-time two-sector economy having an infinitely-lived representative agent with single period linear utility function, i.e. $u(c) = c$. We assume that the labor supply is inelastic. There are two goods: the pure consumption good, c , and the pure capital good, k . Each good is assumed to be produced with a CES technology which contains some sector specific externalities. We denote by c and y the outputs of sectors c and k , and by e_c and e_y the corresponding external effects:

$$c = (\alpha_1 K_c^{-\rho_c} + \alpha_2 L_c^{-\rho_c} + e_c)^{-1/\rho_c}, y = (\beta_1 K_y^{-\rho_y} + \beta_2 L_y^{-\rho_y} + e_y)^{-1/\rho_y}$$

with $\rho_c, \rho_y > -1$ and $\sigma_c = 1/(1+\rho_c) \geq 0$, $\sigma_y = 1/(1+\rho_y) \geq 0$ the elasticities of capital/labor substitution in each sector. The externalities, e_c and e_y , will be equal to $e_c = a_1 \bar{K}_c^{-\rho_c} + a_2 \bar{L}_c^{-\rho_c}$, $e_y = b_1 \bar{K}_y^{-\rho_y} + b_2 \bar{L}_y^{-\rho_y}$, with $a_i, b_i \geq 0$, $i = 1, 2$, and where \bar{K}_i, \bar{L}_i denote the average use of capital and labor in sector $i = c, y$. We will assume that these economy-wide averages are taken as given by the individual firm. At the equilibrium, since all firms of sector $i = c, y$ are identical, we have $\bar{K}_i = K_i$ and $\bar{L}_i = L_i$. Denoting $\hat{\alpha}_i = \alpha_i + a_i$, $\hat{\beta}_i = \beta_i + b_i$, the *social production functions* are thus defined as

$$c = (\hat{\alpha}_1 K_c^{-\rho_c} + \hat{\alpha}_2 L_c^{-\rho_c})^{-1/\rho_c} \text{ and } y = (\hat{\beta}_1 K_y^{-\rho_y} + \hat{\beta}_2 L_y^{-\rho_y})^{-1/\rho_y}$$

with $\hat{\alpha}_1 + \hat{\alpha}_2 = 1$ and $\hat{\beta}_1 + \hat{\beta}_2 = 1$. The returns to scale are therefore constant at the social level, and decreasing at the private level.³

Total labor is normalized to one, $L_c + L_y = 1$, and the total stock of capital is given by $K_c + K_y = k$. We consider that capital partially depreciates in one period so that the capital accumulation equation is $y_t = k_{t+1} - (1 - \mu)k_t$, with $\mu \in [0, 1]$. In order to focus the analysis on the interplay between the rate of depreciation and the elasticity of substitution we will assume in the rest of the paper that both technologies are characterized by the same properties of substitution, i.e. $\rho_c = \rho_y = \rho$. The consumer's optimization program will be given by

$$\begin{aligned} \max_{\{K_{c,t}, L_{c,t}, K_{y,t}, L_{y,t}, k_t, y_t\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \delta^t \left(\alpha_1 K_{ct}^{-\rho} + \alpha_2 L_{ct}^{-\rho} + e_{ct} \right)^{-1/\rho} \\ \text{s.t.} & y_t = \left(\beta_1 K_{yt}^{-\rho} + \beta_2 L_{yt}^{-\rho} + e_{yt} \right)^{-1/\rho} \\ & 1 = L_{ct} + L_{yt} \\ & k_t = K_{ct} + K_{yt} \\ & y_t = k_{t+1} - (1 - \mu)k_t \\ & k_0, (e_{ct})_{t=0}^{+\infty}, (e_{yt})_{t=0}^{+\infty} \text{ given} \end{aligned} \quad (1)$$

with $\delta \in (0, 1]$ the discount factor. Denote by p_t , w_{0t} and w_t respectively the price of the capital good, the wage rate of labor and the rental rate of the capital good at time $t \geq 0$, all in terms of the price of the consumption good. For any sequences $\{e_{ct}\}_{t=0}^{\infty}$ and $\{e_{yt}\}_{t=0}^{\infty}$ of external effects that agents consider as given, the Lagrangian at time $t \geq 0$ is:

$$\begin{aligned} \mathcal{L}_t &= \left(\alpha_1 K_{ct}^{-\rho} + \alpha_2 L_{ct}^{-\rho} + e_{ct} \right)^{\frac{-1}{\rho}} + p_t \left[\left(\beta_1 K_{yt}^{-\rho} + \beta_2 L_{yt}^{-\rho} + e_{yt} \right)^{\frac{-1}{\rho}} \right. \\ &\quad \left. - k_{t+1} + (1 - \mu)k_t \right] + w_{0t}(1 - L_{ct} - L_{yt}) + w_t(k_t - K_{ct} - K_{yt}) \end{aligned} \quad (2)$$

For any (k_t, y_t) , solving the first order conditions w.r.t. $(K_{ct}, L_{ct}, K_{yt}, L_{yt})$ and using $y_t = k_{t+1} - (1 - \mu)k_t$ gives inputs as functions of capital stocks at times t and $t + 1$, and external effects, namely: $K_c =$

³Considering the technology of, say, the consumption good, we have for any $\lambda > 1$

$$\begin{aligned} (\alpha_1(\lambda K_c)^{-\rho_c} + \alpha_2(\lambda L_c)^{-\rho_c} + e_c)^{-1/\rho_c} &= \lambda (\alpha_1 K_c^{-\rho_c} + \alpha_2 L_c^{-\rho_c} + \lambda^{\rho_c} e_c)^{-1/\rho_c} \\ &< \lambda (\alpha_1 K_c^{-\rho_c} + \alpha_2 L_c^{-\rho_c} + e_c)^{-1/\rho_c} \end{aligned}$$

$K_c(k_t, k_{t+1}, e_{ct}, e_{yt})$, $L_c = L_c(k_t, k_{t+1}, e_{ct}, e_{yt})$, $K_y = K_y(k_t, k_{t+1}, e_{ct}, e_{yt})$, $L_y = L_y(k_t, k_{t+1}, e_{ct}, e_{yt})$. We then define the production frontier as

$$T(k_t, k_{t+1}, e_{ct}, e_{yt}) = (\alpha_1 K_c^{-\rho} + \alpha_2 L_c^{-\rho} + e_{ct})^{-\frac{1}{\rho}}$$

Using the envelope theorem we derive:

$$p_t = -T_2(k_t, k_{t+1}, e_{ct}, e_{yt}) \quad (3)$$

$$w_t = T_1(k_t, k_{t+1}, e_{ct}, e_{yt}) + (1 - \mu)T_2(k_t, k_{t+1}, e_{ct}, e_{yt}) \quad (4)$$

where $T_1 = \frac{\partial T}{\partial k_t}$ and $T_2 = \frac{\partial T}{\partial k_{t+1}}$. From the first order conditions w.r.t. k_t we obtain

$$-p_t + \delta[w_{t+1} + (1 - \mu)p_{t+1}] = 0 \quad (5)$$

Mixing equations (3)-(5) gives the Euler equation:

$$T_2(k_t, k_{t+1}, e_{ct}, e_{yt}) + \delta T_1(k_{t+1}, k_{t+2}, e_{ct+1}, e_{yt+1}) = 0$$

Any sequence needs also to satisfy the following transversality condition

$$\lim_{t \rightarrow +\infty} \delta^t k_t T_1(k_t, k_{t+1}, e_{ct}, e_{yt}) = 0$$

