Stochastic Stability in the Best–Shot Game

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Abstract

The best–shot game applied to networks is a discrete model of many processes of contribution in local public goods. It has generally a wide multiplicity of equilibria that we refine through stochastic stability. In this paper we show that, depending on how we define the perturbations, i.e. the possible mistakes that agents can make, we can obtain very different set of stochastically stable equilibria. In particular and non–trivially, if we assume that the only possible source of error is that of an agent contributing that stops doing so, then the only stochastically stable equilibria are the most inefficient ones.

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1 Introduction

Let us start with an example.

Example 1. Ann, Bob, Cindy, Dan and Eve live in a suburb of a big city and they all prefer to take private cars to reach downtown every working day. They could share the car but they are not all friends together: Ann and Eve do not know each other but they both know Bob, Cindy and Dan, who also don't know each other. The network of relations is shown in Figure 1. In a one–shot equilibrium (the first working day) they will end up sharing cars. Any of our characters would be happy to give a lift to a friend, but we assume here that non–linked people do not know each other and would not offer each others a lift. No one would take the car if a friend is doing so, but someone would be forced to take it if none of her/his friends is doing so. There is an efficient equilibrium in which Ann and Eve take the car (and the other three take somehow a lift), and a more polluting one in which Bob, Cindy and Dan take their car (offering a lift to Ann and Eve, who will choose one of them).

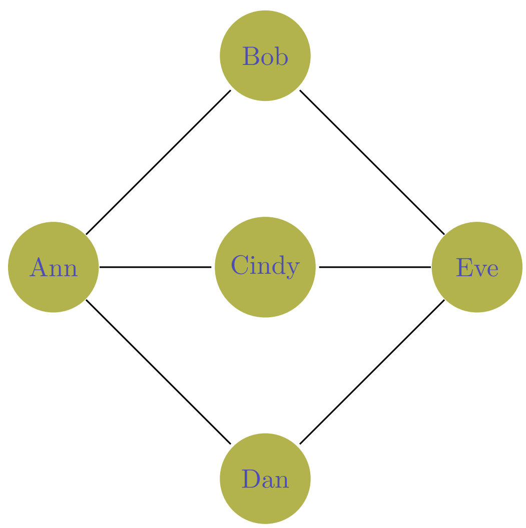


Figure 1: Five potential drivers in a network of relations.

Imagine to be in the efficient equilibrium. Now suppose that, even if they all agreed how to manage the trip, in the morning Ann finds out that her car's engine is broken and she cannot start it. She will call the other three, who are however not planning to take the car and will not be able to offer her a lift. As Ann does not know Eve, and Eve will be the only one left with a car, Ann will have to wait for her own car to be repaired before she can reach her workplace. Only if both cars of Ann end Eve break down, then Bob, Cindy and

Dan will take their cars, and we will shift to the inefficient equilibrium. It is easy to see that if we start instead from the inefficient equilibrium, then we need three cars to break down before we can induce Ann and Eve to get their own. In this sense the *bad* equilibrium is more *stable*, as it needs a less likely event in order to be changed with another equilibrium. □

In this paper we analyze the best shot game:[1] in a fixed exogenous network of binary relations, each node (player) may or may not provide a local public good. The best response for each player is to provide the good if and only if no one of her neighbors is doing so.
In the previous example we have described an equilibrium where each player can take one of two actions: take or not take the car. Then we have included a possible source of *error*: the car may break down and one should pass from action 'take the car' to action 'not take the car'. Clearly we can also imagine a second type of error, e.g. if a player forgets that someone offered her/him a lift and takes her/his own car anyway. We think however that there are situations in which the first type of error is the only plausible one, as well as there can be cases in which the opposite is true, and finally cases where the two are equally likely.

What we want to investigate in the present paper is how the likelihood of different kind of errors may influence the likelihood of different Nash Equilibria. Formally, we will analyze stochastic stability (Young, 1998) of the Nash equilibria of the best shot game, under different assumptions on the perturbed Markov chain that allows the agents to make errors.

What we find is that, if only errors of the type described in the example are possible, that is players can only make a mistake by not providing the public good even if that is their best response in equilibrium, then the only stochastically stable Nash equilibria are those (inefficient ones) that maximize the number of contributors. If instead also (or only) the other type of error (i.e. provide the good even if it is a dominant action to free ride) is admitted, then every Nash equilibrium is stochastically stable.

