Marketing via Friends: Strategic Diffusion of Information in Social Networks with Homophily

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Abstract

The paper studies the impact of homophily on the optimal strategies of a monopolist, whose marketing campaign of new product relies on a word of mouth communication in the network. Homophily is a tendency of people to interact more with those who are similar to them. In the model there are two types of consumers, which differ in linking preferences and the desirable design of the product. Consumers are engaged in word of mouth communication and can learn about the product directly from advertisement or from their neighbors. The monopolist uses product design and price to influence the decision of consumers to buy the product and to select the pattern of communication that takes place within the network. We find a number of results: (i) for low levels of homophily the product, designed to attract both types of consumers is preferred to specialized products even if there is no cost of producing more than one product; (ii) price elasticity of demand is higher for products towards which consumer networks exhibit higher level of homophily; (iii) social welfare is increasing in the level of homophily; and (iv) a product designed to attract two types of consumers may be optimal even if the monopolist benefits only from one type.

JEL Classification numbers: D21, D42, D60, D83, L11, L12
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1 Introduction

In the last decade word of mouth (WOM hereafter) and viral marketing have received considerable amount of attention from mass-media and the scientific community as efficient marketing tools (see for instance Campbell, 2009, Goyal and Galeotti, 2007, Leskovec et al. 2007, and Iribarren and Moro, 2007). The idea that a company can recruit consumers to advertise its products for free is really exciting. WOM marketing takes an advantage of the natural human inclination to spread information. A recent study by Reichheld (2003) shows that willingness of consumers to recommend a company to their friends not just augments sales, but by far is the best predictor of a company's growth.

Apart from being efficient when it works, performance of a WOM campaign is quite uncertain. A report by Riley and Wigder (2007) from Jupiter Research reveals that only 15% of viral campaigns are considered to be successful, moreover 55% of companies planned to reduce the use of this tactic next year. This rises the question why companies that face the same network of consumers show so different performance in terms of success of WOM campaigns? In the paper we argue that volatile behavior of WOM marketing campaigns at least partially can be explained by a phenomenon known as homophily. Homophily is a tendency of people to interact more with those who are similar to them. It has been documented at least since Aristotle's time.[1],[2]

Our paper contributes to the WOM literature in two dimensions. First and most importantly, the paper introduces homophily into the network upon which WOM spreads and studies its impact on the optimal strategy of the monopolist. The notion of homophily enriches network structure by specifying a probability of friendship relationships among groups of consumers. Second, the paper extends the monopolist's problem by including product design that affects further WOM communication. To the author's best knowledge product design has not been the subject of academic research in WOM framework.

The description of the model is following. There is a monopolist that introduces a new product to an initially uninformed population of heterogeneous consumers of two types. Consumers are embedded into a social network, which is represented by a random graph with arbitrary degree distribution. Across types, consumers differ in friendship preferences and desirable design of the product. Within types, consumers differ in a willingness to pay for the product. We model consumers friendship preferences by a linking bias towards types, which represents homophily level of the society. Consumers communicate with their friends and learn about the existence of the product from neighbors who already have acquired it. The monopolist knows the degree distribution and homophily level of the society and strategically chooses the price and design of the product. To induce sales

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[1] In Aristotles Rhetoric and Nichomachean Ethics, he noted that people "love those who are like themselves" (Aristotle 1934, p. 1371).

[2] The term homophily appeared in the sociological literature for the first time in Lazarsfeld and Mertons (1954) who also quoted the proverbial expression - "birds of a feather flock together," which has summarized homophily ever since.

the monopolist advertises the product directly to an infinitesimal part of the population and the rest of the population is expected to find out about the product through WOM communication.

Our analysis begins by examining a necessary conditions on network structure such that WOM can spread over a significant proportion of the population[3]. This was a case of such remarkable examples of WOM campaigns as diffusion of Hotmail accounts[4] and the advertising campaign of tiny budget movie "The Blair Witch Project"[5]. We find that in sparse networks a sufficiently high level of homophily is a necessary condition for success of WOM campaign. High levels of homophily imply that preferences of connected consumers are correlated. This allows monopolist to develop the product attractive for longer chains of connected consumers.

Next, we turn to the optimal design of the product. Networks literature on diffusion assumes that a message to be spread in the network is given, and focuses on the effect of network structure on its propagation (for a survey see Geroski, 2000). In contrast, we assume an active role of the monopolist. In the model the monopolist designs a message to the network by choosing the price and characteristic of the product. In our base-line model we find that for sufficiently high levels of homophily, when people mostly interact with those who are similar to them, specialized products designed to target needs of one type of consumers are optimal. However, for sufficiently low levels of homophily, the product attractive for both types of consumers is preferred to specialized design even if there is no cost of producing more than one product. The latter happens, since majority of the links are between consumers of different types and to insure spreading of the information product should be attractive to both types.

The sociological literature on homophily adopts a view that diversity of individual's contacts is a socially desirable property per se (e.g. Moody, 2001). Although this assertion could be supported by evidence, no rigorous analysis has been made. Perhaps surprisingly, in our model social welfare is increasing in the level of homophily. The result comes from informational and monetary benefits for consumers generated by an increase in the level of homophily. Informational benefits consist in higher awareness of consumers about the product. Monetary benefits come from a lower price charged by the monopolist, which converts a higher awareness of the product into a higher volume of sales.

There is a popular idea in business and academic literature that focusing advertisement efforts on a group of consumers is the efficient strategy. We show that it is indeed true - an advertisement targeted to consumers of one group is optimal for the monopolist. However, the same does not always hold for the product design. In the case when society exhibits low level of homophily the idea holds depending on the density of the social network. If

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[3] In the network literature this phenomenon is known as global cascade

[4] Hotmail spent a mere 50,000 dollars on traditional marketing and still grew from zero to 12 million users in 18 months.

[5] A movie released in 1999 with principal photography budget ranging from \$20,000 to \$25,000.

the density of social network is low then expected demand triggered per advertisement is small and the monopolist specializes the product on a group of consumers targeted by advertisement. If the network density is sufficiently high then it is optimal for the monopolist to choose compromise design. This strategy sacrifices some initial adopters from the targeted group, but insures that initial acquisition leads to higher level of WOM communication.

A term “freakonomics” has firmly entered to vocabularies of many economists. The popular book of the same name[6] with over 3 million copies sold worldwide gained popularity not only among the general public, but became well known in the academic community (e.g. DiNardo, 2006, DiNardo 2007, Rubinstein, 2006). Despite extraordinary success, the case of “Freakonomics” is not unique. One can recall such examples as “Linked: The New Science of Networks” on networks by Barabási, “The Selfish Gene” on evolution by Richard Dawkins etc. All these books provoked numerous discussions in academic circles, while the primary audience was the general public. Influenced by this phenomenon, we consider the monopolist that is interested only in one type of consumer (for instance the academic community). We show that designing the product attractive to both types of consumers may be the optimal strategy even though monopolist benefits only from one type.

1.1 Related Literature

In this section we relate our paper to two streams of network literature. The first one studies strategic diffusion of information in networks (see, for instance, Campbell, 2009, Goyal and Galeotti, 2007, Galeotti and Mattozzi, 2008). In this literature a network is represented by generalized random graph with underlying assumption that nodes are matched randomly. The second stream of literature studies the impact of homophily on various processes unfolding on networks (for instance Golub and Jackson, 2009, Van der Leij et al., 2009, Buhai and Van der Leij, 2008 and Valat, 2009). Our paper bridges these two literatures in a simple model which studies the impact of social structure, given by the homophily, on information diffusion in the network. The model allows us to yield a number of insights into how homophily level shapes optimal strategies of the monopolist and influences social welfare.

There are two recent papers studying strategic diffusion of information that are most related to our research: Campbell (2009) and Goyal and Galeotti (2007). Campbell (2009) studies the optimal pricing and advertisement strategies of a monopolist when consumers are engaged in WOM communication. Goyal and Galeotti (2007) study general model of strategic diffusion where they distinguish between level of interaction and content of interaction. In their paper, the level of interaction is characterized by degree distribution

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[6] The full title of the book: “Freakonomics: A Rogue Economist Explores the Hidden Side of Everything”.

while content of interaction is a way in which actions of others affect individual incentives. There are two key differences between these papers and our paper. First, we extend existing models by relaxing random matching assumption and study how intrinsic property of the network such as who is connected to whom affects strategic diffusion of information. Second, to our best knowledge, we are the first to consider the optimal design of the product in the presence of WOM communication.

The recent paper from second stream of literature Golub and Jackson (2009) studies how different mechanisms of communication operating through network are affected by the homophily level of the society. The principal difference of our paper is that in our setup the monopolist (sender of message) takes an active role and influences WOM spreading by choosing the design of the product.

Within the broader literature that considers epidemic diffusion (Newman, 2002 and Sander et al., 2002) our paper contributes to the analysis of multi-type networks with homophily by extending Newman's generating functions approach. In particular, we consider the case when node is operational (is able to transmit information further in the network) with some probability, which depends on a type of the node.

The rest of the paper is organized as follows. Section 2 presents a stylized model of strategic diffusion. In section 3 we derive the expected size of the cascade of sales per advertisement. Section 4 presents the main results on the optimal price and design strategies and considers welfare implications of homophily. Section 5 examines the optimal product design and advertisement strategies when the monopolist can target advertisements by types of consumer. Section 6 considers robustness of the obtained results to the variation in shape of the preference frontier. Section 7 studies the optimal strategy of the monopolist that is interested only in one group of consumers. Section 8 considers the case of a global cascade of sales. Finally, Section 9 outlines avenues for future research and concludes.

2 Model

In this section we formally present the model. There are three main blocks, which constitute the model: network structure, consumer preferences and the monopolist problem.

2.1 Network Structure

There is a continuum of consumers of two types A and B, which are embedded into undirected social network. Consumers of type A constitute measure γ of the population and the rest are consumers of type B. The main reason why we focus on the case of consumers of two types is because it provides basic intuitions and insights, while keeping the analysis transparent[7].

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[7] For example, the case of three types with the third type that is not interested in the product is the same as the case of two types with degree distribution that does not count links to the third type.

The network is represented by a random graph characterized by a degree distribution $p(k)$ and probabilities of ties among types of consumers (ρ^A, ρ^B). The parameter ρ^i is the probability that a randomly chosen link of consumer of type i leads to a consumer of the same type. Hence, on average consumer of type A has proportion ρ^A of her neighbors of type A and proportion $1 - \rho^A$ of consumers of type B.

Links of randomly selected consumer can be partitioned into two sets: links to consumers of the same type and links to consumers of another type. Each link of randomly selected consumer leads to a consumer of the same type with the probability ρ^i and with complementary probability to consumer of another type. Thus the probability that a consumer of type i with k links has $j \leq k$ links to consumers of the same type is given by a binomial expression:

$$Pr\left(J = j|k, \rho^i\right) = \frac{k!}{j!(k-j)!}(\rho^i)^j(1-\rho^i)^{k-j} \tag{1}$$

The expected number of links connecting type i consumer to consumers of the same type is given by:

$$\mathbb{E}\left(J|k, \rho^i\right) = \sum_{j=0}^{k}\left[j \times Pr\left(J = j|k, \rho^i\right)\right] = k\rho^i$$

A randomly selected consumer of type A with probability $p(k)$ has k links, $k(1 - \rho^A)$ of which connect her to consumers of type B. Taking the expectation we find that on average consumer of type A has $z_1(1 - \rho^A)$ links to consumers of type B, where z_1 is expected number of the first neighbors. Multiplying obtained expression by the measure of consumers of type A in the population we obtain total number of links of type AB, which is $\gamma z_1(1 - \rho^A)$. By analogy the number of links of type BA is equal to $(1 - \gamma)z_1(1 - \rho^B)$. Using the fact that the graph is undirected and number of links of type AB should be equal to number of links of type BA we arrive to the equality $\gamma(1-\rho^A) = (1-\gamma)(1-\rho^B)$. Solving for ρ^B we obtain:

$$\rho^B = 1 - \frac{\gamma}{1-\gamma}(1 - \rho^A) \tag{2}$$

Therefore, without loss of generality, in the case of two types we have just one parameter $\rho = \rho^A$ that characterizes linking preferences of consumers. The parameter ρ represents the level of homophily of the society, since it specifies the probability of friendship relationships among consumers of the same type, for both types.

It is important to underline that friendship relationships among consumers are formed on the basis of many parameters such as geographical proximity, common interests and so on. The network formation itself is beyond scope of this paper. In the analysis we assume that the network of social contacts is exogenously given, and is the same for all products

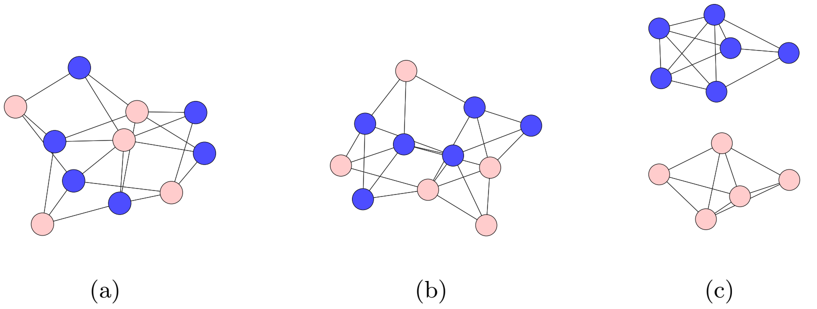

Figure 1: All three graphs have nodes with the same number of neighbors, however they differ in the homophily level. In (a) consumers are linked only to consumers of another type, $\rho = 0$; in (b) we have random mixing of consumers, $\rho = 0.5$; in (c) consumers exhibit extreme homophily, $\rho = 1$.

in question. Spreading of WOM in the network for different types of products, therefore varies due to different levels of homophily that society exhibits towards the product.

Figure 1 illustrates 3 different networks with the same degree distribution, moreover all nodes preserve the same connectivity. The only parameter that changes is the level of homophily of the society. As one can see, depending on ρ networks range from perfectly mixed to two separated graphs, where consumers of type A are completely disjoint from consumers of type B.

To avoid ambiguity we introduce some key definitions concerning measurement of the level of homophily. A benchmark case that we will use extensively is the case when links among consumers are formed with uniform probability independently of type.

Definition 1. *The friendship ties in the society are randomly matched if* $\rho = \gamma$.

One of the examples of network with random matching can be a network of detergent consumers. A plausible assumption would be that preference towards liquid or powdered detergent is not important itself for forming ties among people and thus consumers of detergent are matched randomly.

In the sociological literature, the tendency of friendship to be biased towards own type beyond the relative proportion in the population is referred to as "inbreeding homophily" (see, for example Coleman, 1958, Marsden, 1987 and McPherson et al., 2001). In this case the proportion of links going to consumers of the same type is higher than otherwise would be implied by random matching.

Definition 2. *The society exhibits inbreeding homophily if* $\rho > \gamma$

There are also networks in which the situation can be reversed and social ties are biased towards different-type relationships (e.g. network of sexual contacts).

Definition 3. *The society exhibits heterophily if $\rho < \gamma$*

To illustrate ideas let us consider examples of random matching and society which exhibits homophily. If consumers are matched randomly with uniform probability then consumer of type A has on average proportion $\rho = \gamma$ of neighbors of the same type. At the same time the expression (2) implies that average proportion of neighbors of consumer of type B of the same type is $\rho^B = (1 - \gamma)$, which equals to proportion of consumers of type B in the population. In the case when consumers of type A are linked more often among themselves as compared to the case of random matching, $\rho > \gamma$, by expression (2) the same applies to consumers of type B, since $\rho^B > (1 - \gamma)$.

2.2 Consumer Preferences

The consumers, in addition of having linking preferences, differ in two other respects. First, across types consumers differ in preferences towards the design of the product. Consumers of type A prefer one characteristic of the product while consumers of type B are interested in the opposite feature. Second, within types consumers differ in the reservation price $\bar{P}_j$ that they are willing to pay for the product and the minimal level of the desirable characteristic $\bar{w}_j$, which induces them to buy the product.