The above Euler equation is a difference equation parameterized by the external effects. We consider an economy with identical agents so that the average of capital and labor in each sector coincide with the actual demand of capital and labor by the representative agent, i.e. $\bar{K}_i = K_i$, $\bar{L}_i = L_i$. We will explain now how we define an equilibrium path. From the optimal demand functions defined above, solving with respect to K_i and L_i the following system of four equations

$$K_i = K_i(k_t, k_{t+1}, a_1 K_c^{-\rho} + a_2 L_c^{-\rho}, b_1 K_y^{-\rho} + b_2 L_y^{-\rho})$$

$$L_i = L_i(k_t, k_{t+1}, a_1 K_c^{-\rho} + a_2 L_c^{-\rho}, b_1 K_y^{-\rho} + b_2 L_y^{-\rho})$$

for $i = c, y$, gives equilibrium demand functions for capital and labor $\hat{K}_i = \hat{K}_i(k_t, k_{t+1})$ and $\hat{L}_i = \hat{L}_i(k_t, k_{t+1})$ (see Appendix 5.1 for details).⁴ We thus define $\hat{e}_c = \hat{e}_c(k_t, k_{t+1}) = a_1 \hat{K}_c^{-\rho} + a_2 \hat{L}_c^{-\rho}$ and $\hat{e}_y = \hat{e}_y(k_t, k_{t+1}) =$

⁴Since we deal with an example, the existence of an equilibrium path becomes trivial when we actually solve the Euler equation to get the steady state with local stability. Note that the paths that satisfy the Euler equation and converge to the steady state also satisfy the transversality condition.

$b_1 \hat{K}_y^{-\rho} + b_2 \hat{L}_c^{-\rho}$. An equilibrium path $\{k_t\}_{t=0}^{+\infty}$ is a path along which e_{ct} and e_{yt} coincide with $\hat{e}_c(k_t, k_{t+1})$ and $\hat{e}_y(k_t, k_{t+1})$.

Let us introduce

$$\begin{aligned} p_t(k_t, k_{t+1}) &= -T_2(k_t, k_{t+1}, \hat{e}_{ct}(k_t, k_{t+1}), \hat{e}_{yt}(k_t, k_{t+1})) \\ w_t(k_t, k_{t+1}) &= T_1(k_t, k_{t+1}, \hat{e}_{ct}(k_t, k_{t+1}), \hat{e}_{yt}(k_t, k_{t+1})) \\ &\quad + (1 - \mu)T_2(k_t, k_{t+1}, \hat{e}_{ct}(k_t, k_{t+1}), \hat{e}_{yt}(k_t, k_{t+1})) \end{aligned}$$

We then consider the Euler equation evaluated at \hat{e}_c and \hat{e}_y :

$$-p(k_t, k_{t+1}) + \delta[w(k_{t+1}, k_{t+2}) + (1 - \mu)p(k_{t+1}, k_{t+2})] = 0 \quad (6)$$

Any solution $\{k_t\}_{t=0}^{+\infty}$ of the Euler equation (6) which also satisfies the transversality condition

$$\lim_{t \rightarrow +\infty} \delta^t k_t T_1(k_t, k_{t+1}, \hat{e}_{ct}(k_t, k_{t+1}), \hat{e}_{yt}(k_t, k_{t+1})) = 0$$

is called an equilibrium path.

2.2 Steady state

A steady state is defined by $k_t = k_{t+1} = k^*$, $y_t = y^* = \mu k^*$ and is given by the solving of $\delta w(k^*, k^*) - p(k^*, k^*)[1 - \delta(1 - \mu)] = 0$. The methodology used in this paper consists in approximating the Euler equation (6) in order to compute the steady state and the characteristic polynomial.

To simplify notations let $\theta \equiv \delta(1 - \mu) \in [0, 1]$ which is the discounted value of capital carried over to the next period when one unit of capital is used in the current period. We will assume in the rest of the paper the following restriction on parameters' values that will guarantee positiveness of all the steady state values k , K_c , K_y , L_c and L_y .

Assumption 1 . *The parameters δ and μ satisfy*

$$(1 - \theta)^{\rho/(1+\rho)} < (\delta\beta_1)^{\rho/(1+\rho)} \hat{\beta}_1^{-1}$$

If the elasticity of capital-labor substitution in both sectors $\sigma = (1 + \rho)^{-1}$ is greater than one, i.e. $\rho \in (-1, 0)$, Assumption 1 defines a lower bound for the depreciation rate of capital. We have indeed $m(\delta) = 1 - \delta^{-1} + \beta_1 \hat{\beta}_1^{-(1+\rho)/\rho}$ and $1 > \mu > \max\{m(\delta), 0\}$ for $\rho \in (-1, 0)$. On the contrary, if σ is less than one, i.e. $\rho > 0$, $m(\delta)$ defines an upper bound, i.e. $\min\{m(\delta), 0\} > \mu > 0$

for $\rho \geq 0$. Note that when $\rho = 0$, both technologies are Cobb-Douglas, Assumption 1 becomes $\hat{\beta}_1 < 1$ which always holds. From this we obtain existence and uniqueness of the steady state

Proposition 1 . *Under Assumption 1, there exists a unique stationary capital stock $k^* > 0$, such that:*

$$k^* = \left\{ \left(\frac{\alpha_1 \beta_2}{\alpha_2 \beta_1} \right)^{\frac{-1}{1+\rho}} \left(\frac{\delta \beta_1}{1-\theta} \right)^{\frac{-1}{1+\rho}} \left[\frac{1 - \hat{\beta}_1 \left(\frac{\delta \beta_1}{1-\theta} \right)^{\frac{-\rho}{1+\rho}}}{\hat{\beta}_2} \right]^{-\frac{1}{\rho}} \left[1 - \mu \left(\frac{\delta \beta_1}{1-\theta} \right)^{\frac{1}{1+\rho}} \right] + \mu \left[\frac{1 - \hat{\beta}_1 \left(\frac{\delta \beta_1}{1-\theta} \right)^{\frac{-\rho}{1+\rho}}}{\hat{\beta}_2} \right]^{-\frac{1}{\rho}} \right\}^{-1}$$

Proof: See Appendix 5.1.

2.3 The characteristic roots

Let $V_i(k_t, k_{t+1})$ denote $T_i(k_t, k_{t+1}, \hat{e}_c(k_t, k_{t+1}), \hat{e}_y(k_t, k_{t+1}))$ for $i = 1, 2$. Provided $V_{12}(k^*, k^*) \neq 0$, the linearization of the Euler equation around the steady state gives the following characteristic polynomial:

$$\delta x^2 + x \left(\frac{V_{22}(k^*, k^*)}{V_{12}(k^*, k^*)} + \delta \frac{V_{11}(k^*, k^*)}{V_{12}(k^*, k^*)} \right) + \frac{V_{21}(k^*, k^*)}{V_{12}(k^*, k^*)} = 0 \quad (7)$$

We now provide expressions of the characteristic roots.

Theorem 1 . *Under Assumption 1, the characteristic roots are given by*

$$x_1(\rho, \mu, \delta) = (1 - \mu) + \left\{ \left(\frac{\delta \beta_2}{1-\theta} \right)^{\frac{1}{1+\rho}} \left[\left(\frac{\beta_1}{\beta_2} \right)^{\frac{1}{1+\rho}} - \left(\frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{1+\rho}} \right] \right\}^{-1}$$

$$x_2(\rho, \mu, \delta) = \frac{\left(\frac{\delta \beta_2}{1-\theta} \right)^{\frac{1}{1+\rho}} \frac{\hat{\beta}_2}{\beta_2} \left[\frac{\hat{\alpha}_1}{\hat{\alpha}_2} \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{\rho}{1+\rho}} - \frac{\hat{\beta}_1}{\hat{\beta}_2} \left(\frac{\beta_2}{\beta_1} \right)^{\frac{\rho}{1+\rho}} \right]}{\theta \left(\frac{\delta \beta_2}{1-\theta} \right)^{\frac{1}{1+\rho}} \frac{\hat{\beta}_2}{\beta_2} \left[\frac{\hat{\alpha}_1}{\hat{\alpha}_2} \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{\rho}{1+\rho}} - \frac{\hat{\beta}_1}{\hat{\beta}_2} \left(\frac{\beta_2}{\beta_1} \right)^{\frac{\rho}{1+\rho}} \right] - \delta}$$

Proof: See Appendix 5.2.