The best shot game is very similar to the local public goods game of Bramoullé and Kranton (2007): they motivate their model with a story of neighborhood farmers, with reduced ability to check each others' technology (this is the network constrain), who can invest in experimenting a new fertilizer. They assume that the action set of the players is continuous on the non–negative numbers (how much to invest in the new risky fertilizer), they define stable equilibria as those that survive small perturbations, and they find that stable equilibria are *specialized* ones, in which every agent either contributes an optimal amount (which is the same for all contributors) or free rides, so that their stable equilibria

[1] This name for exactly the same game comes from Galeotti et al. (2008), but it stems back to the non–network application of Hirshleifer (1983)

look like the equilibria of the discrete best shot game.

The main difference between our setup and the one of Bramoullé and Kranton (2007) is that in the best shot games that we study actions are discrete, errors, even if rare, are therefore more dramatic and the concept of stochastic stability naturally arises. We think that this stark model offers a valid intuition of why typical problems of congestion are much more frequently observed in some coordination problems with local externalities. Most of these problems deal with discrete choice. Traffic is an intuitive and appealing example,[2] while others are given in the introduction of Dall'Asta et al. (2009a). In such complex situations we analyze those equilibria which are more likely to be the outcome of convergence, under the effect of local positive externalities and the possibility of errors.

In next section we formalize the best shot game. Section 3 describes the general best response dynamics that we apply to the game. In Section 4 we introduce the possibility of errors through perturbed dynamics. The main theoretical analysis on the effects of different assumptions is there. In Section 5 we enrich our research with numerical simulations that allow for quantitative analysis and to a comparison between different network structures.

2 Best-Shot Game

We consider a finite set of agents I of cardinality n. Players are linked together in a fixed exogenous network which is undirected and irreflexive; this network defines the range of a local externality described below. We represent such network through a $n \times n$ symmetric matrix G, where $G_{ij} = 1$ means that agents i and j are linked together (they are called *neighbors*), while $G_{ij} = 0$ means that they are not. We indicate with N_i the set of i's neighbors (the number of neighbors of a node is called its *degree* and is also its number of links).

Each player can take one of two actions, $x_i = \{0, 1\}$ with x_i denoting i's action. This action is interpreted as contribution, and an agent i such that $x_i = 1$ is called *contributor*. Similarly, action 0 is interpreted as defection, and an agent i such that $x_i = 0$ is called *defector*.[3] We will consider only pure strategies. A state of the system is represented by a

[2] Economic modelling of traffic have shown that simple assumptions can easily lead to congestion, even when agents are rational and utility maximizers (see Arnott and Small, 1994). Moreover, if we consider the discretization of the choice space, the motivation for the Logit model of McFadden (1973) were actually the transport choices of commuting workers.

[3] As will be clear below, we are dealing with a *local* public good game, so probably *free rider* would be a more suitable term than *defector*. Nevertheless, in the public goods game also "defector" is often used.

vector $\mathbf{x}$ which specifies each agent's action, $\mathbf{x} = (x_1, \ldots, x_i, \ldots, x_n)$. The set of all states is denoted with X.

Payoffs are not explicitly specified. We limit ourselves to the class of payoffs that generate the same type of best reply functions.[4] In particular, if we denote with b_i agent i's best reply function that maps a state of the system into a payoff value, then:

$$b_i(\mathbf{x}) = \begin{cases} 1 & \text{if } x_j = 0 \text{ for all } j \in N_i \ , \\ 0 & \text{otherwise.} \end{cases} \tag{P1}$$

We introduce some further notation in order to simplify the following exposition. We define the set of *satisfied agents* at state $\mathbf{x}$ as $S(\mathbf{x}) = \{i \in I : \ x_i = b_i(\mathbf{x})\}$. Similarly, the set of *unsatisfied agents* at state $\mathbf{x}$ is $U(\mathbf{x}) = I \setminus S(\mathbf{x})$. We also refer to the set of contributors as $C(\mathbf{x}) = \{i \in I : \ x_i = 1\}$, and to the set of defectors as $D(\mathbf{x}) = \{i \in I : \ x_i = 0\}$. We also define intersections of the above sets: the set of *satisfied contributors* is $S^C(\mathbf{x}) = S(\mathbf{x}) \cap C(\mathbf{x})$, the set of *unsatisfied contributors* is $U^C(\mathbf{x}) = U(\mathbf{x}) \cap C(\mathbf{x})$, the set of *satisfied defectors* is $S^D(\mathbf{x}) = S(\mathbf{x}) \cap D(\mathbf{x})$, and the set of *unsatisfied defectors* is $U^D(\mathbf{x}) = U(\mathbf{x}) \cap D(\mathbf{x})$. Finally, given any pair of states $(\mathbf{x}, \mathbf{x}')$ we indicate with $K(\mathbf{x}, \mathbf{x}') = \{i \in I : \ x_i = x'_i\}$ the set of agents that keep the same action in both states, and we indicate with $M(\mathbf{x}, \mathbf{x}') = I \setminus K(\mathbf{x}, \mathbf{x}')$ the set of agents whose action is modified between the states.