More formally, in the model two variables affect the decision of consumers: the price $P \in [0, 1]$ and characteristic of the product $w \in [0, 1]$. For concreteness, a consumer j of type A buys the product if characteristic is higher than the threshold characteristic level $w \geq \bar{w}_j$ and the price is lower than the reservation price $P \leq \bar{P}_j$. In contrast, a consumer l of type B buys the product if $w \leq \bar{w}_l$ and $P \leq \bar{P}_l$.

We assume that within a type the reservation price and the characteristic threshold are distributed according to $f^i(\bar{w}, \bar{P})$ probability density function. Hence, a randomly chosen consumer j of type A, which is aware of the product with characteristic w and price P buys it with the probability:

$$q^A = Pr(w \geq \bar{w}_j \cap P \leq \bar{P}_j) = \int_0^w \int_P^1 f^A(\bar{w}, \bar{P}) d\bar{w} d\bar{P}$$

And similarly a randomly chosen consumer l of type B, which knows about the product buys it with the probability:

$$q^B = Pr(w \leq \bar{w}_l \cap P \leq \bar{P}_l) = \int_w^1 \int_P^1 f^B(\bar{w}, \bar{P}) d\bar{w} d\bar{P}$$

To simplify the analysis for major part of it we assume that both types have threshold characteristic and threshold price that are distributed independently and identically according to the uniform distribution $U[0, 1]$. This implies that $f^A(\bar{w}, \bar{P}) = f^B(\bar{w}, \bar{P}) = 1$ and probabilities to buy the product are given by the following expressions:

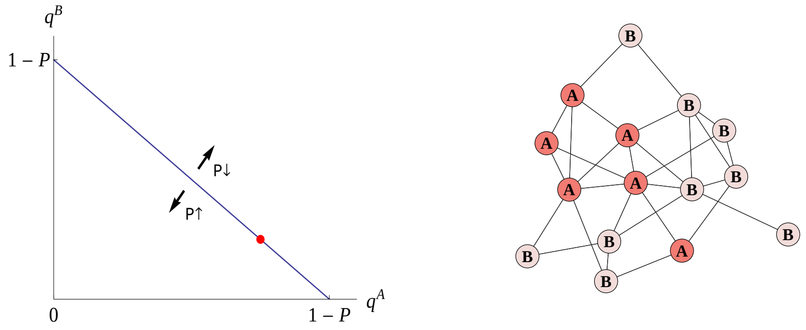

Figure 2: On the left hand side preferences frontier with characteristic of the product being marked by circle. On the right hand side implied social network, with probability to buy the product shown by intensity of the color.

$$\begin{cases} q^A &= (1-P)w \\ q^B &= (1-P)(1-w) \end{cases}$$

For given price the system describes a preference frontier, depicted on the Figure 2, which encompasses all admissible pairs of probabilities for two types of consumers to buy the product. By choosing product design the monopolist identifies a probability pair (q^A, q^B) and fixes network of potential buyers. The network of potential buyers consists of consumers that buy the product conditional that they know about it. The increase in P moves frontier inwards, simultaneously decreasing the probabilities to buy the product for two types, and decrease in P moves frontier outwards.

In the paper we will encounter two special types of product design.

Definition 4. *The design is called symmetric if the product characteristic w is such that two types of consumers acquire the product with the same probability $q^A = q^B$.*

In the case described above, the symmetric design is represented by $w = \frac{1}{2}$.

Definition 5. *The design is called specialized if product characteristic $w \in \{0, 1\}$, which implies that only one type of consumers acquires the product.*

These two types of design represent different marketing strategies. A symmetric design intends to satisfy needs of both types of consumers, without giving preference to any of them, while the specialized one focuses on one type and neglects the other.

2.3 Monopoly Problem

The monopolist develops new product and introduces it to consumers who are engaged in WOM communication. In the model the monopolist chooses design of the product w

and price P to maximize profits. To induce sales the monopolist advertises the product directly to an infinitesimal part of the population. The rest of the population is expected to find out about the product from neighbors who have acquired the product. Diffusion of information stops when there are no new acquisitions of the product.

There are two possible scenarios of information spreading: information propagates to some finite number of consumers and than stops or continues to propagate unboundedly. Let us give precise definition of the latter case:

Definition 6. *We say that the global cascade of sales arises if ultimately some non-infinitesimal proportion of the population buys the product.*

Depending wether the global cascade of sales arises there are two techniques available to study diffusion of information. The main results of the paper are developed for the case when of finite sales, while in Section 8 we study the case of global cascade.

3 Cascade of Sales

In this section we derive the expression for the expected size of cascade of sales generated by one advertisement and study its properties. In the derivation of expression for size of sales cascade we rely on generating functions approach for multi-type nodes based on Newman (2003). The main focus of Newman's paper is heterogeneity of types in terms of degree distribution. Our paper adopts different perspective. While two types of consumers enjoy the same degree distribution, they differ in their willingness to purchase the product. This implies that further transmission of information depends on the way in which different types are linked.

3.1 Generating Functions Approach

In the field of complex networks, generating functions were introduced by Newman et al. (2001) and since than have been widely used. The main idea is that generating function encapsulates all the information about degree distribution, and thus completely characterizes a random graph. The generating functions allow us to calculate various local and global properties, such as average degree, average size of component, etc.

In the case of nodes of two types we need to define two generating functions associated with degree distribution of each type of consumer. Recall that probability of having j links to nodes of the same type for a randomly selected node of type i with k links is given by $Pr(J = j|k, \rho^i)$, which is described in (1). The probability pseudo-generating function $F_0^i(x, y)$, where $i \in \{A, B\}$, is given by:

$$F_0^i(x, y) = \sum_{k=0}^{\infty} p(k) q^i \sum_{j=0}^{k} Pr(J = j|k, \rho^i) x^j y^{k-j} \tag{3}$$

This is polynomial expression in x and y where the coefficient on $x^j y^{k-j}$ is the probability $p(k)q^i Pr(J=j|k,\rho^i)$ that a randomly chosen consumer of type i buys the product, given that she has j links to consumers of the same type and $k-j$ links to consumers of another type. These functions are known as pseudo-generating due to the fact that for $x=y=1$ they do not sum to 1. This happens since not all consumers buy the product. Actually, $F_0^i(1,1)=q^i$, which is the probability that a randomly chosen consumer of type i buys the product given that she is aware about it.

Using binomial identity we can perform the sum over j and the expression reduces to the following:

$$F_0^i(x,y)=\sum_{k=0}^{\infty} p(k)q^i[\rho^i x+(1-\rho^i)y]^k$$

Note, when $x=y$, we obtain the generating function that characterizes the degree distribution of consumers, when there is only one type:

$$F_0^i(x,x)=\sum_{k=0}^{\infty} p(k)q^i x^k$$

This inheritance allows us to calculate number of useful properties applying the same techniques as in the case of one-type nodes. Taking the k-th derivative and normalizing by the factor $k!F_0^i(1,1)$ one can recover degree distribution of consumers:

$$\frac{1}{k!F_0^i(1,1)}\left[\left(\frac{\partial}{\partial x}\right)^k F_0^i(x,x)\right]_{x=0}=p(k)$$

Moment of the probability distribution of order m can be calculated by deriving the generating function m times and multiplying each time by x:

$$\frac{1}{F_0^i(1,1)}\left[\left(x\frac{\partial}{\partial x}\right)^m F_0^i(x,x)\right]_{x=1}=\sum_k k^m p(k)$$

For example average degree of consumer of type i who is equal to $\frac{1}{q^i}\frac{\partial F_0^i(1,1)}{\partial x}$.

Apart of the characteristics intrinsic to all random graphs we also can calculate properties tailored to types of consumer. For example, the degree distribution of links going to nodes of the same type for node of type i is given by:

$$\frac{1}{k!F_0^i(1,1)}\left[\left(\frac{\partial}{\partial x}\right)^k F_0^i(x,1)\right]_{x=0}=Pr(J=j|k,\rho^i)\times p(k)$$

The degree distribution of neighbor of a randomly chosen consumer plays the important role in the analysis to come. Note it is not the same as degree distribution of randomly selected consumer, since the more links neighbor has the more often she is selected from neighborhood. A consumer with k links is found k-times more often through friends than a consumer with one link. Therefore, the probability to have a neighbor with k links is proportional to $kp(k)$. After normalization we obtain that degree distribution of a neighboring node $\xi(k)$ is given by:

$$\xi(k) = \frac{kp(k)}{\sum_{j=1}^{\infty} jp(j)} = \frac{kp(k)}{z_1}$$

Using degree distribution of neighboring consumer we can write the generating function characterizing degree distribution of a neighboring node of consumer of type i:

$$F_1^i(x,y) = \sum_{k=0}^{\infty} \xi(k) q^i [\rho^i x + (1-\rho^i) y]^k = \frac{1}{z_1} [\rho^i x + (1-\rho^i) y] \left(\frac{\partial F_0^i(x,y)}{\partial x} + \frac{\partial F_0^i(x,y)}{\partial y} \right)$$

The important characteristic that affects the process of information diffusion or spreading of a disease in the network is the excess degree of a neighboring node. That is to say we want to find generating functions that characterize the probability that neighboring consumer of type i has k links apart of the link which leads us to this consumer. The excess degree distribution is given by:

$$\hat{\xi}(k) = \xi(k+1) = \frac{(k+1)p(k+1)}{z_1}$$

And associated generating functions are:

$$\begin{aligned} \hat{F}_1^i(x,y) &= \sum_{k=0}^{\infty} \hat{\xi}(k) q^i [\rho^i x + (1-\rho^i) y]^k \\ &= \sum_{k=1}^{\infty} \xi(k) q^i [\rho^i x + (1-\rho^i) y]^{k-1} \\ &= \frac{1}{z_1} \left(\frac{\partial F_0^i(x,y)}{\partial x} + \frac{\partial F_0^i(x,y)}{\partial y} \right) \end{aligned}$$

The generating function characterizing the degree distribution of second neighbors who buy the product is given by:

$$\sum_{k=0}^{\infty} p(k) q^i [\rho^i \hat{F}_1^i(x,y) + (1-\rho^i) \hat{F}_1^{\sim i}(x,y)]^k = F_0^i(\hat{F}_1^i(x,y), \hat{F}_1^{\sim i}(x,y)),$$

where $\sim i$ denotes type of consumer which is different from type i.

Using the expression we can calculate z_2, expected number of second neighbors of randomly selected node. Since we are interested in the expected number of neighbors regardless of type we put $y = x$. To account for all neighbors in the following calculation we assume the probability to buy product for two types q^A and q^B equal to 1. This implies that $\hat{F}_1^i(x,y) = \hat{F}_1^{\sim i}(x,y)$, since now there is no difference between consumers of different types. Applying method described above we arrive to the expression[8]:

$$\begin{aligned} z_2 &= \left[\tfrac{1}{F_0^i(1,1)} \times x\tfrac{\partial}{\partial x}F_0^i(\hat{F}_1^i(x,x))\right]_{x=1,q^i=1} \\ &= \sum_{k=0}^{\infty} kp(k)[\hat{F}_1^i(1,1)]^{k-1} \times \tfrac{\partial \hat{F}_1^i(1,1)}{\partial x} \\ &= \sum_{k=0}^{\infty} kp(k) \sum_{n=1}^{\infty} \xi(n)(n-1) \\ &= \sum_{n=1}^{\infty} n(n-1)p(n) \end{aligned}$$

In the following analysis we assume that the underlying conditions are such that no giant component of consumers who buy the product arises. The case of giant cascade of sales is considered in Section 8. Let us denote by $H_1^i(x,y)$ generating functions characterizing probability distribution of sizes of finite components of buyers, induced by the information flow through a randomly chosen link to a consumer of type i. If the consumer of type i does not buy the product component is empty (since information does not spread any further). This happens with the probability:

$$1 - \sum_{k=1}^{\infty} \xi(k)q^i = 1 - \hat{F}_1^i[1,1],\ i \in \{A,B\}$$

However, with complementary probability a consumer with degree k buys the product and transmits information to neighbors. Further spread of information is subject to analogous considerations for $k-1$ additional links and is described by $\hat{F}_1^i[H_1^i(x,y), H_1^{\sim i}(x,y)]$. This leads us to the following system of self-consistency conditions for $H_1^i(x,y)$:

$$\begin{cases} H_1^A(x,y) &= 1 - \hat{F}_1^A[1,1] + x\hat{F}_1^A[H_1^A(x,y), H_1^B(x,y)] \\ H_1^B(x,y) &= 1 - \hat{F}_1^B[1,1] + y\hat{F}_1^B[H_1^A(x,y), H_1^B(x,y)] \end{cases}$$

where the leading factor x and y account for the fact that the first visited consumer buys the product.

On the basis of $H_1^i(x,y)$ we can define $H_0^i(x,y)$ generating functions of size of buyers components generated by advertisement to a randomly chosen consumer of type i. Since a randomly chosen consumer of type i does not buy the product with the probability $1 - F_0^i(1,1)$ we have:

$$\begin{cases} H_0^A(x,y) &= 1 - F_0^A(1,1) + xF_0^A[H_1^A(x,y), H_1^B(x,y)] \\ H_0^B(x,y) &= 1 - F_0^B(1,1) + yF_0^B[H_1^A(x,y), H_1^B(x,y)] \end{cases}$$

The derivative of the generating function evaluated at the point $(1,1)$ gives us the first moment of the distribution. That is why the number of consumers who eventually

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[8] In the case when $q^i = 1$ for $i \in \{A,B\}$ all generating functions evaluated at 1 are equal to 1.

buy the product if we advertise it to a randomly chosen consumer of type i is the sum $H^i_{0x} + H^i_{0y}$ evaluated at the point $(1, 1)$. Recall that in the population there is proportion γ of consumers of type A and $1 - \gamma$ of type B. Thus if we advertise the product to a randomly chosen consumer the expected number of purchases is given by the expression:

$$s(q^A, q^B, \rho, \gamma, z_1, z_2) = (\gamma \;\; 1 - \gamma) \begin{pmatrix} H^A_{0x} + H^A_{0y} \\ H^B_{0x} + H^B_{0y} \end{pmatrix}$$

Omitting further derivations to the appendix, the resulting expression is given by the following lemma:

Lemma 1. *The expected number of consumers who buy the product if the monopolist advertises it to a randomly chosen consumer is given by an expression:*

$$s(q^A, q^B, \rho, \gamma, z_1, z_2) = (\gamma \;\; 1 - \gamma) \left[\mathbf{I} + \mathbf{F'_0} (\mathbf{I} - \mathbf{\hat{F}'_1})^{-1} \right] \begin{pmatrix} q^A \\ q^B \end{pmatrix},$$

where z_1 and z_2 are expected numbers of first and second neighbors correspondingly, $\rho^A = \rho$, $\rho^B = 1 - \frac{\gamma}{1-\gamma}(1 - \rho)$, $\mathbf{\hat{F}'_0} = \frac{z_1^2}{z_2} \mathbf{\hat{F}'_1}$ and

$$\mathbf{\hat{F}'_1} = \frac{z_2}{z_1} \begin{pmatrix} q^A \rho^A & q^A(1 - \rho^A) \\ q^B(1 - \rho^B) & q^B \rho^B \end{pmatrix}$$

Proof see appendix □

The first term of the expression $(\gamma \;\; 1 - \gamma)\mathbf{I}\binom{q^A}{q^B}$ is the probability that a randomly chosen consumer buys the product and transmits information further. The second term consists of two parts. The first part $(\gamma \;\; 1 - \gamma)\mathbf{F'_0}$ is a vector with components showing the number of the first neighbors of type A and type B who buy the product. The second part is the vector $(\mathbf{I} - \mathbf{\hat{F}'_1})^{-1}\binom{q^A}{q^B}$ with components that represent number of purchases generated by the flow of information through a randomly chosen link to a consumer of type A and B.