We have now to study the stability properties of the steady state depending on the value of the technological parameters. It can be easily proved from the static first order conditions derived from the Lagrangian (2):

Proposition 2 . Under Assumption 1, at the steady state:

i) the investment (consumption) good sector is capital intensive from the private perspective if and only if $\beta_1/\beta_2 > (<) \alpha_1/\alpha_2$;

ii) the investment (consumption) good sector is capital intensive from the social perspective if and only if $\hat{\beta}_1/\hat{\beta}_2 > (<) \hat{\alpha}_1/\hat{\alpha}_2$.

Proof: *i)* We define input coefficients at the private level where a_{1i} is the capital input and a_{2i} is the labor input to produce one unit of good $i = c, y$.

From the first order conditions

$$\begin{aligned} a_{1c} &= K_c/c = (\alpha_1/w)^{1/(1+\rho)}, & a_{2c} &= L_c/c = (\alpha_2/w_0)^{1/(1+\rho)} \\ a_{1y} &= K_y/y = (p\beta_1/w)^{1/(1+\rho)}, & a_{2y} &= L_y/y = (p\beta_2/w_0)^{1/(1+\rho)} \end{aligned}$$

Assuming $\rho = \rho$, it follows easily that

$$a_{1c}a_{2y} - a_{2c}a_{1y} = (p/(w_0))^{1/(1+\rho)} \left((\alpha_1\beta_2)^{1/(1+\rho)} - (\alpha_2\beta_1)^{1/(1+\rho)} \right)$$

The statement of Proposition 2-*i)* is immediate from this.

ii) If we consider that the agents take account of externalities as endogenous variables in the maximisation, the first order conditions become

$$\begin{aligned} \hat{\alpha}_1 (c/K_c)^{1+\rho} - w &= 0 \\ \hat{\alpha}_2 (c/L_c)^{1+\rho} - w_0 &= 0 \\ p\hat{\beta}_1 (y/K_y)^{1+\rho} - w &= 0 \\ p\hat{\beta}_2 (y/L_y)^{1+\rho} - w_0 &= 0 \end{aligned}$$

We may also define input coefficients $\hat{a}_{1i}, \hat{a}_{2i}$ at the social level from the above first order conditions:

$$\hat{a}_{1c}\hat{a}_{2y} - \hat{a}_{2c}\hat{a}_{1y} = (p/(w_0))^{1/(1+\rho)} \left((\hat{\alpha}_1\hat{\beta}_2)^{1/(1+\rho)} - (\hat{\alpha}_2\hat{\beta}_1)^{1/(1+\rho)} \right)$$

The result follows. □

We will therefore discuss the stability properties of equilibria depending on the capital intensity differences at the private and social levels.

Definition 1 . A steady state k^* is called locally indeterminate if there exists $\epsilon > 0$ such that from any k_0 belonging to $(k^* - \epsilon, k^* + \epsilon)$ there are infinitely many equilibrium paths converging to the steady state.

If both roots of the characteristic equation have modulus less than one then the steady state is locally indeterminate. If a steady state is not locally

indeterminate, then we call it locally determinate. We first prove that as in the Cobb-Douglas framework local indeterminacy will require the investment good to be labor intensive at the private level.⁵

Proposition 3 . *Under Assumption 1, if the investment good is capital intensive at the private level, the steady state is locally determinate.*

Proof: If the investment good is capital intensive at the private level then $\alpha_1\beta_2 - \alpha_2\beta_1 < 0$. This implies $x_1(\rho, \mu, \delta) > 0$. Moreover it is easy to check that for any given $\mu \in [0, 1]$, $x_1(\rho, \mu, \delta)$ is a decreasing function of δ . The result will be proved if $x_1(\rho, \mu, \delta) > 1$ when $\delta = 1$. Assume that $x_1(\rho, \mu, 1) \leq 1$. This is equivalent to

$$\begin{aligned} & (\alpha_2\beta_1)^{\frac{1}{1+\rho}} - (\alpha_1\beta_2)^{\frac{1}{1+\rho}} \geq (\alpha_2\mu^{-\rho})^{\frac{1}{1+\rho}} \\ \Leftrightarrow & \alpha_2^{\frac{1}{1+\rho}} \left[\beta_1^{\frac{1}{1+\rho}} - \mu^{\frac{-\rho}{1+\rho}} \right] - (\alpha_1\beta_2)^{\frac{1}{1+\rho}} \geq 0 \end{aligned}$$

Assumption 1 is rewritten as $\mu^{-\rho/(1+\rho)} > \beta_1^{-\rho/(1+\rho)}\hat{\beta}_1$, and this implies $\beta_1^{\frac{1}{1+\rho}} - \mu^{\frac{-\rho}{1+\rho}} < 0$. This is in contradiction with $x_1(\rho, \mu, 1) \leq 1$. Therefore $x_1(\rho, \mu, 1) > 1$ must hold and $x_1(\rho, \mu, \delta) > 1$ for all $\delta \in (0, 1]$. \square

3 Main results

We will be concerned with the analysis of local indeterminacy when δ is sufficiently close to one. For such δ , saddle-point stability is a typical dynamics when externalities are not present. Therefore we will assume that $\delta = 1$ in the analysis throughout the rest of this section. By continuity, the local indeterminacy or saddle-point stability at $\delta = 1$ is carried over to the cases in which δ is less than 1 as long as it is sufficiently close to 1.

We discuss the local stability properties of the steady state when the investment good is labor intensive at the private level, i.e. $\alpha_1\beta_2 - \alpha_2\beta_1 > 0$. Under this condition, $x_1(\rho, \mu, \delta)$ is always less than one but its sign is not

⁵See Benhabib, Nishimura and Venditti [3].

determined. Next we consider the other root $x_2(\rho, \mu, \delta)$. First consider case which implies that $x_2(\rho, \mu, \delta)$ is positive, i.e. $\hat{\alpha}_1\hat{\beta}_2/\hat{\alpha}_2\hat{\beta}_1 < (\alpha_1\beta_2/\alpha_2\beta_1)^{\frac{\rho}{1+\rho}}$. Note that when $\delta = 1$

$$m(1) = \beta_1\hat{\beta}_1^{-(1+\rho)/\rho}$$

We introduce the following assumption.

Assumption 2 . $\beta_2^{\frac{1}{1+\rho}} \left[\left(\frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{1+\rho}} - \left(\frac{\beta_1}{\beta_2} \right)^{\frac{1}{1+\rho}} \right] > \frac{m(1)^{\frac{1}{1+\rho}}}{1-m(1)}$

Under this Assumption, the investment good is always labor intensive at the private level, and when $\delta = 1$, $x_1(\rho, \mu, \delta)$ is positive.