The above game is called *best-shot game*. A state $\mathbf{x}$ is a Nash equilibrium of the best-shot game if and only if $S(\mathbf{x}) = I$ and consequently $U(\mathbf{x}) = \emptyset$. We will call all the possible Nash equilibria, given a particular network, as $\mathcal{N} \subseteq X$.

The set $\mathcal{N}$ is always non–empty but typically very large. It is an NP–hard problem to enumerate all the elements of $\mathcal{N}$, and to identify, among them, those that maximize and minimize the set $C(\mathbf{x})$ of contributors. For extensive discussions on this points see Dall'Asta et al. (2009b) and Dall'Asta et al. (2009a). Here we provide two examples, the second one illustrates how even very homogeneous networks may display a large variability of contributors in different equilibria.

Example 2. Figure 2 shows two of the three possible Nash equilibria of the same 5–nodes network, where the five characters of our introductory example have now a different network of friendships. □

[4] Note that it would be very easy to define specific payoffs that generate the best reply defined by equation (P1): imagine that the cost for contributing is c and the value of a contribution, either from an agent herself and/or from one of her neighbors (players are satiated by one unit of contribution in the neighborhood), is $V > c > 0$.

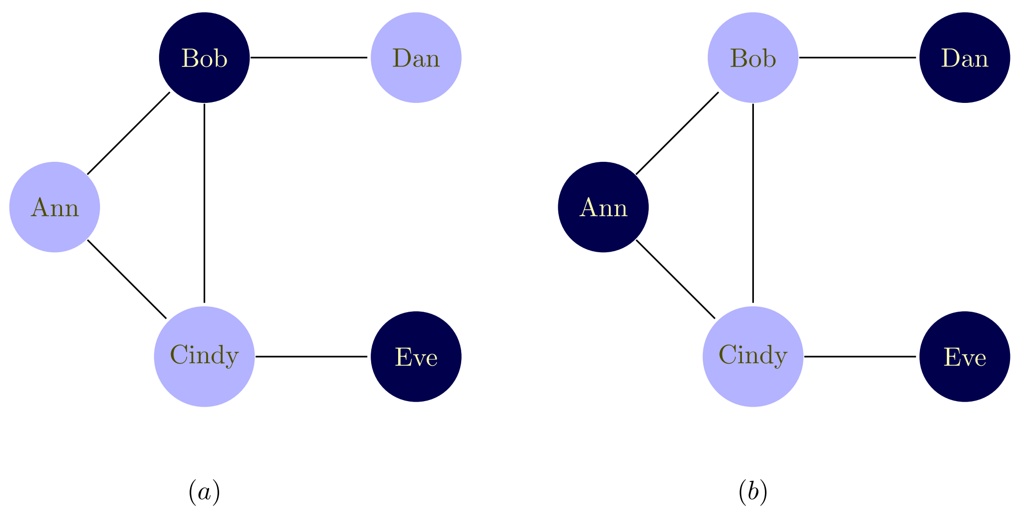


Figure 2: Two Nash equilibria for a 5–nodes network. The dark blue stands for contribution, while the light blue stands for defection.

EXAMPLE 3. Consider the particular regular random network, of 20 nodes and degree 4, that is shown in Figure 3. The relatively small size of this network allows us to count all its Nash equilibria. There exist 132 equilibria: 1 with 4 contributors (Figure 3, left), 17 with 5 contributors, 81 with 6 contributors, 32 with 7 contributors, 1 with 8 contributors (Figure 3, right). □

3 Unperturbed Dynamics

We imagine a dynamic process in which the network G is kept fixed, while the actions $\mathbf{x}$ of the nodes change.