Note, that similarly to the epidemic diffusion literature only first two moments z_1 and z_2 of degree distribution are relevant for the propagation of cascade of sales. This substantially reduces the amount of information about network structure that monopolist needs to possess to make the optimal decision.

In the special case when consumers of both types have the same preferences towards the product and buy it with same probability q the expression of cascade of sales reduces to well known expression of average size of component of operational nodes[9]:

$$s(q^A, q^B, \rho, \gamma, z_1, z_2)|_{q^A = q^B = q} = q + \frac{q^2 z_1}{1 - q(z_2/z_1)}$$

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[9] See Callaway et al. (2000)

In this case size of sales cascade is independent of such network characteristics as population composition γ and homophily level ρ. This points out that for homophily to have the impact on diffusion we need heterogeneity of types in terms of preferences towards both the product and friendship relationships.

4 Main Results

We begin our analysis by establishing conditions under which the global cascade of sales arises. With this condition in mind, we turn to the problem of the monopolist considering the case when sales are finite. We derive the optimal price and product design solving the maximization problem in two steps. In the first step we fix the price and solve the problem for the optimal design of the product. In the second step we relax assumption about exogenous price and allow monopolist to re-optimize with respect to the price. We complete our analysis by studying the implications of homophily level for price elasticity of demand and social welfare.

4.1 Arise of the Global Cascade of Sales

A WOM marketing campaign is regarded as successful if it induces multiple sales per advertisement. However, there are some exceptional cases when information propagates to a significant part of the population. These were the case of advertisement of movie "The Blair Witch Project" and the diffusion of Hotmail accounts. In this section we identify the conditions under which the monopolist acting optimally can sell the product to non-infinitesimal part of the population. We consider two cases, when price is endogenous and forms part of the decision process of the monopolist and when price is exogenous.

From Proposition 1 we know that number of buyers of the product explodes when denominator $\det(\mathbf{I}-\hat{\mathbf{F}}'_{\mathbf{1}})$ goes to 0. Thus the condition of appearance of the global cascade of sales is:

$$1 - q^A q^B \left(\frac{z_2}{z_1}\right)^2 (1-\rho^A-\rho^B) - \frac{z_2}{z_1}(q^A\rho^A + q^B\rho^B) \leq 0$$

In order not to favor any group of consumers in the following analysis we assume that consumers are partitioned into two groups of equal sizes, thus consumers of type A and of type B constitute half of the population. In this case the expression (2) implies that $\rho^A = \rho^B = \rho$. Substituting expressions for q^A and q^B and incorporating assumptions we obtain following quadratic inequality:

$$w^2(1-P)^2\left(\frac{z_2}{z_1}\right)^2(1-2\rho) - w(1-P)^2\left(\frac{z_2}{z_1}\right)^2(1-2\rho) + 1 - (1-P)\frac{z_2}{z_1}\rho \leq 0$$

In the expression, degree distribution is summarized by the ratio of expected number of second neighbors to expected number of first neighbors. This ratio tells us how efficient is the network in information diffusion. In particular, it shows how many second neighbors on average become aware of the product if a consumer shares the information with one of her first neighbor.

The following proposition summarizes the result:

Proposition 1. *For endogenous price P, if $z_2/z_1 \geq \min\left\{2, \rho^{-1}\right\}$ there exists non empty set $E(z_2/z_1, \rho)$ such that for any $(w, P) \in E(z_2/z_1, \rho)$ global cascade of sales arises.* □

Proof see appendix □

In the framework of one-type nodes, paper by Molloy and Reed (1995) for the first time derives the condition for appearance of the giant component of connected nodes, which is $z_2/z_1 > 1$. In our case it is a necessary condition for global cascade of sales to occur. One can easily check that $z_2/z_1 < 1$ does not satisfy condition in Proposition 1, since $\rho \in [0, 1]$. Intuitively, for the information to spread unboundedly, there should exist giant component of connected consumers upon which spreading takes place.

The condition in Proposition 1 is stronger than $z_2/z_1 > 1$ since not all consumers buy the product and consequently retransmit WOM further. One can separate the condition into two parts: $z_2/z_1 \geq 2$ and $z_2/z_1 \geq \rho^{-1}$. The first part of the condition tells us that independently of homophily level ρ, if z_2/z_1 is higher than 2 then global cascade of sales occurs. This part of condition comes from the case when maximal spread of WOM is attained for symmetric characteristic $\left(w = \frac{1}{2}\right)$, which mitigates differences between nodes and makes ρ irrelevant. Moreover, it resembles the condition by Callaway et al. (2000) for appearance of the giant component of operational nodes. The condition is $z_2/z_1 \geq \frac{1}{p}$, where p is a probability that randomly taking node is operational. In our case $p = \frac{1}{2}$, since all consumers buy the product with the same probability $w = \frac{1}{2}$.

The second part of the condition comes from the case when $\rho > \frac{1}{2}$ and the maximal spread of WOM is attained when the monopolist chooses specialized design ($w \in \{0, 1\}$). In this case information propagates only via consumers of one type and we need to adjust expected numbers of neighbors to count just one type of nodes. To fix ideas assume that the monopolist chooses $w = 1$ and thus only consumers of type A buy the product. The expected number of first neighbors of type A is ρz_1 and of the second is $\rho^2 z_2$. Applying condition from Molloy and Reed (1995) we obtain $z_2/z_1 > \rho^{-1}$ which is the same as the second part of the condition in Proposition 1.

In analysis to come we also consider the case when price is exogenously given and the monopolist only chooses characteristic of the product w. Following lemma establishes condition for the global cascade of sales to occur for exogenously given price:

Lemma 2. *For exogenously given price P, if $z_2/z_1 \geq \frac{1}{1-P} \min\left\{2, \rho^{-1}\right\}$ there exists non-empty interval $[\underline{w}, \overline{w}]$ such that for any $w \in [\underline{w}, \overline{w}]$ the global cascade of sales arises.*

Proof see appendix □

Not surprisingly, in the case of exogenous price the condition of appearance of giant cascade of sales in the Lemma 2 is more strict, since not all consumers are willing to pay the price P for the product.

4.2 Optimal Design

In this section we consider the problem of the monopolist who takes the price as given and chooses the design of the product to maximize profits. The profits are given by the product of price and size of sales cascade, since cost of production is zero. If the price is exogenously given by Lemma 2 we know that there is no global cascade if $z_2/z_1 \leq \frac{1}{1-P}\min\left\{2, \rho^{-1}\right\}$. Thus the monopolist profits maximization problem subject to preferences frontier is following:

$$\max_{w} P \times s(q^A, q^B, \rho, \frac{1}{2}, z_1, z_2)$$

$$\begin{aligned} \text{s.t.:}\quad & q^A = (1-P)w \\ & q^B = (1-P)(1-w) \end{aligned}$$

Before going to results we develop some intuition. We already have seen in the proof of Lemma 2 the importance of homophily level. In the following exercise let us assume that society exhibits heterophily, which implies that nodes of type A more often connected to nodes of type B. Assume further that consumer of type A has bought the product. Since most of her neighbors are of type B, the necessary condition for further spread of the information is attractiveness of the product to consumers of type B. However once they buy the product, most of their neighbors are of type A and process reiterates. Thus we can conclude that for sufficiently low homophily level optimal product design should be appealing to both groups of consumers. Assume now that homophily is sufficiently high and consumers of both types have majority of their links to consumers of the same type. Would it be optimal to focus on consumers of one type, and forget about others? The question is non-trivial since there are components of consumers of both types and if the monopolist focuses on one type all components of another type will not be reached.

The following proposition summarizes the results:

Proposition 2. *For any exogenously given price P following holds:*

(a) if $\rho = \frac{1}{2}$ the function $s(\cdot)$ is horizontal line and all $w \in [0, 1]$ are solutions to the maximization problem.

(b) if $\rho < \frac{1}{2}$ the function $s(\cdot)$ is quasi-concave and has its unique maximum at the point $w = \frac{1}{2}$

(c) if $\rho > \frac{1}{2}$ the function $s(\cdot)$ is quasi-convex and its unique minimum is at the point $w = \frac{1}{2}$ and maxima are situated at points $w \in \{0, 1\}$

□

The first result states that for two groups of consumers of equal sizes if $\rho = \frac{1}{2}$ (which for $\gamma = \frac{1}{2}$ implies random mixing) the size of sales cascade is not affected by the product characteristic w. That is why heterogeneity of consumers preferences towards the product and towards linking both constitute key ingredients of the model.

The Proposition 2 confirms our intuition for the case of low homophily level and most importantly states that maximization problem has threshold solution. More precisely, independently of a degree distribution and price, if ρ becomes higher than $\frac{1}{2}$, the optimal product design abruptly changes from symmetric $w^* = \frac{1}{2}$ to specialized $w^* \in \{0, 1\}$. The explanation is following: when ρ is higher than $\frac{1}{2}$ the majority of consumer's neighbors are of the same type as a consumer. That is why the design most attractive for the consumer is the one that induces the highest sales of the product among the neighbors. The situation repeats for every consumer that buys the product reinforcing the optimality of specialized design.

The result does not depend on the degree distribution and the price, which makes it easy to apply. The monopolist just needs to know the homophily level of the society to choose the optimal design of the product.

4.3 Demand

Incorporating the optimal design of the product into expression for cascade of sales we obtain the following demand function:

$$Q(P, \rho, z_1, z_2) = \begin{cases} \frac{1-P}{2}\left(1 + \frac{z_1(1-P)}{2 - z_2/z_1(1-P)}\right), & \rho \leq \frac{1}{2} \\ \frac{1-P}{2}\left(1 + \frac{z_1(1-P)}{\frac{1}{\rho} - z_2/z_1(1-P)}\right), & \rho > \frac{1}{2} \end{cases}$$

Note that for $\rho \leq \frac{1}{2}$ the demand is independent of homophily level ρ, since in this case optimal design is given by symmetric characteristic $w^* = \frac{1}{2}$. Symmetric design implies that both types of consumers buy the product with the same probability and mixing does not matter. The following proposition summarizes main properties of the demand:

Proposition 3. *The demand is given by the function $Q(P, \rho, z_1, z_2)$, which has following properties:*

(i) $Q(P, \rho, z_1, z_2)$ is continuous in ρ and for $\rho > \frac{1}{2}$ is increasing and convex in ρ.

(ii) $Q(P, \rho, z_1, z_2)$ is decreasing and convex in P.

(iii) Price elasticity of demand is increasing in ρ, for $\rho > \frac{1}{2}$.

Proof see appendix $\square$

The first result states that for homophily level $\rho > \frac{1}{2}$ classical demand increases in ρ. In the Proposition 2 we have seen that for $\rho > \frac{1}{2}$ the optimal design is specialized with characteristic w^* belonging to the set $\{0, 1\}$. In this case a randomly chosen consumer has the majority of neighbors of the same type and further increase of homophily increases this subset. To fix the ideas assume that $w^* = 1$ and $P = 0$. Thus if consumer of type A gets the information about the product she and all neighbors of type A acquire it. Thus increase in homophily level leads to higher number of acquisitions in the neighborhood of consumer of type A.

The result is in disagreement with McPherson (2001), which argues that for higher homophily levels information flows tend to be localized and status quo of individuals is maintained. We show that if further transmission of information depends on the adoption decision, homophily actually induces higher spread of the information. This happens because it is easier for the monopolist to construct the message that penetrates homogenous medium.

The convexity part of the result *(i)* comes from the fact that increase in homophily expands the subset of same type consumers in the neighborhood of consumer. Moreover, each consequent increase in homophily level operates on the neighborhood of higher number of consumers, which acquire the product. This gives rise to increasing marginal returns of number of buyers on homophily level.

The result *(ii)* has similar nature as the convexity of demand in ρ. A price increase affects the decision of all consumers to acquire the product regardless of their position in chain of buying consumers. Consider the case when due to a price increase a consumer which is situated early in the buying chain does not buy the product. In this case whole branch of consumers that receive information about the product through this consumer stop to acquire the product. A further price increase has smaller impact on the demand, since length of chains of buying consumers decreases.

Having two previous results at hand we are equipped to understand the third one. The result *(i)* implies that when $\rho > \frac{1}{2}$ an increase in ρ leads to higher sales and awareness of consumers about the product. Hence, the price increase affects decision of increased number of consumers, which translates into increased price elasticity of demand.

4.4 Optimal Price

We have seen solution of the maximization problem with respect to the optimal design of the product. In this subsection we relax assumption of exogenous price and allow the monopolist to re-optimize with respect to price. The optimal product design is independent of the price chosen by the monopolist, thus all results derived in previous sections hold for the optimal price as well. The monopolist maximizes profits and solves following problem with respect to price:

$$\max_{0\leq P\leq 1} P \times \frac{1-P}{2}\left(1+\frac{z_1(1-P)}{\frac{1}{\rho}-\frac{z_2}{z_1}(1-P)}\right)$$

In the price setting the monopolist faces usual trade-off: increase in the price increases profits from each unit sold, but simultaneously decreases demand for the product. However, in the presence of WOM communication there is additional informational component of the trade-off. Since a consumer spreads information about the product only if she acquires it, the price increase lowers product awareness of consumers. The optimal price behavior is summarized in the following proposition:

Proposition 4. *The optimal price P^* for $\rho \geq \frac{1}{2}$ is decreasing function in the homophily level, while for $\rho < \frac{1}{2}$, P^* is independent of the homophily level. The optimal price P^* is always lower than $\frac{1}{2}$.*

Proof see appendix □

The result is a direct consequence of the fact that price elasticity of demand is increasing in the homophily level. As we have seen in Proposition 3 for $\rho > \frac{1}{2}$ an increase in homophily implies that more consumers become aware about the product and some of them buy it. The monopolist reduces the price capturing part of the informed consumers. The result implies that informational component outweighs the increase of profits per purchase generated by higher price.

4.5 Social Welfare

The model allows us to address welfare implications of the homophily explicitly. The majority of the literature on the topic adopts the view that diversity of contacts is socially desirable property per se (e.g. Moody, 2001). Although this assertion could be supported by evidence, no rigorous analysis has been made. A recent paper by Currarini et al. (2009) shows that welfare implications of homophily crucially depend on the structure of consumers preferences. In the following analysis we try to contribute to this end.

Consumer surplus is given by the following expression:

$$CS(P^*(\rho),\rho,z_1,z_2) = \int_{P^*(\rho)}^{1} Q(P,\rho,z_1,z_2)dP$$

We already have seen that for $\rho \geq \frac{1}{2}$, demand is increasing in homophily, and thus increase in ρ shifts demand curve upwards. This happens, since more consumers become aware about the product. At the same time an increase in the homophily level by Proposition 4 leads to price decrease. As a consequence more consumers buy the product for lower price and thus both effects lead to increase in consumer surplus.

Producer surplus is the area below the price of the product and marginal cost curve (MC) of producer. In our case $MC = 0$ and thus:

$$PS(P^*(\rho), \rho, z_1, z_2) = P^*(\rho) \times Q(P^*(\rho), \rho, z_1, z_2)$$

Proposition 5. *Both consumer surplus and monopolistic profits are increasing in the level of homophily.* $\square$

The Proposition 5 states that if information retransmission is subject to adoption decision then society is better-off when homophily level is high. There are two driving forces of the result. First, the optimally constructed message propagates better in homogenous groups and more consumers become aware about the product. Second, the price reduction more effective in facilitating diffusion of WOM in the case of higher homophily level. These two effects are beneficial for both consumers and the monopolist.

4.6 Example of Classical Random Graph

In further analysis we will often refer to a special case of network structure known as classical random graph. The notion of random graph has been introduced by Paul Erdõs and Alfréd Rényi and since than it is the most studied model of graphs. Nodes connectivity in random graph follows Poisson degree distribution and arises in infinite networks where each node has uniform probability to create a link to any other node in the network.