Theorem 2 . *Let Assumptions 1, 2 and $\hat{\alpha}_1\hat{\beta}_2/\hat{\alpha}_2\hat{\beta}_1 < (\alpha_1\beta_2/\alpha_2\beta_1)^{\frac{\rho}{1+\rho}}$ hold. The following cases hold:*

- i) If $(\alpha_1\beta_2)^{\frac{1}{1+\rho}} - (\alpha_2\beta_1)^{\frac{1}{1+\rho}} > \alpha_2^{\frac{1}{1+\rho}}$, the steady state is locally indeterminate for any $\mu \in [0, 1]$ when $\rho \geq 0$ or for any $\mu \in (m(1), 1]$ when $\rho \in (-1, 0)$;*
- ii) If $(\alpha_1\beta_2)^{\frac{1}{1+\rho}} - (\alpha_2\beta_1)^{\frac{1}{1+\rho}} \leq \alpha_2^{\frac{1}{1+\rho}}$, there exists $\mu_{F1} \in (0, 1)$ such that the steady state is locally indeterminate for any $\mu \in [0, \mu_{F1})$ when $\rho \geq 0$, or there exists $\mu_{F2} \in (m(1), 1)$ such that the steady state is locally indeterminate for any $\mu \in (m(1), \mu_{F2})$ when $\rho \in (-1, 0)$.⁶*

Proof: We start by studying $x_2(\rho, \mu, 1)$. Under the hypothesis $\hat{\alpha}_1\hat{\beta}_2/\hat{\alpha}_2\hat{\beta}_1 < (\alpha_1\beta_2/\alpha_2\beta_1)^{\frac{\rho}{1+\rho}}$ of the Theorem, $x_2(\rho, \mu, 1)$ is always positive. Moreover $x_2(\rho, \mu, 1) < 1$ if and only if

$$\frac{\hat{\beta}_1}{\hat{\beta}_2} \left(\frac{\beta_2}{\beta_1} \right)^{\frac{\rho}{1+\rho}} - \frac{\hat{\alpha}_1}{\hat{\alpha}_2} \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{\rho}{1+\rho}} < \left(\frac{\beta_2}{\mu} \right)^{\frac{\rho}{1+\rho}} \frac{1}{\hat{\beta}_2}$$

Notice that this may be rewritten as

$$\frac{\beta_2^{\frac{\rho}{1+\rho}}}{\hat{\beta}_2} \left[m(1)^{\frac{-(1+\rho)}{\rho}} - \mu^{\frac{-\rho}{1+\rho}} \right] - \frac{\hat{\alpha}_1}{\hat{\alpha}_2} \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{\rho}{1+\rho}} < 0 \quad (8)$$

When $\rho \geq 0$, Assumption 1 implies that $m(\delta)$ is an upper bound for μ . But when $\delta = 1$, $m(1)$ exceeds 1. So $m(1) > \mu$ and the first term in (8) is negative. Therefore (8) holds for every $\mu \in [0, 1]$. When $\rho \in (-1, 0)$,

⁶Note that if μ crosses μ_{F1} or μ_{F2} from below the steady state becomes saddle-point stable, and there generically exist equilibrium period-two cycles either in a left or in a right neighbourhood of μ_{F1} or μ_{F2} .

Assumption 1 implies $\mu \in (m(1), 1]$ and the first term in (8) is also negative. Therefore (8) holds for every $\mu \in (m(1), 1]$.

Now we study $x_1(\rho, \mu, 1)$. Since the investment good is labor intensive at the private level, $x_1(\rho, \mu, 1)$ is always less than 1. Therefore we only need to check if $x_1(\rho, \mu, 1)$ is larger than -1 or not. We have $x_1(\rho, \mu, 1) > -1$ iff

$$\beta_2^{\frac{1}{1+\rho}} \left[\left(\frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{1+\rho}} - \left(\frac{\beta_1}{\beta_2} \right)^{\frac{1}{1+\rho}} \right] > \frac{\mu^{\frac{1}{1+\rho}}}{2-\mu} \quad (9)$$

Consider first the case in which $(\alpha_1\beta_2)^{\frac{1}{1+\rho}} - (\alpha_2\beta_1)^{\frac{1}{1+\rho}} > \alpha_2^{\frac{1}{1+\rho}}$. This is equivalent to

$$\beta_2^{\frac{1}{1+\rho}} \left[\left(\frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{1+\rho}} - \left(\frac{\beta_1}{\beta_2} \right)^{\frac{1}{1+\rho}} \right] > 1$$

Note that

$$1 \geq \frac{\mu^{\frac{1}{1+\rho}}}{2-\mu}$$

for all $\mu \in [0, 1]$. Therefore (9) holds for all $\mu \in [0, 1]$. However $x_2(\rho, \mu, 1) < 1$ does not require any restriction on μ when $\rho \geq 0$, but requires $\mu \in (m(1), 1]$ when $\rho \in (-1, 0)$. Case *i*) of the Theorem immediately follows.

Consider now the converse case in which $(\alpha_1\beta_2)^{\frac{1}{1+\rho}} - (\alpha_2\beta_1)^{\frac{1}{1+\rho}} \leq \alpha_2^{\frac{1}{1+\rho}}$. $x_1(\rho, \mu, 1) > 0$ if and only if

$$\beta_2^{\frac{1}{1+\rho}} \left[\left(\frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{1+\rho}} - \left(\frac{\beta_1}{\beta_2} \right)^{\frac{1}{1+\rho}} \right] > \frac{\mu^{\frac{1}{1+\rho}}}{1-\mu} \quad (10)$$

The right-hand-side of (10) is increasing in μ , becomes zero when $\mu = 0$ and arbitrarily large when μ approaches to 1. Therefore there exists $\mu^* \in (0, 1)$ that makes the right-hand-side of (10) equal to its left-hand-side. For μ^* , (9) is always satisfied. Therefore $x_1(\rho, \mu, 1) > -1$ when μ is close to μ^* . The hypothesis $(\alpha_1\beta_2)^{\frac{1}{1+\rho}} - (\alpha_2\beta_1)^{\frac{1}{1+\rho}} \leq \alpha_2^{\frac{1}{1+\rho}}$ implies that (9) cannot hold when μ is equal to 1. When $\rho \geq 0$, Assumption 1 does not impose any lower bound on μ . Therefore there exists $\mu_{F1} \in (\mu^*, 1)$ such that $x_1(\rho, \mu, 1) > -1$ when $\mu \in (0, \mu_{F1})$. When $\rho \in (-1, 0)$, Assumption 1 implies that $\mu > m(1)$. Therefore there exists $\mu_{F2} \in (\mu^*, 1)$ such that $x_1(\rho, \mu, 1) > -1$ when $\mu \in (m(1), \mu_{F2})$. Case *ii*) of the Theorem immediately follows. \square

Remark: When $\rho \in (-1, 0]$, inequality $\hat{\alpha}_1\hat{\beta}_2/\hat{\alpha}_2\hat{\beta}_1 < (\alpha_1\beta_2/\alpha_2\beta_1)^{\frac{\rho}{1+\rho}}$ requires the investment good to be capital intensive at the social level. When

$\rho > 0$, this inequality is also compatible with an investment good labor intensive at the social level.

Theorem 2 clearly shows that when the elasticity of capital-labor substitution is less than 1, local indeterminacy may arise for low values of the rate of capital depreciation. More precisely, if the investment good is strongly labor intensive at the private level, i.e. case *i*), local indeterminacy does not require any restriction on the rate of depreciation when $\rho \geq 0$. If the investment good is weakly labor intensive at the private level, i.e. case *ii*), local indeterminacy cannot hold with complete depreciation. It appears also that when the elasticity of capital-labor substitution is greater than 1, local indeterminacy is more likely if the rate of depreciation is high. Notice also that equilibrium period-two cycles may appear through Flip bifurcations. In the super-critical case, the cycles exist in a right neighbourhood of μ_{F1} or μ_{F2} and are locally indeterminate.

We need now to deal with the converse configuration to the hypothesis of Theorem 2, i.e. $\hat{\alpha}_1\hat{\beta}_2/\hat{\alpha}_2\hat{\beta}_1 > (\alpha_1\beta_2/\alpha_2\beta_1)^{\frac{\rho}{1+\rho}}$. In order to focus on the case in which $x_2(\rho, \mu, 1)$ may be positive we introduce an additional restriction that will also define an upper bound for the ratio $\hat{\alpha}_1\hat{\beta}_2/\hat{\alpha}_2\hat{\beta}_1$. These two conditions are summarized into the following assumption:

Assumption 3 . $\left(1 + m(1)^{\frac{1+\rho}{\rho}}\right) \left(\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)^{\frac{\rho}{1+\rho}} > \frac{\hat{\alpha}_1\hat{\beta}_2}{\hat{\alpha}_2\hat{\beta}_1} > \left(\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)^{\frac{\rho}{1+\rho}}$

Theorem 3 . Under Assumptions 1, 2, 3, let $(\alpha_1\beta_2)^{\frac{1}{1+\rho}} - (\alpha_2\beta_1)^{\frac{1}{1+\rho}} > \alpha_2^{\frac{1}{1+\rho}}$ hold. Then there exists $\tilde{\mu} \in]0, 1[$ such that the steady state is locally indeterminate for any $\mu \in]\tilde{\mu}, 1[$.

Proof: From the proof of Theorem 2, if $(\alpha_1\beta_2)^{\frac{1}{1+\rho}} - (\alpha_2\beta_1)^{\frac{1}{1+\rho}} > \alpha_2^{\frac{1}{1+\rho}}$, then $0 > x_1(\rho, \mu, 1) > -1$. Next we study $x_2(\rho, \mu, 1)$. Since $\hat{\alpha}_1\hat{\beta}_2/\hat{\alpha}_2\hat{\beta}_1 > (\alpha_1\beta_2/\alpha_2\beta_1)^{\frac{\rho}{1+\rho}}$, the numerator of $x_2(\rho, \mu, 1)$ is positive. Its denominator is negative if and only if:

$$\beta_2^{\frac{-\rho}{1+\rho}}\hat{\beta}_2 \left[\frac{\hat{\alpha}_1}{\hat{\alpha}_2} \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{\rho}{1+\rho}} - \frac{\hat{\beta}_1}{\hat{\beta}_2} \left(\frac{\beta_2}{\beta_1}\right)^{\frac{\rho}{1+\rho}} \right] < \frac{\mu^{1/(1+\rho)}}{1-\mu}$$

This inequality is satisfied if μ is sufficiently close to 1. Therefore there exists $\hat{\mu} \in (0, 1)$ such that $x_2(\rho, \mu, 1) < 0$ for $\mu \in (\hat{\mu}, 1]$. In this case $x_2(\rho, \mu, 1) > -1$ if and only if:⁷

$$\beta_2^{-\frac{\rho}{1+\rho}} \hat{\beta}_2 \left[\frac{\hat{\alpha}_1}{\hat{\alpha}_2} \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{\rho}{1+\rho}} - \frac{\hat{\beta}_1}{\hat{\beta}_2} \left(\frac{\beta_2}{\beta_1} \right)^{\frac{\rho}{1+\rho}} \right] < \frac{\mu^{1/(1+\rho)}}{2-\mu} \quad (11)$$

Under Assumption 3 this inequality holds when $\mu = 1$. When $\mu = \hat{\mu}$ the left-hand-side of (11) is equal to $\hat{\mu}^{1/(1+\rho)}/(1-\hat{\mu})$ and (11) becomes

$$\frac{\hat{\mu}^{1/(1+\rho)}}{1-\hat{\mu}} < \frac{\hat{\mu}^{1/(1+\rho)}}{2-\hat{\mu}}$$

which is not possible. Then, there exists $\mu_F \in (\hat{\mu}, 1)$ such that $x_2(\rho, \mu_F, 1) = -1$ and $x_2(\rho, \mu, 1) > -1$ if and only if $\mu \in (\mu_F, 1]$.

The proof is completed by taking $\tilde{\mu} = \max\{\mu(1), \mu_F\}$. □

Contrary to the previous case in which the inequality $\hat{\alpha}_1 \hat{\beta}_2 / \hat{\alpha}_2 \hat{\beta}_1 < (\alpha_1 \beta_2 / \alpha_2 \beta_1)^{\frac{\rho}{1+\rho}}$ holds, this Theorem shows that for any given value of the elasticity of capital-labor substitution, local indeterminacy is more likely if the rate of depreciation is high.

4 Concluding comments

The main objective of this paper has been to study the interplay between the elasticity of capital-labor substitution and the rate of capital depreciation, and its influence on the local determinacy properties of the steady state. We have proved the following results: when the elasticity of substitution is less than one, local indeterminacy may arise for any value of the rate of depreciation of capital. When the elasticity of substitution is greater than one, local indeterminacy is more likely if the rate of depreciation is high.

We have assumed that both sectors have the same elasticity of substitution. It would be interesting to study how our results are modified if some heterogeneity is introduced on this parameter.

⁷We have indeed $x_2(\rho, \mu, 1) \geq 0$ if $\mu \in [0, \hat{\mu})$ and it is immediate from Theorem 1 that $x_2(\rho, \mu, 1) > 1$ for any $\mu \in [0, \hat{\mu})$.

5 Appendix

5.1 Proof of Proposition 1

lemma 5.1 . Let $y = k_1 - (1 - \mu)k_0$. Along an equilibrium path, the partial derivatives of $T(k_0, k_1, e_c, e_y)$ with respect to k_0 and k_1 are given by

$$T_1(k_0, k_1, \hat{e}_c(k_0, k_1), \hat{e}_y(k_0, k_1)) = (1 - \mu) \frac{w}{\beta_1} \left(\frac{g}{y} \right)^{1+\rho} + \alpha_1 \left\{ \hat{\alpha}_1 + \hat{\alpha}_2 \left(\frac{\alpha_1 \beta_2}{\alpha_2 \beta_1} \right)^{\frac{\rho}{1+\rho}} \left(\frac{(g/y)^\rho}{\hat{\beta}_2} - \frac{\hat{\beta}_1}{\hat{\beta}_2} \right) \right\}^{-\frac{1+\rho}{\rho}}$$

$$T_2(k_0, k_1, \hat{e}_c(k_0, k_1), \hat{e}_y(k_0, k_1)) = -\frac{w}{\beta_1} \left(\frac{g}{y} \right)^{1+\rho}$$

where

$$g = g(k_0, k_1) = \left\{ K_y \in [0, k_0] / \left(\frac{k_0 - K_y}{1 - L_y(K_y, y)} \right)^{1+\rho} \left(\frac{L_y(K_y, y)}{K_y} \right)^{1+\rho} = \left(\frac{\alpha_1 \beta_2}{\alpha_2 \beta_1} \right) \right\}$$

and

$$L_y(K_y, y) = \left(\frac{y^{-\rho} - \hat{\beta}_1 K_y^{-\rho}}{\hat{\beta}_2} \right)^{-1/\rho}$$

Proof: From the Lagrangian (2) we derive the first order conditions:

$$\alpha_1 (c/K_c)^{1+\rho} - w = 0 \quad (12)$$

$$\alpha_2 (c/L_c)^{1+\rho} - w_0 = 0 \quad (13)$$

$$p\beta_1 (y/K_y)^{1+\rho} - w = 0 \quad (14)$$

$$p\beta_2 (y/L_y)^{1+\rho} - w_0 = 0 \quad (15)$$

Using $K_c = k_0 - K_y$, $L_y = 1 - L_c$, and manipulating (12)-(15) give

$$L_y = \left((y^{-\rho} - \hat{\beta}_1 K_y^{-\rho}) / \hat{\beta}_2 \right)^{-1/\rho} \quad (16)$$

$$\left(\frac{\alpha_1 \beta_2}{\alpha_2 \beta_1} \right) = \left(\frac{k_0 - K_y}{1 - L_y} \right)^{1+\rho} \left(\frac{L_y}{K_y} \right)^{1+\rho} \quad (17)$$