In our case probability that two consumers of same type are linked is different from the probability of link between consumers of different types. One can think about a network of N consumers of two types where each consumer creates a link to any other consumer of the same type with probability $\frac{\rho z_1}{N}$ and to consumer of another type with probability $\frac{(1-\rho)z_1}{N}$. When N goes to infinity we obtain infinite network with two Poisson degree distributions. One for links among consumers of the same type with mean ρz_1 and another for links among different types with mean $(1-\rho)z_1$. Recall, that the sum of two Poisson variables also follows Poisson distribution with mean equal to the sum of means. That is why overall connectivity of a randomly chosen node follows Poisson distribution and the network is classical random graph with average connectivity given by z_1.

In the case of Poisson degree distribution average connectivity z_1 is a sufficient characteristic of the network, since $z_2 = z_1^2$. This property allows us to study the effect of network density on propagation of WOM and the optimal strategies of the monopolist.

The optimal design of the product does not depend on the degree distribution and is given by the Proposition 2. Incorporating relation between expected number of first and second neighbors to the demand function we obtain:

$$Q(P, \rho, z_1, z_2) = \begin{cases} \frac{1-P}{2}\left(1 + \frac{z_1(1-P)}{2-z_1(1-P)}\right), & \rho \leq \frac{1}{2} \\ \frac{1-P}{2}\left(1 + \frac{z_1(1-P)}{\frac{1}{\rho}-z_1(1-P)}\right), & \rho > \frac{1}{2} \end{cases}$$

Note that demand is continuous and does not depend on ρ if $\rho \leq \frac{1}{2}$. In the derivations we use the demand function for the case of $\rho > \frac{1}{2}$. In order to get results for the case

$\rho \leq \frac{1}{2}$ one needs to substitute $\rho = \frac{1}{2}$. Taking derivative with respect to z_1 of the demand function one can show that the denser is the network the higher is the demand:

$$\frac{\partial}{\partial z_1} Q(P, \rho, z_1, z_2) = \frac{(1-P)^2 \rho}{2(1-(1-P)z_1\rho)^2} > 0$$

Maximizing profits we obtain an expression for the optimal price,[10,11] which is given by:

$$P^* = \begin{cases} 1 - \frac{2-\sqrt{4-2z_1}}{z_1}, & \rho \leq \frac{1}{2} \\ 1 - \frac{1-\sqrt{1-z_1\rho}}{z_1\rho}, & \rho > \frac{1}{2} \end{cases}$$

The derivative of optimal price with respect to homophily level ρ is negative, thus optimal price is decreasing with homophily of the society:

$$\frac{\partial P^*}{\partial \rho} = -\frac{2 - z_1\rho - 2\sqrt{1-z_1\rho}}{2z_1\rho^2\sqrt{1-z_1\rho}} < 0$$

To study the effect of network density on the optimal price we take derivative with respect to z_1:

$$\frac{\partial P^*}{\partial z_1} = -\frac{2 - z_1\rho - 2\sqrt{1-z_1\rho}}{2z_1^2\rho\sqrt{1-z_1\rho}} < 0$$

The derivative is negative, which implies that the optimal price P^* is decreasing in both average connectivity and homophily parameter ρ.

Turning to welfare implications of the homophily and using the same line of arguments as we have outlined before one can show that increase in z_1 leads to higher consumer surplus. This happens since denser network implies higher diffusion of information about the product and results in higher awareness of consumers. Denser network also increases benefits for the monopolist to reduce the price, since now information spreads further. Both effects benefit consumers.

5 Targeted Advertisement

In the previous section we have considered the problem of the monopolist, which cannot distinguish consumers by type. The monopolist, restricted by anonymity assumption, was advertising the product to a randomly chosen subset of the population. This formulation is relevant for an advertisement through the mass media, when the monopolist can not control who is watching or hearing an advertisement. However, in the case of direct

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[10] In the case of Poisson degree distribution the first order condition for optimal price when $\rho > \frac{1}{2}$ reduces to $z_1\rho P^2 + 2P(1-z_1\rho) + z_1\rho - 1 = 0$.

[11] Condition of absence of global cascade of sales in the case of Poisson distribution reduces to $z_1 < \min\{2, \rho^{-1}\}$.

advertisement there is a possibility to target chosen group of the population. For example, monopolist that is interested in students' community, can distribute an advertisement at a university or specifically ask for student's id-card.

In this section we are going to relax anonymity assumption and allow the monopolist to observe types of consumers. More precisely, we assume that monopolist chooses the design of the product w and proportion α of consumers of type A in the subset selected for targeted advertisement. Note, before proportion of consumers of type A which receive advertisement was fixed exogenously at the level γ, which is proportion of nodes of type A in the society. In analytical part, for tractability of the problem, we assume that the price P is exogenously given. Thus the maximization problem of the monopolist becomes:

$$\max_{w,\alpha} P \times s(q^A, q^B, \rho, \alpha, z_1, z_2)$$

$$\text{s.t.:} \quad \begin{aligned} q^A &= (1-P)w \\ q^B &= (1-P)(1-w) \end{aligned}$$

The expression for sales cascade $s(q^A, q^B, \rho, \alpha, z_1, z_2)$ can be rewritten as a linear combination of number of purchases resulted from an advertisement to consumer of type A and of type B: $\alpha \times s^A(q^A, q^B, \rho, z_1, z_2) + (1-\alpha) \times s^B(q^A, q^B, \rho, z_1, z_2)$. Given the linear structure of the problem in terms of α it is easy to see that if $q^A \neq q^B$ the optimal targeting proportion has a corner solution. Namely, the solution depends whether cascade of sales induced by advertisement is higher if we advertise to a consumer of type A or of type B. In the case when $q^A = q^B$ both types of consumers buy the product with the same probability and thus all values of α on the interval $[0, 1]$ are optimal.

Proposition 6. *Targeting one type of consumers in advertisement is always optimal strategy for the monopolist.* □

Since preference frontier is symmetric, without loss of generality assume that the monopolist targets consumers of type A and hence $\alpha^* = 1$. Moreover by symmetric nature of the problem if $\alpha^* = 1$ and some w^* is the solution then $\alpha^* = 0$ and $1 - w^*$ is a solution too. For the following analysis we assume that $\alpha^* = 1$.

The characterization of the optimal product design for arbitrary degree distribution when monopolist can target advertisement quickly becomes intractable. In the following analysis we focus on the famous case of classical random graphs with Poisson degree distribution.

The intuition tells us that possibility of consumers targeting in advertisement would unavoidably bring bias towards the characteristic of the product favorable for consumers of targeted type. The bias itself can be of two forms. The first form is that the threshold level of homophily, which separates specialized design ($w \in \{0, 1\}$) and symmetric ($w = \frac{1}{2}$) moves to new level, which is lower than $\frac{1}{2}$. The second form is that for ρ lower than new threshold level the optimal design belongs to the interval $\left(\frac{1}{2}, 1\right)$ instead of being symmetric. The following proposition shows that both types of bias are present:

Proposition 7. *For Poisson degree distribution and exogenously given price, there is a threshold level* $\hat{\rho}_T(z_1, P) = \frac{3}{4} + \frac{1-\sqrt{9-2z_1(1-P)+(1-P)^2 z_1^2}}{4(1-P)z_1}$ *such that for* $\rho \geq \hat{\rho}_T(z_1, P)$ *the monopolist will advertise and specialize only on one type of consumers* ($\alpha^* = 1$ *and* $w^* = 1$)*. For* $\rho < \hat{\rho}_T(z_1, P)$ *the optimal advertisement strategy is still* $\alpha^* = 1$*, but the optimal characteristic is given by:*

$$w^* = \rho - \frac{1 - 2\rho - \sqrt{(1-\rho)(1-2\rho)(1-(1-P)z_1\rho)[2+(1-P)z_1(1-2\rho)]}}{(1-P)z_1(1-2\rho)} > \frac{1}{2}$$

Proof see appendix □

The success of WOM campaign to high extend depends on the effectiveness of the direct advertisement in inducing acquisition of the product. To convert consumers of type A who receive direct advertisement into initial adopters the monopolist designs the product more attractive to them. The bias in product design persist even when society exhibits heterophily ($\rho < \frac{1}{2}$), and propagation of WOM depends on attractiveness of the product to both types of consumers.

In the case of the Poisson distribution the rise in z_1 implies higher diffusion of WOM in the network, since there are more channels for information to spread on. In the case when we approach the global cascade phase (z_1 goes to $2(1-P)^{-1}$) the threshold level $\hat{\rho}_T(z_1, P)$ goes to $\frac{1}{2}$ and the optimal characteristic for $\rho < \frac{1}{2}$ is $w^* = \frac{1}{2}$. Thus the optimal design becomes exactly the same as in the case of non-targeted advertisement. In this case the monopolist optimally sacrifices some initial adopters, and design the product in a way that WOM can penetrate further in the network.

The result implies that when we approach the global cascade phase option for the monopolist to target advertisement to some group of consumers does not influence neither the optimal design of the product nor the demand. We regard this result as indication of robustness of the optimal design strategy that we obtained in our base-line model.

6 Non-Linear Shape of the Preference Frontier

The previous analysis develops results for the case of linear preference frontier. However, one can think about many examples when relationship between acquisition probabilities for two groups of consumer is not linear. These situations occur when attractiveness of the product for two types of consumers non-linearly varies in characteristic. For example, longer guarantee on the product is attractive for both types, while pink color of the product may be very welcome by one type and be unattractive for another. In this situations shape of the frontier can vary from concave to convex depending on the product in question.

In this section we want to address the robustness of results to the change in curvature of the preference frontier that the monopolist faces. We consider CES functional form of the preference frontier which allows to model variety of shapes. Thus probabilities to buy the product for two types of consumers are related in the following manner:

$$\left(q^A\right)^r + \left(q^B\right)^r = (1-P)^r$$

By varying parameter r we can obtain shapes of the preferences frontier that include bend inward circle ($r = 0.562$), linear function ($r = 1$), bend outward circle ($r = 2$) and everything in between.

6.1 Arise of the Global Cascade of Sales

Similar to analysis in the case of linear frontier an important question to address is conditions under which the global cascade of sales occurs. An essential parameter for appearance of the global cascade of sales is r, degree of curvature. For example, if r goes to ∞, as we know from properties of CES function, we have frontier which is a step function and the best design of the product is $w = 1$ which implies $q^A = q^B = 1 - P$. In this case two types of consumers always are fully satisfied by the design of the product and thus effectively there is only one type of consumers. The condition for appearance of global cascade in this case is well known in networks literature, namely $\frac{z_2}{z_1} \geq \frac{1}{1-P}$.

For arbitrary values of r the condition of existence of global cascade is given by the following inequality:

$$1 - (1-P)^2 w (1-w^r)^{\frac{1}{r}} \left(\frac{z_2}{z_1}\right)^2 (1-2\rho) - (1-P)\left(w + (1-w^r)^{\frac{1}{r}}\right)\frac{z_2}{z_1}\rho \leq 0$$

Due to the arbitrary power of the polynomial equation it is impossible to have analytical solution, however we can identified broad set of parameters for which global cascade of sales occurs. The results are summarized in the following proposition:

Proposition 8. *In the case of CES preference frontier with curvature parameter r global cascade arises if $z_2/z_1 > 2^{\frac{1}{r}}$ or $\rho > \hat{\rho}_{NL}(z_2/z_1, r, P)$, where*

$$\hat{\rho}_{NL}(z_2/z_1, r, P) = \min_{0 \leq w \leq 1} \frac{1 - w(z_2/z_1)^2(1-P)^2(1-w^r)^{\frac{1}{r}}}{(1-P)(z_2/z_1)\left(w + (1-w^r)^{\frac{1}{r}}(1 - 2w(z_2/z_1)(1-P))\right)}.$$

Proof see appendix □

6.2 Problem of the Monopolist

Now we are going to address robustness of the optimal design obtained for linear case to the change in the curvature of the preference frontier. This subsection consists of two parts: analytical and numerical. In analytical part we focus on the case of exogenous price and Poisson degree distribution and find conditions such that symmetric and specialized designs are solutions to the maximization problem. In the second part we endogenize price and solve problem numerically for arbitrary type of degree distribution.

Similarly to analysis of the Section 5 we assume Poisson degree distribution and exogenous price. The monopolist faces non-linear preferences frontier and chooses design of the product to maximize profits. The monopolist solves following maximization problem:

$$\max_{w} P \times s(q^A, q^B, \rho, \frac{1}{2}, z_1, z_2)$$

$$\text{s.t.:} \quad \begin{aligned} q^A &= (1-P)w \\ q^B &= (1-P)(1-w^r)^{\frac{1}{r}} \end{aligned}$$

We identify homophily level such that symmetric and specialized designs are solutions. The results are summarized by the following proposition:

Proposition 9. *Given that exogenous price and Poisson degree distribution following holds:*

- *(a) For $r \leq 1$ there is $\hat{\rho}_{NL}(r, P, z_1) = \frac{1}{2} - \frac{2^{\frac{1}{r}}-2}{2z_1(1-P)}$ such that the optimal design is symmetric $w^* = \left(\frac{1}{2}\right)^{\frac{1}{r}}$ if $\rho < \hat{\rho}_{NL}(r, P, z_1)$ and otherwise the optimal design is specialized $w^* \in \{0, 1\}$.*
- *(b) For $r > 1$ the optimal design is symmetric $w^* = \left(\frac{1}{2}\right)^{\frac{1}{r}}$ if $\rho < \hat{\rho}_{NL}(r, P, z_1)$ otherwise the optimal design belongs to the interval $\left(\left(\frac{1}{2}\right)^{\frac{1}{r}}, 1\right)$.*

Proof see appendix □

The Proposition 9 states that the optimal design preserves the same structure as in the case of linear preferences frontier. More precisely, for $r \leq 1$, only symmetric and specialized designs are optimal. They are separated by new threshold value $\hat{\rho}_{NL}(r, P, z_1)$. For $r > 1$, and low levels of ρ, symmetric design is optimal, however for high levels of ρ solution gradually changes from symmetric to specialized.

In the case of bend outward shape of the frontier ($r > 1$) the optimal design is biased towards the symmetric design. The explanation is following: while corner solutions are still as attractive as they were in the problem with $r = 1$ (endpoints are fixed), symmetric solution becomes more appealing. This happens since in the case of $r > 1$ by moving to the center of the symmetry from $q^A = 1$ we are gaining more of q^B while sacrificing the same amount of q^A as compared to the linear case with $r = 1$. For the case when $r < 1$ we have the opposite situation.

The threshold level of homophily $\hat{\rho}_{NL}(r, P, z_1)$ is increasing in z_1. Thus the denser the network the more appealing becomes symmetric design. The intuition is the same as for

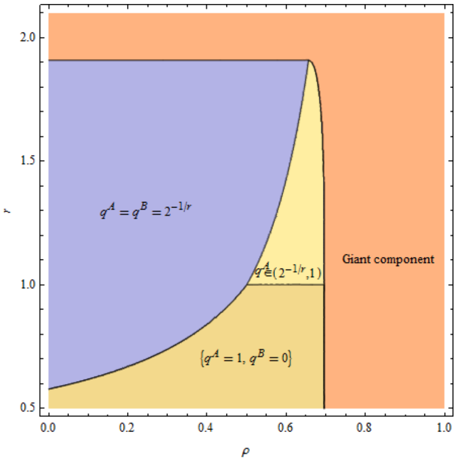


Figure 3: Diagram depicts the solution for $z_1 = 1.23$ and $z_2 = 1.77$. Areas represent: symmetric solution(blue), asymmetric (light yellow), specialized (yellow) and area where global cascade arises (red).