By solving (16)-(17) with respect to K_y and substituting $y = k_1 - (1 - \mu)k_0$, we get $K_y = g(k_0, k_1)$. From (12) we have

$$w = \alpha_1 \left\{ \hat{\alpha}_1 + \hat{\alpha}_2 \left(\frac{\alpha_1 \beta_2}{\alpha_2 \beta_1} \right)^{\frac{\rho}{1+\rho}} \left(\frac{g}{L_y} \right)^\rho \right\}^{-\frac{1+\rho}{\rho}}$$

Substituting an expression of (g/L_y) taken from (16) gives

$$w = \alpha_1 \left\{ \hat{\alpha}_1 + \hat{\alpha}_2 \left(\frac{\alpha_1 \beta_2}{\alpha_2 \beta_1} \right)^{\frac{\rho}{1+\rho}} \left(\frac{(g/y)^\rho}{\hat{\beta}_2} - \frac{\hat{\beta}_1}{\hat{\beta}_2} \right) \right\}^{-\frac{1+\rho}{\rho}} \quad (18)$$

Moreover we have from (14)

$$p = \frac{w}{\beta_1} \left(\frac{g}{y} \right)^{1+\rho}$$

The result follows from the envelope theorem which gives

$$T_1 = p(1 - \mu) + w, \quad T_2 = -p \quad \square$$

We may now prove Proposition 1. A steady state k^* is defined as $T_2(k^*, k^*, e_c^*, e_y^*) + \delta T_1(k^*, k^*, e_c^*, e_y^*) = 0$ with $e_c^* = \hat{e}_c(k^*, k^*)$ and $e_y^* = \hat{e}_y(k^*, k^*)$. Denote $g^* = g(k^*, k^*)$ and $y^* = \mu k^*$. Using the derivatives of T given in Lemma 5.1 in the definition of k^* gives

$$g^* = \left(\frac{\delta \beta_1}{1 - \theta} \right)^{\frac{1}{1+\rho}} \mu k^* \quad (19)$$

with $\theta = \delta(1 - \mu)$. When $k_0 = k^*$, equations (16)-(17) and (19) give

$$K_c^* = k^* \left(1 - \mu \left(\frac{\delta \beta_1}{1 - \theta} \right)^{\frac{1}{1+\rho}} \right) \quad (20)$$

$$L_y^* = \mu k^* \left(\frac{1 - \hat{\beta}_1 \left(\frac{\delta \beta_1}{1 - \theta} \right)^{-\frac{\rho}{1+\rho}}}{\hat{\beta}_2} \right)^{-\frac{1}{\rho}} \quad (21)$$

$$L_c^* = 1 - \mu k^* \left(\frac{1 - \hat{\beta}_1 \left(\frac{\delta \beta_1}{1 - \theta} \right)^{-\frac{\rho}{1+\rho}}}{\hat{\beta}_2} \right)^{-\frac{1}{\rho}} \quad (22)$$

Now we substitute these input demand functions into equation (17)

$$\left(\frac{K_c^*}{L_c^*} \right)^{1+\rho} \left(\frac{L_y^*}{g^*} \right)^{1+\rho} = \left(\frac{\alpha_1 \beta_2}{\alpha_2 \beta_1} \right) \quad (23)$$

and after some algebra we get:

$$k^* = \left\{ \left(\frac{\alpha_1 \beta_2}{\alpha_2 \beta_1} \right)^{\frac{-1}{1+\rho}} \left(\frac{\delta \beta_1}{1 - \theta} \right)^{\frac{-1}{1+\rho}} \left[\frac{1 - \hat{\beta}_1 \left(\frac{\delta \beta_1}{1 - \theta} \right)^{-\frac{\rho}{1+\rho}}}{\hat{\beta}_2} \right]^{-\frac{1}{\rho}} \left[1 - \mu \left(\frac{\delta \beta_1}{1 - \theta} \right)^{\frac{1}{1+\rho}} \right] + \mu \left[\frac{1 - \hat{\beta}_1 \left(\frac{\delta \beta_1}{1 - \theta} \right)^{-\frac{\rho}{1+\rho}}}{\hat{\beta}_2} \right]^{-\frac{1}{\rho}} \right\}^{-1}$$

We need now to show that k^* is positive. Direct inspection of its expression shows that $k^* > 0$ if

$$\left(\frac{\delta\beta_1}{1-\theta}\right)^{\frac{1}{1+\rho}} < \mu^{-1} \quad \text{and} \quad \left(\frac{\delta\beta_1}{1-\theta}\right)^{\frac{-\rho}{1+\rho}} < \hat{\beta}_1^{-1}$$

Assumption 1 implies the second inequality, and the second inequality implies the first one. It follows that $k^* > K_c^* > 0$ and $1 > L_y^* > 0$. \square

5.2 Proof of Theorem 1

Denote in what follows $g_i = \partial g(k_0, k_1)/\partial k_i$ for $i = 1, 2$.

lemma 5.2 . Under Assumption 1, at the steady state the following hold

$$g_1 = \frac{\frac{1+\rho}{K_c} - (1-\mu) \left[(1+\rho)L_y^\rho + (1+\rho)\frac{L_y^{1+\rho}}{L_c} \right] \frac{y^{-1-\rho}}{\hat{\beta}_2}}{\Delta}$$

$$g_2 = \frac{\left[(1+\rho)L_y^\rho + (1+\rho)\frac{L_y^{1+\rho}}{L_c} \right] \frac{y^{-1-\rho}}{\hat{\beta}_2}}{\Delta}$$

with g , K_c , L_y , L_c respectively given by equations (19)-(22) and

$$\Delta = \frac{1+\rho}{g} + \frac{1+\rho}{K_c} + \left((1+\rho)L_y^\rho + (1+\rho)\frac{L_y^{1+\rho}}{L_c} \right) \frac{\hat{\beta}_1}{\hat{\beta}_2} g^{-1-\rho}$$

Proof: From equation (23) we get

$$\frac{\alpha_1\beta_2}{\alpha_2\beta_1} = g^{-1-\rho}(k_0 - g)^{1+\rho} \left(\frac{y^{-\rho} - \hat{\beta}_1 g^{-\rho}}{\hat{\beta}_2} \right)^{-\frac{1+\rho}{\rho}} \left\{ 1 - \left(\frac{y^{-\rho} - \hat{\beta}_1 g^{-\rho}}{\hat{\beta}_2} \right)^{-\frac{1}{\rho}} \right\}^{-1-\rho}$$

Totally differentiating this expression gives after simplification

$$\begin{aligned} & \left[(1+\rho)g^{-1} + (1+\rho)(x-g)^{-1} + (1+\rho) \left(\frac{y^{-\rho} - \hat{\beta}_1 g^{-\rho}}{\hat{\beta}_2} \right)^{-1} \frac{\hat{\beta}_1}{\hat{\beta}_2} g^{-1-\rho} \right. \\ & + (1+\rho) \left\{ 1 - \left(\frac{y^{-\rho} - \hat{\beta}_1 g^{-\rho}}{\hat{\beta}_2} \right)^{-\frac{1}{\rho}} \right\}^{-1} \left(\frac{y^{-\rho} - \hat{\beta}_1 g^{-\rho}}{\hat{\beta}_2} \right)^{-\frac{1+\rho}{\rho}} \frac{\hat{\beta}_1}{\hat{\beta}_2} g^{-1-\rho} \left. \right] dg \\ & = (1+\rho)(x-g)^{-1} dk_0 + (1+\rho) \left(\frac{y^{-\rho} - \hat{\beta}_1 g^{-\rho}}{\hat{\beta}_2} \right)^{-1} \frac{y^{-1-\rho}}{\hat{\beta}_2} dy \\ & + (1+\rho) \left\{ 1 - \left(\frac{y^{-\rho} - \hat{\beta}_1 g^{-\rho}}{\hat{\beta}_2} \right)^{-\frac{1}{\rho}} \right\}^{-1} \left(\frac{y^{-\rho} - \hat{\beta}_1 g^{-\rho}}{\hat{\beta}_2} \right)^{-\frac{1+\rho}{\rho}} \frac{y^{-1-\rho}}{\hat{\beta}_2} dy \end{aligned}$$