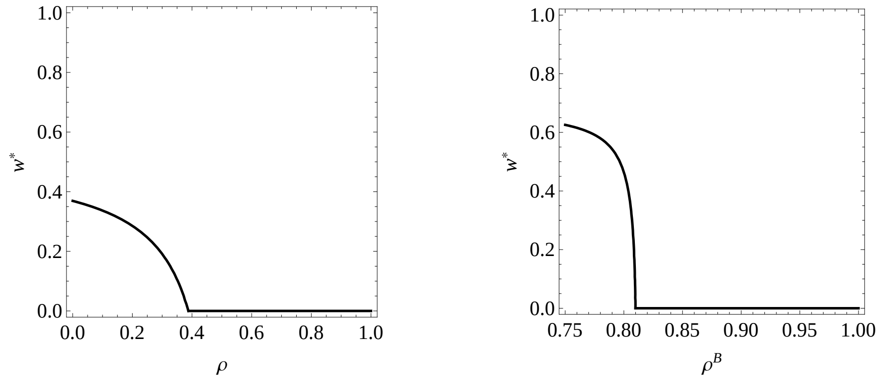
the case of targeted advertisement. Dense network implies potentially higher diffusion of the information and to penetrate further the product should be appealing for both types.

To check the generality of results obtained for the case of Poisson degree distribution we consider numerical solution for the scale free distribution. Figure 3 shows a diagram of the structure of the solution. The diagram is made for scale free distribution with pdf $Ck^{-3.34}$, where C is normalizing constant. In this case number of first and second neighbors are $z_1 = 1.23$ and $z_2 = 1.77$.

One can see from the diagram that for $r < 1$ we have similar result as in linear case. Namely, there is the threshold level of $\hat{\rho}_{NL}(r, z_1, z_2, P)$ such that for $\rho < \hat{\rho}_{NL}(r, z_1, z_2, P)$ the optimal solution is symmetric and for values of $\rho > \hat{\rho}_{NL}(r, z_1, z_2, P)$ we have specialized solution. In the case of $r > 1$ the structure stays the same but after $\rho = \hat{\rho}_{NL}(r, z_1, z_2, P)$ solution gradually changes from symmetric to some intermediate value which lies in the interval $\left(\left(\frac{1}{2}\right)^{\frac{1}{r}}, 1\right)$.

7 Targeting One Type of Consumers

In this section we address the problem of the monopolist who is has interest only in one type of consumers, for concreteness lets assume that this is type B. This situation could arise if the monopolist believes that consumers differ in their post purchasing behavior. For example, once consumer of type B buys the product she becomes loyal customer and makes further purchases of the same brand, while consumers of type A are accidental buyers. For the sake of simplicity we assume that monopolist completely ignores consumers of type A. Assume further that the monopolist maximizes awareness of the brand and chooses price



(a) (b)

Figure 4: The optimal design when $z_1 = 1.7$, $z_2 = 2$. Figure (a) for the case of $\gamma = \frac{1}{2}$, figure (b) for the case of $\gamma = 0.8$

equal to 0. The main question is than: what is the optimal product design that maximizes purchases of the product by consumers of type B?

The first guess could be that the monopolist should completely forget about consumers of type A and design the product as attractable as possible to consumers of type B. The first guess however turns out to be wrong for broad set of parameters. Assume for example that homophily level of the society is low, which implies that consumers of type B are mostly connected to consumers of type A. Hence to spread information should be able to pass through consumers of type A. A Figure 4a illustrates the optimal product design for the case of groups of consumers of equal size ($\gamma = \frac{1}{2}$) and expected number of neighbors $z_1 = 1.7$ and $z_2 = 2$. Note that for $\rho \in [0, 0.39]$ the optimal design is such that there is non zero probability for consumers of type A to buy the product. The result requires low levels of homophily and actually implies heterophily of the society. We already have seen similar answer in the case when the monopolist gets profits from both types.

The more surprising result is one where although society exhibits homophily it is still optimal to make product attractive for consumers of type A. The only requirement is that proportion of consumers of type A should be sufficiently high. A Figure 4b illustrates the optimal product design for the case when consumers of type A constitute 80% of the population ($\gamma = 0.8$) and expected number of neighbors are as before $z_1 = 1.7$ and $z_2 = 2$. Note that $\rho \in [0.8, 1]$ implies that society exhibits homophily and there is a range of $\rho \in [0.8, 0.81]$ such that the optimal product characteristic w is not zero. Another surprising feature of the result is that for sufficiently small ρ it is optimal to construct the product more attractive to consumers of type A than B.

8 The Global Cascade Phase

In the previous sections we have seen what happens when WOM marketing campaign does not trigger global cascade of sales. This is the case for majority of WOM campaigns. However, WOM campaigns such as diffusion of Hotmail accounts and "The Blair Witch Project" (tiny budget movie) were so successful that actually a considerable fraction of the population became aware of the product. In this cases we can no longer apply techniques from the previous sections.

So let us assume that conditions are such that global cascade of sales arises, by Lemma 2 this happens when $\frac{z_2}{z_1} > \frac{1}{1-P}\min\{2, \rho^{-1}\}$. To determine the fraction of the population that buys the product we turn back to the generating functions, but instead of looking on the distribution of sizes of cascades we estimate fractional size of the global cascade. Assume that by following randomly chosen link we arrive to a consumer of type i then lets denote by u^i the probability that this consumer does not have a link to the giant component of consumers who buy the product. We can write system of self-consistency conditions for u^i as:

$$\begin{aligned}
u^A &= 1 - \sum_{k=1}^{\infty} \xi(k) q_k^A + \sum_{k=1}^{\infty} \xi(k) q_k^A \left[\rho^A u^A + (1-\rho^A) u^B\right]^{k-1} \\
u^B &= 1 - \sum_{k=1}^{\infty} \xi(k) q_k^B + \sum_{k=1}^{\infty} \xi(k) q_k^B \left[\rho^B u^B + (1-\rho^B) u^A\right]^{k-1}
\end{aligned}$$

Using generating functions we can rewrite them as:

$$\begin{aligned}
u^A &= 1 - \hat{F}_1^A(1,1) + \hat{F}_1^A(u^A, u^B) \\
u^B &= 1 - \hat{F}_1^B(1,1) + \hat{F}_1^B(u^A, u^B)
\end{aligned}$$

Having at hand u^A and u^B one can find probability that a randomly chosen consumer of type i does not form a part of the global cascade. This the probability that a randomly chosen consumer does not have links to the giant component of buyers and is equal to $F_0^i(u^A, u^B)$. A randomly selected consumer of type i is not in the giant component if she does not like the product or the price or she would buy the product but she is not aware of it. This happens with the probabilities:

$$\begin{aligned}
v^A &= 1 - F_0^A(1,1) + F_0^A(u^A, u^B) \\
v^B &= 1 - F_0^B(1,1) + F_0^B(u^A, u^B)
\end{aligned}$$

Taking the linear combination with waits equal to proportion of consumers in the society and subtracting from 1 we obtain the probability that a randomly selected consumer is in the giant component:

$$s = 1 - \gamma v^A - (1-\gamma) v^B = \gamma F_0^A(1,1) + (1-\gamma) F_0^B(1,1) - \gamma F_0^A(u^A, u^B) - (1-\gamma) F_0^B(u^A, u^B),$$

where

$$\begin{aligned} u^A &= 1 - \hat{F}_1^A(1,1) + \hat{F}_1^A(u^A, u^B) \\ u^B &= 1 - \hat{F}_1^B(1,1) + \hat{F}_1^B(u^A, u^B) \\ \gamma(1-\rho^A) &= (1-\gamma)(1-\rho^B) \end{aligned}$$

Note that last expression insures that a number of links going from consumers of type A to consumers of type B should be equal of number of links going from consumers of type B to consumers of type A. If we choose γ, ρ^A and ρ^B exogenously we can insure that this condition holds.

Assume that there are equal numbers of consumers of type A and type B, $\gamma = \frac{1}{2}$. The last equation in turn implies that $\rho^A = \rho^B = \rho$ and maximization problem of the monopolist is summarized by the following lemma:

Lemma 3. *For two groups of equal sizes maximization problem of the monopolist becomes:*

$$\max_{q^A, q^B} \frac{1}{2} \left(q^A[1 - G_0(x)] + q^B[1 - G_0(y)] \right)$$

$$\begin{aligned} \textit{s.t.:} \quad & x = 1 - \rho q^A[1 - \hat{G}_1(x)] - (1-\rho) q^B [1 - \hat{G}_1(y)] \\ & y = 1 - (1-\rho) q^A [1 - \hat{G}_1(x)] - \rho q^B [1 - \hat{G}_1(y)] \\ & 0 \le q^A, q^B, x, y \le 1 \end{aligned}$$

where $x = \rho^A u^A + (1-\rho^A)u^B$ is the probability that randomly chosen link of consumer of type A leads to the giant component of buyers and $y = \rho^B u^B + (1-\rho^B)u^A$ is the same for consumers of type B. The functions $G_0(x) = \sum_{k=0}^{\infty} p(k)x^k$ and $\hat{G}_1(x) = \sum_{k=0}^{\infty} \xi(k)x^{k-1}$.
***Proof** see appendix* □

Assuming linear preferences frontier, $q^A = (1-P)w$ and $q^B = (1-P)(1-w)$ the maximization problem becomes:

$$\max_{w} \frac{1-P}{2} [1 - wG_0(x) - (1-w)G_0(y)]$$

$$\begin{aligned} \text{s.t.:} \quad & x = 1 - (1-P) \Big(\rho w [1 - \hat{G}_1(x)] + (1-\rho)(1-w)[1 - \hat{G}_1(y)] \Big) \\ & y = 1 - (1-P) \Big((1-\rho) w [1 - \hat{G}_1(x)] + \rho(1-w)[1 - \hat{G}_1(y)] \Big) \\ & 0 \le w, x, y \le 1 \end{aligned}$$

Solution to the maximization problem is summarized in the following proposition:

Proposition 10. *In the case when population is divided into two equally sized groups, $\gamma = \frac{1}{2}$, for any degree distribution following hold:*

- *For* $\rho < \frac{1}{2}$, $w = \frac{1}{2}$ *is local maximum, which gives higher value then* $w \in \{0, 1\}$.
- *For* $\rho > \frac{1}{2}$, $w = \{0, 1\}$ *are local maxima, which give higher value then* $w = \frac{1}{2}$.
- *For* $\rho = \frac{1}{2}$, *all interval* $[0, 1]$ *is solution to the problem.*

The Proposition 10 indicates that the optimal design of the product has the same structural form as in the case where there is no global cascade of sales.

9 Conclusion

The importance of word of mouth communication for a company's performance is well documented by growing number of research. However, success of WOM marketing campaign varies enormously between product categories and within. We show that high variation in the performance of WOM campaigns can be explained by different homophily levels of the network towards different products.

A key innovation of our paper is two-fold. First, we enrich the network structure by incorporating notion of homophily and study its impact on the optimal strategies of the monopolist. Second, monopolist is allowed to construct a message to network by choosing the design of the product. We found a number of results: (i) for low levels of homophily the product, designed to attract both types of consumers is preferred to specialized products even if there is no cost of producing more than one product; (ii) price elasticity of demand is higher for products towards which consumer networks exhibit higher level of homophily; (iii) social welfare is increasing in the level of homophily; and (iv) a product designed to attract two types of consumers may be optimal even if the monopolist benefits only from one type.

Flexibility of the model allows to outline several avenues for future research. The first one consists in introduction of influencers, consumers whose opinion affect opinion of many others. In the extension we want to match two observations: influencers on average enjoy higher average degree and their proportion in the population is small. The extension aimed to study the impact of homophily on information spreading in a network with hubs and the effect of hubs on the optimal design of the product and price. In the case when society exhibits homophily influencers will be linked among themselves and will constitute core with access to a large share of consumers.

In the second extension we want to consider the optimal strategy for an entrant who faces presence of the incumbent firm on the market. We assume that the product is durable and consumer buys it only once. Hence the optimal strategies for the monopolist that we have studied in the paper are optimal for incumbent firm as well. We want to study how homophily affects optimal product design of an entrant and its effect on the variety of products that are produced. Finally we plan to compare predictions of the model with observations from the real market.

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10 APPENDIX

Proof of Lemma 1

Let us find first what is the number of consumers of type A that buy the product if we advertise it to consumer of type A. The answer is:

$$\frac{\partial}{\partial x} H_0^A(x,y)\bigg|_{x=1,y=1} = H_{0x}^A(1,1)$$

With abuse of notation we assume that all function are being evaluated at point $(1,1)$:

$$H_{0x}^A = F_0^A + F_{0x}^A H_{1x}^A + F_{0y}^A H_{1x}^B$$

We can find H_{1x}^i by solving linear system of self-consistency conditions:

$$\begin{cases} H_{1x}^A = \hat{F}_1^A + \hat{F}_{1x}^A H_{1x}^A + \hat{F}_{1y}^A H_{1x}^B \\ H_{1x}^B = \hat{F}_{1x}^B H_{1x}^A + \hat{F}_{1y}^B H_{1x}^B \end{cases}$$

In vector form:

$$\begin{pmatrix} 1-\hat{F}_{1x}^A & -\hat{F}_{1y}^A \\ -\hat{F}_{1x}^B & 1-\hat{F}_{1y}^B \end{pmatrix} \begin{pmatrix} H_{1x}^A \\ H_{1x}^B \end{pmatrix} = \begin{pmatrix} \hat{F}_1^A \\ 0 \end{pmatrix}$$

or in more compact way

$$\left(\mathbf{I} - \hat{\mathbf{F}}'_{\mathbf{1}}\right) \begin{pmatrix} H_{1x}^A \\ H_{1x}^B \end{pmatrix} = \begin{pmatrix} \hat{F}_1^A \\ 0 \end{pmatrix},$$

where $\hat{F}_1^i = \sum_{k=1}^{\infty} \xi(k) q_k^i$ and

$$\hat{\mathbf{F}}'_{\mathbf{1}} = \begin{pmatrix} \hat{F}_{1x}^A & \hat{F}_{1y}^A \\ \hat{F}_{1x}^B & \hat{F}_{1y}^B \end{pmatrix} = \sum_{k=1}^{\infty} \xi(k)(k-1) \begin{pmatrix} q_k^A \rho^A & q_k^A(1-\rho^A) \\ q_k^B(1-\rho^B) & q_k^B \rho^B \end{pmatrix}$$

The number of consumers of type A who buy the product if consumer of type i finds out about the product from one of her friends H_{1x}^i goes to infinity when determinant of the matrix $\mathbf{I} - \hat{\mathbf{F}}'_{\mathbf{1}}$ goes to zero:

$$\Delta = \det \begin{pmatrix} 1-\hat{F}_{1x}^A & -\hat{F}_{1y}^A \\ -\hat{F}_{1x}^B & 1-\hat{F}_{1y}^B \end{pmatrix}$$

The system has following solution:

$$\begin{pmatrix} H_{1x}^A \\ H_{1x}^B \end{pmatrix} = (\mathbf{I} - \hat{\mathbf{F}}'_{\mathbf{1}})^{-1} \begin{pmatrix} \hat{F}_1^A \\ 0 \end{pmatrix}$$

Thus we can find the number of consumers of type A who buy the product if consumer of type A receives direct advertisement:

$$H_{0x}^A = F_0^A + (F_{0x}^A \; F_{0y}^A)\begin{pmatrix} H_{1x}^A \\ H_{1x}^B \end{pmatrix} = F_0^A + (F_{0x}^A \; F_{0y}^A)(\mathbf{I} - \mathbf{\hat{F}'_1})^{-1}\begin{pmatrix} \hat{F}_1^A \\ 0 \end{pmatrix}$$

By doing analogous calculations we can get:

$$H_{0x}^B = (F_{0x}^B \; F_{0y}^B)\begin{pmatrix} H_{1x}^A \\ H_{1x}^B \end{pmatrix} = (F_{0x}^B \; F_{0y}^B)(\mathbf{I} - \mathbf{\hat{F}'_1})^{-1}\begin{pmatrix} \hat{F}_1^A \\ 0 \end{pmatrix}$$

$$H_{0y}^A = (F_{0x}^A \; F_{0y}^A)\begin{pmatrix} H_{1y}^A \\ H_{1y}^B \end{pmatrix} = (F_{0x}^A \; F_{0y}^A)(\mathbf{I} - \mathbf{\hat{F}'_1})^{-1}\begin{pmatrix} 0 \\ \hat{F}_1^B \end{pmatrix}$$

$$H_{0y}^B = F_0^B + (F_{0x}^B \; F_{0y}^B)\begin{pmatrix} H_{1y}^A \\ H_{1y}^B \end{pmatrix} = F_0^B + (F_{0x}^B \; F_{0y}^B)(\mathbf{I} - \mathbf{\hat{F}'_1})^{-1}\begin{pmatrix} 0 \\ \hat{F}_1^B \end{pmatrix}$$

The total number of purchases resulted from direct advertisement to a consumer of type A is following:

$$H_{0x}^A + H_{0y}^A = F_0^A + (F_{0x}^A \; F_{0y}^A)(\mathbf{I} - \mathbf{\hat{F}'_1})^{-1}\begin{pmatrix} \hat{F}_1^A \\ 0 \end{pmatrix} + (F_{0x}^A \; F_{0y}^A)(\mathbf{I} - \mathbf{\hat{F}'_1})^{-1}\begin{pmatrix} 0 \\ \hat{F}_1^B \end{pmatrix} =$$

$$= F_0^A + (F_{0x}^A \; F_{0y}^A)(\mathbf{I} - \mathbf{\hat{F}'_1})^{-1}\begin{pmatrix} \hat{F}_1^A \\ \hat{F}_1^B \end{pmatrix}$$

If the monopolist advertises the product to consumer of type B:

$$H_{0x}^B + H_{0y}^B = F_0^B + (F_{0x}^B \; F_{0y}^B)(\mathbf{I} - \mathbf{\hat{F}'_1})^{-1}\begin{pmatrix} \hat{F}_1^A \\ \hat{F}_1^B \end{pmatrix}$$

Let us define:

$$F'_0 = \begin{pmatrix} F_{0x}^A & F_{0y}^A \\ F_{0x}^B & F_{0y}^B \end{pmatrix} = \sum_{k=1}^{\infty} kp(k)\begin{pmatrix} q_k^A \rho^A & q_k^A(1-\rho^A) \\ q_k^B(1-\rho^B) & q_k^B \rho^B \end{pmatrix}$$

The resulting number of purchases resulting from advertisement to consumers of type A and B in vector form is:

$$\mathbf{s} = \begin{pmatrix} H_{0x}^A + H_{0y}^A \\ H_{0x}^B + H_{0y}^B \end{pmatrix} = \begin{pmatrix} F_0^A \\ F_0^B \end{pmatrix} + \mathbf{F'_0}(\mathbf{I} - \mathbf{\hat{F}'_1})^{-1}\begin{pmatrix} \hat{F}_1^A \\ \hat{F}_1^B \end{pmatrix}$$

Thus the number of purchases resulting from advertisement to a randomly drawn consumer is:

$$s = (\gamma\ 1-\gamma)\mathbf{s} = (\gamma\ 1-\gamma)\left[\begin{pmatrix}F_0^A\\ F_0^B\end{pmatrix} + \mathbf{F_0'}(\mathbf{I}-\mathbf{\hat{F}_1'})^{-1}\begin{pmatrix}\hat{F}_1^A\\ \hat{F}_1^B\end{pmatrix}\right]$$

Assuming that the probability to purchase the product does not depend on the number of neighbors that consumer has, namely $q_k^A = q^A$ and $q_k^B = q^B$ we obtain:

$$s = (\gamma\ 1-\gamma)\mathbf{s} = (\gamma\ 1-\gamma)\left[\mathbf{I} + \mathbf{F_0'}(\mathbf{I}-\mathbf{\hat{F}_1'})^{-1}\right]\begin{pmatrix}q^A\\ q^B\end{pmatrix}$$

Note that expression depends on the linear combination of probability to infect initial node $(\gamma\ 1-\gamma)\binom{q^A}{q^B}$ and $(\gamma\ 1-\gamma)\left(\mathbf{I}-\mathbf{\hat{F}_1'}\right)^{-1}\binom{q^A}{q^B}$ which is number of infected nodes if we follow randomly chosen link, with weight given buy $\frac{z_1^2}{z_2}$.

Proof of Proposition 1

The global cascade of sales arises when inequality holds:

$$w^2(1-P)^2\left(\frac{z_2}{z_1}\right)^2(1-2\rho) - w(1-P)^2\left(\frac{z_2}{z_1}\right)^2(1-2\rho) + 1 - (1-P)\frac{z_2}{z_1}\rho \le 0$$

We want to identify a condition such that there exists characteristic of the product w which satisfies the inequality and hence the global cascade of sales may arise. To this end we find the minimum of the expression and check when it is less than zero. A derivative of the expression with respect to w is $-(1-P)^2(z_2/z_1)^2(1-2\rho)(1-2w)$. Note that if $\rho < \frac{1}{2}$ coefficient of the term w^2 is positive and thus we have upward sloping parabola. In this case function has its minimum at the point $w = \frac{1}{2}$. Substituting to the expression and taking positive root we obtain a condition $z_2/z_1 > 2(1-P)^{-1}$. On the other hand, if $\rho > \frac{1}{2}$ we have downward parabola with maximum at $w = \frac{1}{2}$ and minima on the ends of the interval $[0,1]$, which implies that we have cascade if $\rho > \frac{z_1}{z_2(1-P)}$. Combining both parts we arrive to the following condition: $\frac{z_2}{z_1} > \frac{1}{(1-P)}\min\{2, \rho^{-1}\}$.

Note that the condition on network structure becomes less restrictive when price decreases. Thus if price is a part of decision process the monopolist can achieve highest diffusion when $P = 0$ and condition becomes $\frac{z_2}{z_1} > \min\{2, \rho^{-1}\}$.

Proof of Proposition 2

Substituting constraints to the objective function and deriving with respect to w we find:

$$(1-P)^2\frac{z_1^4(1-2w)(1-2\rho)(2z_1 + z_2(1-P)(1-2\rho))}{2(z_1^2 - z_1z_2\rho(1-P) - wz_2^2(1-P)^2(1-w)(1-2\rho))^2}$$

A denominator of the condition is always positive and thus sign depends on the numerator. Recall that we assume that we are in sub-critical phase with $\frac{z_2}{z_1} < 2(1-P)^{-1}$ and thus term $2z_1 + z_2(1-P)(1-2\rho)$ is always positive. The sign of the condition depends exclusively on values of ρ and w. Namely if $\rho < \frac{1}{2}$ derivative is positive for $w < \frac{1}{2}$ and negative afterwards. Thus, we can conclude that for $\rho < \frac{1}{2}$ objective function has unique maximum at the point $w = \frac{1}{2}$. In the case when $\rho > \frac{1}{2}$ results are reversed and the objective function has its minimum at a point $w = \frac{1}{2}$ and maxima lie on the boundaries, namely $w^* \in \{0,1\}$. If $\rho = \frac{1}{2}$ all interval $[0,1]$ satisfies first order condition.

Proof of Proposition 3

We analyze the second part of the functional form of demand. Results for the first part can be obtained by substituting $\rho = \frac{1}{2}$. The demand is decreasing and convex in P:

$$\frac{\partial}{\partial P} Q(P,\rho,z_1,z_2) = -\frac{1}{2}\left(1 + \frac{(1-P)z_1^2\rho(2z_1 - z_2(1-P)\rho)}{(z_1-(1-P)z_2\rho)^2}\right) < 0$$

The second derivative:

$$\frac{\partial^2}{\partial P^2} Q(P,\rho,z_1,z_2) = \frac{z_1^4\rho}{(z_1-(1-P)z_2\rho)^3} > 0$$

It is positive since by condition of no global cascade from Lemma 2 we know that $z_1 > (1-P)z_2\rho$. Moreover cross derivative of $Q(P,\rho,z_1,z_2)$ with respect to P and ρ is $-\frac{(1-P)z_1^4}{(z_1-(1-P)z_2\rho)^3}$, which is negative.

Lets turn to the elasticity of demand:

$$\begin{aligned} E_d &= -\frac{\partial_P \log Q(P,\rho,z_1,z_2)}{\partial_P \log P} \\ &= \frac{P}{1-P}\left(1 + z_1\left(\frac{1}{z_1-(1-P)z_2\rho} - \frac{1}{z_1+(1-P)(z1^2-z2)\rho}\right)\right) \end{aligned}$$

Taking derivative of E_d with respect to ρ we obtain:

$$\frac{\partial}{\partial \rho} E_d = \frac{z_1^3 z_2 P(1-P)^2(z1^2-z2)\rho^2 + z_1^5 P}{(z1-(1-P)z_2\rho)^2(z_1-(1-P)z_2\rho + z_1^2\rho(1-P))^2} > 0$$

Which implies that elasticity of demand is increasing in ρ.

$$\frac{\partial}{\partial \rho} s^*(P,\rho,z_1,z_2) = \frac{(1-P)^2 z_1^3}{2(z_1-(1-P)z_2\rho)^2} > 0$$

Thus for $\rho > \frac{1}{2}$ function is increasing in ρ.

Proof of Proposition 4

Price is decreasing in the homophily level

The first order condition for P is:

$$\frac{(1-2P)z_1^2-(1-P)z_1[(1-P)z_2+(1-3P)(z_2-z_1^2)]\rho}{2(z_1-(1-P)z_2\rho)^2}-$$
$$-\frac{(1-P)^2(1-2P)(z_1^2-z_2)z_2\rho^2}{2(z_1-(1-P)z_2\rho)^2}\quad=0$$

Let us fixing expected number of friends z_1 and z_2 and call expression on the left hand side $F(P,\rho)$. The second derivative of $F(P,\rho)$ with respect to P is:

$$F''_P(P,\rho)=\frac{3z_1^4\rho(z_1-z_2\rho)}{(z_1-(1-P)z_2\rho)^4}>0$$

It is positive since by conditions of no giant component we have $z_1>z_2\rho$. Thus function is convex. Evaluating function on the ends of the interval we have $F(0,\rho)=\frac{z_1+z_1^2\rho-z_2\rho}{2(z_1-z_2\rho)}>0$ and $F(1,\rho)=-\frac{1}{2}$. The first derivative with respect to P is negative at 0:

$$F'_P(0,\rho)=-1-\frac{z_1^2\rho(2z_1-z_2\rho)}{(z_1-z_2\rho)^2}<0$$

If $F(P,\rho)$ is convex in P, positive at 0 and negative at 1, we can conclude that function should intersect x-axis from above on the interval $[0,1]$. Hence, the derivative of the $F(P,\rho)$ evaluated for the optimal price $P=P^*$ is negative, $\frac{\partial}{\partial P}F(P^*,\rho)<0$.

Moreover $F(\frac{1}{2},\rho)=\frac{z_1^3\rho}{2(2z_1-z_2\rho)^2}>0$, which implies that the optimal price is always less than $\frac{1}{2}$. The derivative with respect to ρ is:

$$F'_\rho(P,\rho)=-\frac{(1-P)z_1^3\left[(1-P)^2z_2\rho-(1-3P)z_1\right]}{2(z_1-(1-P)z_2\rho)^3}$$

The sign of the derivative depends on the sign of the term in square brackets. The derivative is negative if $P>1-\frac{3z_1}{2z_2\rho}$. By the condition of no giant component we know that $\frac{z_2}{z_1}<\frac{1}{\rho}$ and thus $1-\frac{z_1}{z_2\rho}$ is negative. Since price is positive we can safely conclude that $P>1-\frac{z_1}{z_2\rho}>1-\frac{3z_1}{2z_2\rho}$. This implies that derivative of the expression is negative.

By implicit function theorem we know:

$$\frac{\partial P^*}{\partial\rho}=-\left.\frac{\frac{\partial}{\partial\rho}F(P,\rho)}{\frac{\partial}{\partial P}F(P,\rho)}\right|_{P=P^*}$$

Taking into account that $F'_P(P^*,\rho)<0$ and $F'_\rho(P^*,\rho)<0$ we can conclude that $\frac{\partial P^*}{\partial\rho}$ is negative and consequently the optimal price P^* is decreasing in ρ.

Profits are increasing in the level of homophily

Lets take two levels of homophily $\rho_2 > \rho_1$. By the Proposition 3 we know that for any fixed price P following holds $Q(P, \rho_2, z_1, z_2) > Q(P, \rho_1, z_1, z_2)$. Thus for any given price P the same is true for profits, namely $PQ(P, \rho_2, z_1, z_2) > PQ(P, \rho_1, z_1, z_2)$. Assume further that P_1^* is optimal price for ρ_1. The previous result states that $\pi(\rho_2, P_1^*) > \pi(\rho_1, P_1^*)$ and thus by optimality we know that $\pi(\rho_2, P_2^*) > \pi(\rho_2, P_1^*) > \pi(\rho_1, P_1^*)$, where P_2^* is optimal price for ρ_2.

Proof of Proposition 7

A derivative of sales function with respect to product characteristic is given by:

$$s'(w) = (1-P)\times$$

$$\times \frac{1 + z_1(1-P)(1-3\rho-(1-2\rho)[w^2(1-P)z_1 + 2w(1-\rho(1-P)z_1) + \rho(1-P)z_1])}{(1-wz_1^2(1-P)^2(1-2\rho)(1-w) - z_1\rho(1-P))^2}$$

Note that the denominator is always positive. It is easy to verify that for $\rho > \frac{1}{2}$ all terms in the numerator involving w are positive too. Thus if we prove that $s'(0) > 0$ then the derivative of sales function with respect to w is positive on the whole interval $[0, 1]$ and we can conclude that the optimal design is $w^* = 1$. Substituting $w = 0$ into the derivative and taking into account that $z_1 < \frac{1}{\rho(1-P)}$ we have:

$$s'(0) = \frac{1 + (1-P)z_1(1-2\rho)}{1-(1-P)z_1\rho} = 1 + \frac{(1-P)z_1(1-\rho)}{1-(1-P)z_1\rho} > 0$$

Thus for $\rho > \frac{1}{2}$ characteristic $w^* = 1$ is the solution to the problem.

When $\rho < \frac{1}{2}$ then all terms involving w are negative and numerator is decreasing function in w. Thus numerator has its minimum at $w = 1$ and condition for $s'(w) > 0$ on the interval $w \in [0, 1]$ is simply $s'(1) > 0$. This in turn implies that if $w^* = 1$ is maximum it is also the global maximum. The derivative at 1 is greater than zero if:

$$-2(1-P)^2z_1^2\rho^2 + z_1(1-P+3(1-P)^2z_1)\rho + 1 - (1-P)z_1 - (1-P)^2z_1^2 > 0$$

An expression on the left describes downward sloping parabola. The solution is:

$$\rho_1 = \frac{3}{4} + \frac{1-\sqrt{9-2z_1(1-P)+(1-P)^2z1^2}}{4(1-P)z_1}$$

$$\rho_2 = \frac{3}{4} + \frac{1+\sqrt{9-2z_1(1-P)+(1-P)^2z1^2}}{4(1-P)z_1}$$

It can be shown that $\rho_2 > 1$ and thus condition reduces to:

$$\frac{3}{4}+\frac{1-\sqrt{9-2z_1(1-P)+(1-P)^2z1^2}}{4(1-P)z_1}<\rho<\frac{1}{2}$$

Combining the previous condition with the case of $\rho > \frac{1}{2}$, we know that $w^* = 1$ is solution if:

$$\rho > \hat{\rho}_T(z_1, P) = \frac{3}{4}+\frac{1-\sqrt{9-2z_1(1-P)+(1-P)^2z1^2}}{4(1-P)z_1}$$

On the other hand for $\rho < \hat{\rho}_T(z_1, P)$ there is an interior solution which is given by:

$$w^* = \rho - \frac{1-2\rho-\sqrt{(1-\rho)(1-2\rho)(1-(1-P)z_1\rho)[2+(1-P)z_1(1-2\rho)]}}{(1-P)z_1(1-2\rho)}$$

It is interesting to note that $w = \frac{1}{2}$ is never solution for $\rho < \frac{1}{2}$ since $s'(\frac{1}{2}) = \frac{4(1-P)}{(2-(1-P)z_1)(2+(1-P)z_1(1-2\rho))} > 0$, which implies that $w^* > \frac{1}{2}$.