Notice from (16) and (23) that

$$\frac{(g/y)^\rho}{\hat{\beta}_2} - \frac{\hat{\beta}_1}{\hat{\beta}_2} = \left(\frac{g}{L_y} \right)^\rho \quad \text{and} \quad \left(\frac{\alpha_1\beta_2}{\alpha_2\beta_1} \right)^{\frac{1}{1+\rho}} = \frac{K_c}{L_c} \left(\frac{g}{L_y} \right)^{-1} \quad (24)$$

Substituting (24) into the previous total differentiation and considering $dy = dk_1 - (1 - \mu)dk_0$ together with

$$\Delta = \frac{1 + \rho}{g} + \frac{1 + \rho}{K_c} + \left((1 + \rho)L_y^\rho + (1 + \rho)\frac{L_y^{1+\rho}}{L_c} \right) \frac{\hat{\beta}_1}{\hat{\beta}_2} g^{-1-\rho}$$

we derive

$$\begin{aligned} \Delta dg &= dk_0 \left\{ \frac{1+\rho}{K_c} - (1-\mu) \left((1+\rho)L_y^\rho + (1+\rho)\frac{L_y^{1+\rho}}{L_c} \right) \frac{y^{-1-\rho}}{\hat{\beta}_2} \right\} \\ &\quad + dk_1 \left((1+\rho)L_y^\rho + (1+\rho)\frac{L_y^{1+\rho}}{L_c} \right) \frac{y^{-1-\rho}}{\hat{\beta}_2} \end{aligned}$$

This completes the proof. \square

lemma 5.3 . Under Assumption 1, at the steady state the following hold

$$g_1y + (1 - \mu)g = (g - g_2y) \left\{ (1 - \mu) + \frac{y}{g} \left(1 - \frac{K_c L_y}{L_c g} \right)^{-1} \right\}$$

with g , K_c , L_y , L_c respectively given by equations (19)-(22).

Proof: From Lemma 5.2 and equation (16) we have

$$g - g_2y = \frac{(1 + \rho)y}{K_c \Delta} g \left(1 - \frac{K_c L_y}{L_c g} \right)$$

and

$$g_1y + (1 - \mu)g = \frac{(1+\rho)y}{K_c \Delta} + \frac{(1-\mu)}{\Delta} \left[\Delta g - \left((1 + \rho)L_y^\rho + (1 + \rho)\frac{L_y^{1+\rho}}{L_c} \right) \frac{y^{-\rho}}{\hat{\beta}_2} \right]$$

Substituting the expression of Δ into the term between brackets and using again equation (16), we obtain

$$\begin{aligned} g_1y + (1 - \mu)g &= \frac{(1+\rho)y}{K_c \Delta} + \frac{(1-\mu)}{\Delta} (1 + \rho) \frac{g}{K_c} \left(1 - \frac{K_c L_y}{L_c g} \right) \\ &= \frac{(1+\rho)y}{K_c \Delta} g \left(1 - \frac{K_c L_y}{L_c g} \right) \left\{ (1 - \mu) + \frac{y}{g} \left(1 - \frac{K_c L_y}{L_c g} \right)^{-1} \right\} \quad \square \end{aligned}$$

lemma 5.4 . Under Assumption 1, at the steady state, $k_0 = k_1 = k^*$, $y = \mu k^*$ and the following hold:

$$\begin{aligned} \frac{V_{11}(k^*, k^*)}{V_{12}(k^*, k^*)} &= -\frac{y}{g} \left(1 - \frac{K_c L_y}{L_c g} \right)^{-1} - (1 - \mu) \\ \frac{V_{22}(k^*, k^*)}{V_{12}(k^*, k^*)} &= -\frac{g}{\beta_1 y} \frac{\mathcal{A} - \left(\frac{g}{y}\right)^\rho}{\frac{(1-\mu)g}{\beta_1 y} \mathcal{A} - \left[1 + \frac{1-\mu}{\beta_1} \left(\frac{g}{y}\right)^{1+\rho} \right]} \\ \frac{V_{21}(k^*, k^*)}{V_{12}(k^*, k^*)} &= \frac{V_{22}(k^*, k^*) V_{11}(k^*, k^*)}{V_{12}(k^*, k^*) V_{12}(k^*, k^*)} \end{aligned}$$

with

$$\mathcal{A} = \frac{\alpha_2 \beta_1 \hat{\alpha}_1 \hat{\beta}_2 K_c L_y}{\alpha_1 \beta_2 \hat{\alpha}_2 L_c g} + \hat{\beta}_2 \left(\frac{g}{L_y} \right)^\rho$$

and g, K_c, L_y, L_c respectively given by equations (19)-(22).

Proof: Let $V_i(k_0, k_1)$ denote $T_i(k_0, k_1, \hat{e}_c(k_0, k_1), \hat{e}_y(k_0, k_1))$ for $i = 1, 2$. By definition we have

$$\begin{aligned} V_{11}(k^*, k^*) &\equiv V_{11}^* = \frac{\partial p}{\partial k_0}(1 - \mu) + \frac{\partial w}{\partial k_0}, & V_{21}(k^*, k^*) &\equiv V_{21}^* = -\frac{\partial p}{\partial k_0} \\ V_{12}(k^*, k^*) &\equiv V_{12}^* = \frac{\partial p}{\partial k_1}(1 - \mu) + \frac{\partial w}{\partial k_1}, & V_{22}(k^*, k^*) &\equiv V_{22}^* = -\frac{\partial p}{\partial k_1} \end{aligned}$$

Some simple computations give

$$\begin{aligned} \frac{\partial w}{\partial k_0} &= -(1 + \rho) \alpha_1^{-\frac{\rho}{1+\rho}} w^{\frac{1+2\rho}{1+\rho}} \frac{\hat{\alpha}_2}{\beta_2} \left(\frac{\alpha_1 \beta_2}{\alpha_2 \beta_1} \right)^{\frac{\rho}{1+\rho}} \left(\frac{g}{y} \right)^\rho \left(\frac{g_1 y + (1-\mu)g}{y g} \right) \\ \frac{\partial p}{\partial k_0} &= \frac{1}{\beta_1} \frac{\partial w}{\partial k_0} \left(\frac{g}{y} \right)^{1+\rho} + (1 + \rho) \frac{w}{\beta_1} \left(\frac{g}{y} \right)^{1+\rho} \left(\frac{g_1 y + (1-\mu)g}{y g} \right) \\ \frac{\partial w}{\partial k_1} &= -(1 + \rho) \alpha_1^{-\frac{\rho}{1+\rho}} w^{\frac{1+2\rho}{1+\rho}} \frac{\hat{\alpha}_2}{\beta_2} \left(\frac{\alpha_1 \beta_2}{\alpha_2 \beta_1} \right)^{\frac{\rho}{1+\rho}} \left(\frac{g}{y} \right)^\rho \left(\frac{g_2 y - g}{y g} \right) \\ \frac{\partial p}{\partial k_1} &= \frac{1}{\beta_1} \frac{\partial w}{\partial k_1} \left(\frac{g}{y} \right)^{1+\rho} + (1 + \rho) \frac{w}{\beta_1} \left(\frac{g}{y} \right)^{1+\rho} \left(\frac{g_2 y - g}{y g} \right) \end{aligned}$$