Proof of Proposition 8

Lets denote by $\theta = \frac{z_2}{z_1}$ and by $\lambda(w)$ the following polynom:

$$\lambda(w) = 1-(1-P)^2w(1-w^r)^{\frac{1}{r}}\theta^2(1-2\rho)-(1-P)\left(w+(1-w^r)^{\frac{1}{r}}\right)\theta\rho$$

The global cascade of sales occurs if there is $\bar{w}$ such that $\lambda(\bar{w}) \leq 0$. One can readily obtain condition for ρ. We just need to take the derivative with respect to ρ and to show that it is negative. Thus there is the global cascade of sales if $\rho > \hat{\rho}(\theta, r)$, where

$$\hat{\rho}(\theta, r) = \min_{0\leq w\leq 1}\frac{1-w\theta^2(1-P)^2(1-w^r)^{\frac{1}{r}}}{(1-P)\theta\left(w+(1-w^r)^{\frac{1}{r}}(1-2w\theta(1-P))\right)}$$

From the previous analysis we know that candidates for maxima are extreme values and such that $q^A = q^B$, which in our case is $\left(\frac{1}{2}\right)^{\frac{1}{r}}$. Evaluating polynomial at 0 we have $\lambda(0) = 1-(1-P)\theta\rho$ and at the point $\left(\frac{1}{2}\right)^{\frac{1}{r}}$ we have:

$$\lambda\left(2^{-\frac{1}{r}}\right) = 4^{-\frac{1}{r}}(2^{\frac{1}{r}}-(1-P)\theta)\left[2^{\frac{1}{r}}+(1-P)\theta(1-2\rho)\right]$$

From the first condition we can conclude that if $\rho > \frac{1}{\theta}$ then there is global cascade. From the second we see that if $\theta > 2^{\frac{1}{r}}(1-P)^{-1}$ and $\rho < \hat{\rho} = \frac{1}{2}+\frac{2^{\frac{1}{r}}}{2\theta(1-P)}$ then global cascade of sales arises. Lets consider a case when $\theta > 2^{\frac{1}{r}}(1-P)^{-1}$, but $\rho < \frac{1}{2}+\frac{2^{\frac{1}{r}}}{2\theta(1-P)}$.

From the first condition we know that there is global cascade if $\rho > \frac{1}{\theta(1-P)}$ thus to insure existence of the global cascade we should prove that $\frac{1}{2} + \frac{2^{\frac{1}{r}}}{2\theta(1-P)} > \frac{1}{\theta(1-P)}$.

$$\frac{1}{2}(1-P) + 2^{\frac{1-r}{r}}\theta^{-1} > \theta^{-1}$$

$$\frac{1}{2}(1-P) > \theta^{-1}\left(1 - 2^{\frac{1-r}{r}}\right)$$

$$\theta > (1-P)^{-1}\left(2 - 2^{\frac{1}{r}}\right)$$

$$\theta > 2^{\frac{1}{r}}(1-P)^{-1}\left(2^{\frac{r-1}{r}} - 1\right)$$

There is the global cascade if the former condition holds. However, we have assumed that $\theta > 2^{\frac{1}{r}}(1-P)^{-1}$, which implies that former condition holds, since $\left(2^{\frac{r-1}{r}} - 1\right) < 1$. Thus we have shown that if $\theta > 2^{\frac{1}{r}}$ there is global cascade independently of the homophily level ρ.

Proof of Proposition 9

For the case of Poisson degree distribution size of sales cascade is given by:

$$s(w, P, r, z_1) = \frac{(1-P)(w + (1-w^r)^{\frac{1}{r}}(1 + 2wz_1(1-P)(1-2\rho)))}{2(1-(1-P)z_1(w\rho + (1-w^r)^{\frac{1}{r}}(\rho + wz_1(1-P)(1-2\rho))))}$$

The product characteristic $w^* = 0$ is global maximum if for any w, $s(0) \geq s(w)$:

$$\frac{1-P}{2(1-z_1\rho(1-P))} \geq \frac{(1-P)(w + (1-w^r)^{\frac{1}{r}}(1 + 2wz_1(1-P)(1-2\rho)))}{2(1-(1-P)z_1(w\rho + (1-w^r)^{\frac{1}{r}}(\rho + wz_1(1-P)(1-2\rho))))}$$

since $1 - P \geq 0$ by definition, we have

$$\frac{1}{2(1-z_1\rho(1-P))} - \frac{(w + (1-w^r)^{\frac{1}{r}}(1 + 2wz_1(1-P)(1-2\rho)))}{2(1-(1-P)z_1(w\rho + (1-w^r)^{\frac{1}{r}}(\rho + wz_1(1-P)(1-2\rho))))} \geq 0$$

Note further that denominators of two fractions are positive due to condition of no global cascade, thus the sign of the expression depends on the numerator of combined terms, which is:

$$\begin{aligned}-4(1-P)^2 w(1-w^r)^{\frac{1}{r}} z_1^2 \rho^2 + 4(1-P)w(1-w^r)^{\frac{1}{r}} z_1(1+(1-P)z_1)\rho + \\ +1-w-(1-w^r)^{\frac{1}{r}}(1+(1-P)wz_1(2+(1-P)z_1)) \quad \geq 0\end{aligned}$$

Note that expression describes downward sloping parabola and thus our condition will be in the form $\rho_1 \leq \rho \leq \rho_2$, where ρ_1 and ρ_2 are solutions to the quadratic equation:

$$\rho_1 = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{z_1(1-P)} - \frac{1}{z_1(1-P)}\sqrt{\frac{(1-w)(1-(1-w^r)^{\frac{1}{r}})}{w(1-w^r)^{\frac{1}{r}}}}\right)$$

$$\rho_2 = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{z_1(1-P)} + \frac{1}{z_1(1-P)}\sqrt{\frac{(1-w)(1-(1-w^r)^{\frac{1}{r}})}{w(1-w^r)^{\frac{1}{r}}}}\right)$$

The condition should hold for all w and thus we should find the maximum of ρ_1 and the minimum of ρ_2. In order to do this we should identify maximum and minimum of the term with w. Taking derivative of this term with respect to w we have:

$$-\frac{(1-w^r)^{-\frac{1+r}{r}}\left[(1-2w^r)+\left(w^{r+1}-(1-w^r)^{\frac{1+r}{r}}\right)\right]}{w^2}$$

Independently of r one can see that first derivative is zero at the point $w = \left(\frac{1}{2}\right)^{\frac{1}{r}}$. It can be proved that for $r < 1$, $-(1-2w^r) - \left(w^{r+1}-(1-w^r)^{\frac{1+r}{r}}\right)$ is negative for $w < \left(\frac{1}{2}\right)^{\frac{1}{r}}$ and positive afterwards, which implies that minimum is at $w = \left(\frac{1}{2}\right)^{\frac{1}{r}}$ and maxima lay at borders.

Substituting back values of w into condition for ρ we obtain:

$$\frac{1}{2} - \frac{2^{\frac{1}{r}}-2}{2z_1(1-P)} < \rho < \frac{1}{2} + \frac{2^{\frac{1}{r}}}{2z_1(1-P)}$$

Since $z_1 \leq 2^{\frac{1}{r}}$ the minimum of last term is 1. Taking it into account we can rewrite the condition as:

$$\frac{1}{2} - \frac{2^{\frac{1}{r}}-2}{2z_1(1-P)} < \rho < 1$$

On the other hand for $r > 1$, one can show that $w^* = 0$ is never a solution. Lets evaluate derivative of sales at $w = 0$ for the case when $r > 1$ we have:

$$\left.\frac{\partial s}{\partial w}\right|_{w=0} = \frac{(1-P)(1+(1-P)z_1(1-2\rho))^2}{2(1-(1-P)z_1\rho])^2} > 0$$

It is always positive which implies that $w^* = 0$ is never solution for $r > 1$.

The symmetric design $w = \left(\frac{1}{2}\right)^{\frac{1}{r}}$ is global maximum if for any w, $s\left(\left(\frac{1}{2}\right)^{\frac{1}{r}}\right) \geq s(w)$:

$$\frac{1-P}{2^{\frac{1}{r}} - (1-P)z_1} \geq \frac{(1-P)(w + (1-w^r)^{\frac{1}{r}}(1 + 2wz_1(1-P)(1-2\rho)))}{2(1-(1-P)z_1(w\rho + (1-w^r)^{\frac{1}{r}}(\rho + wz_1(1-P)(1-2\rho))))}$$

The denominators are positive due to no global cascade condition and thus sign of the expression depends on the numerator of combined fraction, which is:

$$\begin{aligned} -2(1-P)(w + (1-2^{\frac{1+r}{r}}w)(1-w^r)^{\frac{1}{r}})z_1\rho + 2 - (2-2w^r)^{\frac{1}{r}} + \\ +(1-P)(1-2^{\frac{1+r}{r}}w)(1-w^r)^{\frac{1}{r}}z_1 - w(2^{\frac{1}{r}} - (1-P)z_1) \quad \geq 0 \end{aligned}$$

Note that the line is downwards sloping if for any w, $(w + (1-2^{\frac{r+1}{r}}w)(1-w^r)^{\frac{1}{r}}) > 0$. Thus to prove that it has downward slope we should prove that the minimum of the term $w + (1-2^{\frac{r+1}{r}}w)(1-w^r)^{\frac{1}{r}}$ is greater or equal to 0. The first derivative is:

$$1 - 2^{\frac{1+r}{r}}(1-2w^r)(1-w^r)^{\frac{1-r}{r}} - \left(\frac{(1-w^r)^{\frac{1}{r}}}{w}\right)^{1-r} \tag{4}$$

It is zero at point $w = \left(\frac{1}{2}\right)^{\frac{1}{r}}$. For $r < 1$ the expression is negative for $w < \left(\frac{1}{2}\right)^{\frac{1}{r}}$, since term $2^{\frac{1+r}{r}}(1-2w^r)(1-w^r)^{\frac{1-r}{r}} > 0$ and term $w^{-(1-r)}(1-w^r)^{\frac{1-r}{r}} > 1$ (by properties of frontier). The expression is positive for $w > \left(\frac{1}{2}\right)^{\frac{1}{r}}$, because term $2^{\frac{1+r}{r}}(1-2w^r)(1-w^r)^{\frac{1-r}{r}} < 0$ and term $w^{-(1-r)}(1-w^r)^{\frac{1-r}{r}} < 1$. This implies that minimum lies at the point $w = \left(\frac{1}{2}\right)^{\frac{1}{r}}$ where the expression equals to zero. Thus the line has negative slope. And condition becomes:

$$\rho < \hat{\rho}_1 = \min_w \left\{ \frac{1}{2}\left(1 - \frac{2^{\frac{1}{r}}(w + (1-w^r)^{\frac{1}{r}}) - 2}{z_1(1-P)(w + (1-2^{\frac{r+1}{r}}w)(1-w^r)^{\frac{1}{r}})}\right) \right\}$$

we can show that for $r < 1$ the expression with w has its maxima on the borders and thus, evaluating at $w = 0$ we have:

$$\rho < \frac{1}{2} - \frac{2^{\frac{1}{r}} - 2}{2z_1(1-P)}$$

The case when $r > 1$

Lets rewrite the expression (4):

$$1 - (2^{\frac{1+r}{r}}(1-2w^r) + w^{r-1})(1-w^r)^{\frac{1-r}{r}}$$

It is zero at point $w = \left(\frac{1}{2}\right)^{\frac{1}{r}}$. For $r > 1$ and $w \leq \left(\frac{1}{2}\right)^{\frac{1}{r}}$, $\min\{(1-w^r)^{\frac{1-r}{r}}\} = 2^{\frac{r-1}{r}}$ and $\min\{(2^{\frac{1+r}{r}}(1-2w^r)+w^{r-1})\} = 2^{\frac{1-r}{r}}$. Thus minimum of the product of two terms is equal to 1 and this implies that for $w \leq \left(\frac{1}{2}\right)^{\frac{1}{r}}$ expression is negative. For $w > \left(\frac{1}{2}\right)^{\frac{1}{r}}$, $\max\{(1-w^r)^{\frac{1-r}{r}}\} = 2^{\frac{r-1}{r}}$ and $\max\{(2^{\frac{1+r}{r}}(1-2w^r)+w^{r-1})\} = 2^{\frac{1-r}{r}}$, which implies that expression is positive. Thus minimum of the expression is at the point $w = \left(\frac{1}{2}\right)^{\frac{1}{r}}$ line has negative slope.

This leads us again to the condition:

$$\rho < \hat{\rho}_2 = \min_{w} \left\{ \frac{1}{2} \left(1 - \frac{2^{\frac{1}{r}}(w + (1-w^r)^{\frac{1}{r}}) - 2}{z_1(1-P)(w + (1-2^{\frac{r+1}{r}}w)(1-w^r)^{\frac{1}{r}})} \right) \right\}$$

Thus we can establish that there is $\hat{\rho}_2$ such that if $\rho < \hat{\rho}_2$ than the optimal characteristic is $w = \left(\frac{1}{2}\right)^{\frac{1}{r}}$. Note that condition $\hat{\rho}_1 \neq \hat{\rho}_2$ since we optimize for different values of r. If $\rho > \hat{\rho}_2$ then solution belongs to $(\left(\frac{1}{2}\right)^{\frac{1}{r}}, 1)$, since as we have seen $w = 1$ is always not optimal for the case of $r > 1$.

Proof of Lemma 3

Assume that probabilities to get infection do not depend on the number of links that node has: $q_k^A = q^A$ and $q_k^B = q^B$. We obtain:

$$s = \gamma q^A + (1-\gamma)q^B - \gamma q^A \sum_{k=0}^{\infty} p(k)[\rho^A u^A + (1-\rho^A)u^B]^k - (1-\gamma)q^B \sum_{k=0}^{\infty} p(k)[\rho^B u^B + (1-\rho^B)u^A]^k$$

$$u^A = 1 - q^A + q^A \sum_{k=1}^{\infty} \xi(k) \left[\rho^A u^A + (1-\rho^A)u^B\right]^{k-1}$$

$$u^B = 1 - q^B + q^B \sum_{k=1}^{\infty} \xi(k) \left[\rho^B u^B + (1-\rho^B)u^A\right]^{k-1}$$

$$\gamma(1-\rho^A) = (1-\gamma)(1-\rho^B)$$

Note that last expression insures that a number of links going from nodes of type A to nodes of type B should be equal of number of links going from nodes of type B to nodes of type A. However since we choose γ, ρ^A and ρ^B exogenously we can insure that condition holds.

Or we can rewrite it as:

$$s = \gamma q^A + (1-\gamma)q^B - \gamma q^A G_0[\rho^A u^A + (1-\rho^A)u^B] - (1-\gamma)q^B G_0[\rho^B u^B + (1-\rho^B)u^A]$$

$$u^A = 1 - q^A + q^A \hat{G}_1[\rho^A u^A + (1-\rho^A)u^B]$$

$$u^B = 1 - q^B + q^B \hat{G}_1[\rho^B u^B + (1-\rho^B)u^A]$$

$$\gamma(1-\rho^A) = (1-\gamma)(1-\rho^B)$$

Let us denote by $x = \rho^A u^A + (1-\rho^A)u^B$ and by $y = \rho^B u^B + (1-\rho^B)u^A$ thus

$$s = \gamma q^A + (1-\gamma)q^B - \gamma q^A G_0[x] - (1-\gamma)q^B G_0[y]$$

$$u^A = 1 - q^A + q^A \hat{G}_1[x]$$

$$u^B = 1 - q^B + q^B \hat{G}_1[y]$$

$$\gamma(1-\rho^A) = (1-\gamma)(1-\rho^B)$$

Or equivalently:

$$s = \gamma q^A + (1-\gamma)q^B - \gamma q^A G_0[x] - (1-\gamma)q^B G_0[y]$$

$$x = \rho^A[(1-q^A) + q^A \hat{G}_1(x)] + (1-\rho^A)[(1-q^B) + q^B \hat{G}_1(y)]$$

$$y = \rho^B[(1-q^B) + q^B \hat{G}_1(y)] + (1-\rho^B)[(1-q^A) + q^A \hat{G}_1(x)]$$

$$\gamma(1-\rho^A) = (1-\gamma)(1-\rho^B)$$

After simplification:

$$s = \gamma q^A + (1-\gamma)q^B - \gamma q^A G_0[x] - (1-\gamma)q^B G_0[y]$$

$$x = 1 - \rho^A q^A - (1-\rho^A)q^B + \rho^A q^A \hat{G}_1(x) + (1-\rho^A)q^B \hat{G}_1(y)$$

$$y = 1 - \rho^B q^B - (1-\rho^B)q^A + \rho^B q^B \hat{G}_1(y) + (1-\rho^B)q^A \hat{G}_1(x)$$

$$\gamma(1-\rho^A) = (1-\gamma)(1-\rho^B)$$

Further assume that we have the same number of nodes of two types, $\gamma = \frac{1}{2}$, thus last condition implies that $\rho = \rho^A = \rho^B$.

$$s = \frac{1}{2}\left[q^A + q^B - q^A G_0(x) - q^B G_0(y)\right]$$

$$x = \rho[1 - q^A + q^A \hat{G}_1(x)] + (1-\rho)[1 - q^B + q^B \hat{G}_1(y)]$$

$$y = \rho[1 - q^B + q^B \hat{G}_1(y)] + (1-\rho)[1 - q^A + q^A \hat{G}_1(x)]$$

OR

$$s = \frac{1}{2}\left[q^A + q^B - q^A G_0(x) - q^B G_0(y)\right]$$

$$x = 1 - \rho q^A - (1-\rho)q^B + \rho q^A \hat{G}_1(x) + (1-\rho)q^B \hat{G}_1(y)$$

$$y = 1 - (1-\rho)q^A - \rho q^B + (1-\rho)q^A \hat{G}_1(x) + \rho q^B \hat{G}_1(y)$$

So we can write maximization problem of the firm as:

$$\max_{q^A, q^B} \frac{1}{2}\left[q^A + q^B - q^A G_0(x) - q^B G_0(y)\right]$$

s.t.

$$x = 1 - \rho q^A - (1-\rho)q^B + \rho q^A \hat{G}_1(x) + (1-\rho)q^B \hat{G}_1(y)$$

$$y = 1 - (1-\rho)q^A - \rho q^B + (1-\rho)q^A \hat{G}_1(x) + \rho q^B \hat{G}_1(y)$$

Proof of Proposition 10

FOC for the problem are:

$$-G_0(x) - wG_0'(x)\frac{\partial x}{\partial w} + G_0(y) - (1-w)G_0'(y)\frac{\partial y}{\partial w} = 0$$

To find derivatives let us derive constraints with respect to w:

$$\frac{\partial x}{\partial w} = 1 - 2\rho + \rho\hat{G}_1(x) + \rho w \hat{G}_1'(x)\frac{\partial x}{\partial w} - (1-\rho)\hat{G}_1(y) + (1-\rho)(1-w)\hat{G}_1'(y)\frac{\partial y}{\partial w}$$

$$\frac{\partial y}{\partial w} = -1 + 2\rho + (1-\rho)\hat{G}_1(x) + (1-\rho)w\hat{G}_1'(x)\frac{\partial x}{\partial w} - \rho\hat{G}_1(y) + \rho(1-w)\hat{G}_1'(y)\frac{\partial y}{\partial w}$$

The case when firm's action has no effect.

Interesting case arises when $\rho = \frac{1}{2}$. It seems that w has no effect on the ultimate outbreak of the infection. Let us rewrite the problem:

$$\max_w 1 - wG_0(x) - (1-w)G_0(y)$$

s.t.

$$x = \frac{1}{2} + \frac{1}{2}w\hat{G}_1(x) + \frac{1}{2}(1-w)\hat{G}_1(y)$$

$$y = \frac{1}{2} + \frac{1}{2}w\hat{G}_1(x) + \frac{1}{2}(1-w)\hat{G}_1(y)$$

$$0 \leq w \leq 1, 0 \leq x \leq 1, 0 \leq y \leq 1$$

Note that in this case equations for x and y are the same and thus in equilibrium $x = y$. This in turn implies that w disappears from the maximization problem:

$$\max_w 1 - G_0(x)$$

s.t.

$$x = \frac{1}{2} + \frac{1}{2}\hat{G}_1(x)$$

$$0 \leq x \leq 1$$

Thus eventual outbreak is the same for all values of w and moreover it's size is equal to the giant component for homogenous nodes.

The case when specialized design is optimal.

Assume that we want to check when it is optimal to focus on first group or equivalently when $w = 1$ is the solution. Note that $w = 1$ is corner solution that is why it is enough to show that derivative of $\frac{\partial s}{\partial w}\Big|_{w=1}$ is positive:

$$-G_0(x) - G_0'(x)\frac{\partial x}{\partial w} + G_0(y) > 0$$

s.t.

$$x = 1 - \rho + \rho\hat{G}_1(x)$$

$$y = \rho + (1-\rho)\hat{G}_1(x)$$

and

$$\frac{\partial x}{\partial w} = 1 - 2\rho + \rho\hat{G}_1(x) + \rho\hat{G}_1'(x)\frac{\partial x}{\partial w} - (1-\rho)\hat{G}_1(y)$$

Thus we have

$$\frac{\partial x}{\partial w} = \frac{1 - 2\rho + \rho\hat{G}_1(x) - (1-\rho)\hat{G}_1(y)}{1 - \rho\hat{G}_1'(x)}$$

Substituting to the maximization problem we obtain:

$$G_0(y) - G_0(x) - G_0'(x)\frac{1 - 2\rho + \rho\hat{G}_1(x) - (1-\rho)\hat{G}_1(y)}{1 - \rho\hat{G}_1'(x)} > 0$$

s.t.

$$x = 1 - \rho + \rho\hat{G}_1(x)$$

$$y = \rho + (1-\rho)\hat{G}_1(x)$$

Let us rewrite first equation:

$$[G_0(y) - G_0(x)] + G_0'(x)\frac{\rho(1 - \hat{G}_1(x)) - (1-\rho)[1 - \hat{G}_1(y)]}{1 - \rho\hat{G}_1'(x)} \geq 0$$

It is obvious that first term is positive when y is greater than x, recall that $G_0'[x] > 0$. Thus the condition for $y \geq x$ is following:

$$\rho + (1-\rho)\hat{G}_1(x) \geq 1 - \rho + \rho\hat{G}_1(x)$$

$$(2\rho - 1)[1 - \hat{G}_1(x)] \geq 0$$

since $\hat{G}_1(x) \leq 1$ for all values of probability $x \in [0,1]$ the condition is $\rho \geq \frac{1}{2}$.

The same happens with second term when $\rho > \frac{1}{2}$. Note that $\rho > \frac{1}{2}$ implies that $\hat{G}_1[x] < \hat{G}_1[y]$ consequently $1 - \hat{G}_1[x] > 1 - \hat{G}_1[y]$ and at last multiplying left side by ρ and right side by $1-\rho$ and taking into account that $\rho \geq \frac{1}{2}$ we have:

$$\rho[1 - \hat{G}_1(x)] > (1-\rho)[1 - \hat{G}_1(y)]$$

Thus we have proved that $w = 1$ is locally optimal if $\rho > \frac{1}{2}$ independently of connectivity distribution!!!

The case when symmetric design is optimal

That is to say we are looking for conditions under which $w = \frac{1}{2}$ is solution to our system of equations.

$$-G_0(x) - \frac{1}{2}G_0'(x)\frac{\partial x}{\partial w} + G_0(y) - \frac{1}{2}G_0'(y)\frac{\partial y}{\partial w} = 0$$

s.t.

$$x = \frac{1}{2} + \frac{1}{2}\rho\hat{G}_1(x) + \frac{1}{2}(1-\rho)\hat{G}_1(y)$$

$$y = \frac{1}{2} + \frac{1}{2}(1-\rho)\hat{G}_1(x) + \frac{1}{2}\rho\hat{G}_1(y)$$

and

$$\frac{\partial x}{\partial w} = 1 - 2\rho + \rho\hat{G}_1(x) + \rho w\hat{G}_1'(x)\frac{\partial x}{\partial w} - (1-\rho)\hat{G}_1(y) + (1-\rho)(1-w)\hat{G}_1'(y)\frac{\partial y}{\partial w}$$

$$\frac{\partial y}{\partial w} = -1 + 2\rho + (1-\rho)\hat{G}_1(x) + (1-\rho)w\hat{G}_1'(x)\frac{\partial x}{\partial w} - \rho\hat{G}_1(y) + \rho(1-w)\hat{G}_1'(y)\frac{\partial y}{\partial w}$$

It is easy to check that $x = y$ satisfies our conditions on x and y, thus we have:

$$x = \frac{1}{2} + \frac{1}{2}\hat{G}_1(x)$$

Moreover it is possible to show (it should be) that system of equations for x and y has only two solution. The first one is $x = y = 1$. Thus we have:

$$G_0'(x)\left[\frac{\partial x}{\partial w} + \frac{\partial y}{\partial w}\right] = 0$$

s.t.

$$x = \frac{1}{2} + \frac{1}{2}\hat{G}_1(x)$$

The first solution to the FOC equation is $G_0'(x) = 0$ which implies $x = 0$. This obviously does not satisfy second equation, thus the only possibility left is $\frac{\partial x}{\partial w} = 0$

$$\frac{\partial x}{\partial w} = 1 - 2\rho + \rho\hat{G}_1(x) + \rho w\hat{G}_1'(x)\frac{\partial x}{\partial w} - (1-\rho)\hat{G}_1(y) + (1-\rho)(1-w)\hat{G}_1'(y)\frac{\partial y}{\partial w}$$

$$\frac{\partial y}{\partial w} = -1 + 2\rho + (1-\rho)\hat{G}_1(x) + (1-\rho)w\hat{G}_1'(x)\frac{\partial x}{\partial w} - \rho\hat{G}_1(y) + \rho(1-w)\hat{G}_1'(y)\frac{\partial y}{\partial w}$$

Solving previous system and substituting $y = x$ we have:

$$\frac{\partial x}{\partial w} = -\frac{1 - 2\rho - \hat{G}_1(x) + 2\rho\hat{G}_1(x) - \rho\hat{G}_1(x)\hat{G}_1'(x) + \rho\hat{G}_1'(x) + \frac{1}{2}\hat{G}_1(x)\hat{G}_1'(x) - \frac{\hat{G}_1'(x)}{2}}{-\frac{1}{2}\rho[\hat{G}_1'(x)]^2 + \frac{1}{4}[\hat{G}_1'(x)]^2 + \rho\hat{G}_1'(x) - 1}$$

$$\frac{\partial y}{\partial w} = \frac{1 - 2\rho - \hat{G}_1(x) + 2\rho\hat{G}_1(x) - \rho\hat{G}_1(x)\hat{G}_1'(x) + \rho\hat{G}_1'(x) + \frac{1}{2}\hat{G}_1(x)\hat{G}_1'(x) - \frac{\hat{G}_1'(x)}{2}}{-\frac{1}{2}\rho[\hat{G}_1'(x)]^2 + \frac{1}{4}[\hat{G}_1'(x)]^2 + \rho\hat{G}_1'(x) - 1}$$

Note that $\frac{\partial x}{\partial w} = -\frac{\partial y}{\partial w}$ and thus we have that:

$$G_0'(x)\left[\frac{\partial x}{\partial w} + \frac{\partial y}{\partial w}\right] = 0$$

This implies that $w = \frac{1}{2}$ is always the critical point. What is left to proof is that it is maximum when $\rho < \frac{1}{2}$.

SOC of the problem

$$-2G_0'(x)\frac{\partial x}{\partial w} - wG_0''(x)\left(\frac{\partial x}{\partial w}\right)^2 - wG_0'(x)\frac{\partial^2 x}{\partial w^2} + 2G_0'(y)\frac{\partial y}{\partial w} - (1-w)G_0''(y)\left(\frac{\partial y}{\partial w}\right)^2 - (1-w)G_0'(y)\frac{\partial^2 y}{\partial w^2}$$

SOC when $w = \frac{1}{2}$

$$-\frac{4z(1-2\rho)(1-\hat{G}_1(x))}{\left(2-\hat{G}_1'(x)\right)\left(2+(1-2\rho)\hat{G}_1'(x)-2\right)^2}\times$$

$$\times\left((1-2\rho)\left(\hat{G}_1'(x)^2 + 2\hat{G}_1'(x) + \hat{G}_1''(x)(1-\hat{G}_1(x))\right)\hat{G}_1(x) + 8\hat{G}_1(x) + (1-2\rho)\left(2-\hat{G}_1'(x)\right)\hat{G}_1'(x)\right)$$

Thus we can conclude that $w = \frac{1}{2}$ is local maximum for $\rho < \frac{1}{2}$ if:

$$2 - \hat{G}_1'(x) > 0$$

s.t.

$$x = \frac{1}{2} + \frac{1}{2}\hat{G}_1(x)$$

Let us denote by $F(x) = \frac{1}{2} + \frac{1}{2}\hat{G}_1(x)$ then solution to equation x^* is such that $F(x)$ crosses 45 degree line in x^* from above since $F(0) = \frac{1}{2}$ Thus we can conclude that $F'(x^*) < 1$. Thus $\frac{1}{2}\hat{G}_1'(x) < 1$ and consequently $\hat{G}_1'(x) < 2$, which in turn implies that our condition always holds.

When $w = \frac{1}{2}$ should be preferred to $w = 1$?

Recall that in the case when $w = \frac{1}{2}$ the size of the giant component is given by:

$$S(\frac{1}{2}) = \frac{1}{2} - \frac{1}{2}G_0(x_m^*)$$

$$x_m^* = \frac{1}{2} + \frac{1}{2}\hat{G}_1(x_m^*)$$

On the other hand if $w = 1$ we have:

$$S(1) = \frac{1}{2} - \frac{1}{2}G_0(x_b^*)$$

$$x_b^* = 1 - \rho + \rho\hat{G}_1(x_b^*)$$

Due to the monotonicity of the $G_0(x)$ we know that $S(\frac{1}{2}) > S(1)$ whenever $x_m^* < x_b^*$. So basically we should see how ρ affects solution to fixed point problem, since first equation is just particular case of the second. Using IFT we have:

$$\frac{\partial x}{\partial \rho} = -1 + \hat{G}_1(x) + \rho\hat{G}_1'(x)\frac{\partial x}{\partial \rho}$$

$$\frac{\partial x}{\partial \rho} = -\frac{1 - \hat{G}_1(x)}{1 - \rho\hat{G}_1'(x)}$$

Note that x is solution to fixed point of $1 - \rho + \rho\hat{G}_1(x)$ at x^* it should cross the 45 degree line and this in turn implies that $\rho\hat{G}_1'(x) < 1$ thus we have shown that $\frac{\partial x}{\partial \rho} < 0$. Thus in turn implies that if $\rho < \frac{1}{2}$ $x_b^* > x_m^*$ and thus $S(\frac{1}{2}) > S(1)$ on the other hand if $\rho > \frac{1}{2}$ thus $S(\frac{1}{2}) < S(1)$