It follows therefore

$$\begin{aligned} V_{11}^* &= \frac{\partial w}{\partial k_0} \left(1 + \frac{1-\mu}{\beta_1} \left(\frac{g}{y} \right)^{1+\rho} \right) + (1 + \rho) w \frac{1-\mu}{\beta_1} \left(\frac{g}{y} \right)^{1+\rho} \left(\frac{g_1 y + (1-\mu)g}{y g} \right) \\ &= (1 + \rho) w \frac{1-\mu}{\beta_1} \left(\frac{g}{y} \right)^\rho \left(\frac{g_1 y + (1-\mu)g}{y g} \right) \left\{ \frac{1-\mu}{\beta_1} \frac{g}{y} - \left(1 + \frac{1-\mu}{\beta_1} \left(\frac{g}{y} \right)^{1+\rho} \right) \right. \\ &\quad \left. \times \left(\frac{w}{\alpha_1} \right)^{\frac{\rho}{1+\rho}} \frac{\hat{\alpha}_2}{\beta_2} \left(\frac{\alpha_1 \beta_2}{\alpha_2 \beta_1} \right)^{\frac{\rho}{1+\rho}} \right\} \end{aligned}$$

Substituting (w/α_1) from (18), and using (24) we get

$$\begin{aligned} V_{11}^* &= (1 + \rho) w \left(\frac{g}{y} \right)^\rho \left(\frac{g_1 y + (1-\mu)g}{y g} \right) \left\{ \frac{1-\mu}{\beta_1} \frac{g}{y} - \left(1 + \frac{1-\mu}{\beta_1} \left(\frac{g}{y} \right)^{1+\rho} \right) \right. \\ &\quad \left. \times \left[\frac{\hat{\alpha}_1 \hat{\beta}_2}{\hat{\alpha}_2} \left(\frac{\alpha_2 \beta_1}{\alpha_1 \beta_2} \right) \frac{K_c L_y}{L_c g} + \hat{\beta}_2 \left(\frac{g}{L_y} \right)^\rho \right]^{-1} \right\} \end{aligned}$$

Following the same procedure we obtain

$$\begin{aligned}
V_{12}^* &= (1 + \rho)w \left(\frac{g}{y}\right)^\rho \left(\frac{g_2y-g}{yg}\right) \left\{ \frac{1-\mu}{\beta_1} \frac{g}{y} - \left(1 + \frac{1-\mu}{\beta_1} \left(\frac{g}{y}\right)^{1+\rho}\right) \right. \\
&\quad \left. \times \left[\frac{\hat{\alpha}_1 \hat{\beta}_2}{\hat{\alpha}_2} \left(\frac{\alpha_2 \beta_1}{\alpha_1 \beta_2}\right) \frac{K_c}{L_c} \frac{L_y}{g} + \hat{\beta}_2 \left(\frac{g}{L_y}\right)^\rho \right]^{-1} \right\} \\
V_{21}^* &= -(1 + \rho) \frac{w}{\beta_1} \left(\frac{g}{y}\right)^{1+\rho} \left(\frac{g_1y+(1-\mu)g}{yg}\right) \left\{ 1 - \left(\frac{g}{y}\right)^\rho \right. \\
&\quad \left. \times \left[\frac{\hat{\alpha}_1 \hat{\beta}_2}{\hat{\alpha}_2} \left(\frac{\alpha_2 \beta_1}{\alpha_1 \beta_2}\right) \frac{K_c}{L_c} \frac{L_y}{g} + \hat{\beta}_2 \left(\frac{g}{L_y}\right)^\rho \right]^{-1} \right\} \\
V_{22}^* &= -(1 + \rho) \frac{w}{\beta_1} \left(\frac{g}{y}\right)^{1+\rho} \left(\frac{g_2y-g}{yg}\right) \left\{ 1 - \left(\frac{g}{y}\right)^\rho \right. \\
&\quad \left. \times \left[\frac{\hat{\alpha}_1 \hat{\beta}_2}{\hat{\alpha}_2} \left(\frac{\alpha_2 \beta_1}{\alpha_1 \beta_2}\right) \frac{K_c}{L_c} \frac{L_y}{g} + \hat{\beta}_2 \left(\frac{g}{L_y}\right)^\rho \right]^{-1} \right\}
\end{aligned}$$

Then we get

$$\begin{aligned}
\frac{V_{11}^*}{V_{12}^*} &= \frac{g_1y + (1 - \mu)g}{g_2y - g} \tag{25} \\
\frac{V_{22}^*}{V_{12}^*} &= -\frac{g}{\beta_1 y} \frac{\mathcal{A} - \left(\frac{g}{y}\right)^\rho}{\frac{(1-\mu)g}{\beta_1 y} \mathcal{A} - \left[1 + \frac{1-\mu}{\beta_1} \left(\frac{g}{y}\right)^{1+\rho}\right]} \\
\frac{V_{21}^*}{V_{12}^*} &= \frac{V_{22}^*}{V_{12}^*} \frac{V_{11}^*}{V_{12}^*}
\end{aligned}$$

with

$$\mathcal{A} = \frac{\hat{\alpha}_1 \hat{\beta}_2}{\hat{\alpha}_2} \left(\frac{\alpha_2 \beta_1}{\alpha_1 \beta_2}\right) \frac{K_c}{L_c} \frac{L_y}{g} + \hat{\beta}_2 \left(\frac{g}{L_y}\right)^\rho$$

Considering Lemmas 5.2 and 5.3 with equation (25) completes the proof. \square

We may now prove Theorem 1. From Lemma 5.4 the characteristic polynomial may be written as

$$\mathcal{P}(x) = \left(x + \frac{V_{11}^*}{V_{12}^*}\right) \left(\delta x + \frac{V_{22}^*}{V_{12}^*}\right)$$

and the characteristic roots are

$$x_1 = -V_{11}^*/V_{12}^* \quad \text{and} \quad x_2 = -V_{22}^*/(\delta V_{12}^*) \tag{26}$$

We then evaluate terms in the expressions of V_{11}^*/V_{12}^* and V_{22}^*/V_{12}^* :

$$\begin{aligned}\frac{K_c L_y}{L_c g} &= \left(\frac{\alpha_1 \beta_2}{\alpha_2 \beta_1} \right)^{\frac{1}{1+\rho}} \\ \frac{g}{y} &= \left(\frac{\delta \beta_1}{1-\theta} \right)^{\frac{1}{1+\rho}} \\ \left(\frac{g}{L_y} \right)^\rho &= \frac{(g/y)^\rho}{\beta_2} - \frac{\hat{\beta}_1}{\beta_2}\end{aligned}$$

It follows

$$\mathcal{A} = \frac{\alpha_2 \beta_1 \hat{\alpha}_1 \hat{\beta}_2}{\alpha_1 \beta_2 \hat{\alpha}_2} \left(\frac{\alpha_1 \beta_2}{\alpha_2 \beta_1} \right)^{\frac{1}{1+\rho}} - \hat{\beta}_1 + \left(\frac{g}{y} \right)^\rho$$

Let $\mathcal{B} = [\mathcal{A} - (g/y)^\rho]/\beta_1$. We also obtain after substitution

$$\frac{(1-\mu)g}{\beta_1 y} \mathcal{A} - \left[1 + \frac{1-\mu}{\beta_1} \left(\frac{g}{y} \right)^{1+\rho} \right] = (1-\mu) \left(\frac{\delta \beta_1}{1-\theta} \right)^{\frac{1}{1+\rho}} \mathcal{B} - 1$$

Considering the roots (26), the rest of the proof follows from the substitution of all the above equations into the expressions given in Lemma 5.4. \square

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