At each time, which is assumed discrete and denoted with t, a play of the best-shot game occurs. The state of the system at time $t+1$ is supposed to depend, possibly in a probabilistic way, only on the state of the system at time t. We can therefore define a Markov chain (X, T), where X is the finite state space and T is the transition matrix, where $T_{\mathbf{x}\mathbf{x}'}$ denotes the probability to pass from state $\mathbf{x}$ to state $\mathbf{x}'$. We assume that T satisfies the following property, which formalizes the idea that all and only the unsatisfied agents can have the possibility to change action:

$$T_{\mathbf{x}\mathbf{x}'} > 0 \text{ if and only if } (i \in M(\mathbf{x}, \mathbf{x}') \Rightarrow i \in U(\mathbf{x})) \tag{P2}$$

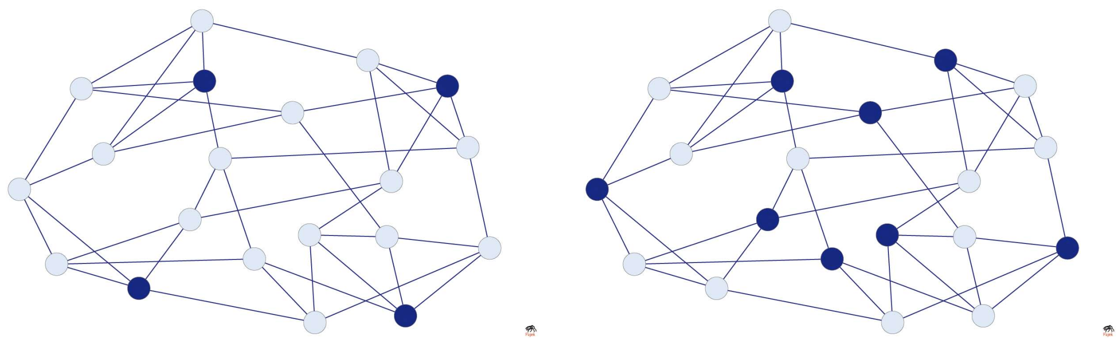


Figure 3: Two Nash equilibria for the same regular random network of 20 nodes and degree 4. The dark blue stands for contribution, while the light blue stands for defection. Picture is obtained by means of the software Pajek (http://pajek.imfm.si/).

We adopt most of the terminology and the notation from Young (1998). A state $\mathbf{x}'$ is called *accessible* from a state $\mathbf{x}$ if there exists a sequence of states, with $\mathbf{x}$ as first state and $\mathbf{x}'$ as last state, such that the system can move with positive probability from each state in the sequence to the next state in the sequence. A set $\mathcal{E}$ of states is called *ergodic set* (or *recurrent class*) when each state in $\mathcal{E}$ is accessible from any other state in $\mathcal{E}$, and no state out of $\mathcal{E}$ is accessible from any state in $\mathcal{E}$. If $\mathcal{E}$ is an ergodic set and $\mathbf{x} \in \mathcal{E}$, then $\mathbf{x}$ is called *recurrent*. Let $\mathcal{R}$ denote the set of all reccurrent states of (X, T). If $\{\mathbf{x}\}$ is an ergodic set, then $\mathbf{x}$ is called *absorbing*. Equivalently, $\mathbf{x}$ is absorbing when $T_{\mathbf{x}\mathbf{x}'} = 1$. Let $\mathcal{A}$ denote the set of all absorbing states of (X, T). Clearly, an absorbing state is recurrent, hence $\mathcal{A} \subseteq \mathcal{R}$. Note finally that the Markov chain defined in (P2) is aperiodic because, as $M(\mathbf{x}, \mathbf{x}) = \emptyset$ for all $\mathbf{x}$, then $T_{\mathbf{x}\mathbf{x}} > 0$ for all $\mathbf{x}$.

In the next two propositions we show that in our setup the set $\mathcal{N}$ of Nash equilibria is equivalent to all and only the absorbing states, and that there are no other recurrent states

Proposition 1. $\mathcal{A} = \mathcal{N}$.

Proof. We prove double inclusion, first we show that $\mathcal{N} \subseteq \mathcal{A}$.
Suppose $\mathbf{x} \in \mathcal{N}$. Since by (P2) we have that $T_{\mathbf{x}\mathbf{x}'} > 0$ with $\mathbf{x}' \neq \mathbf{x}$ only if $U(\mathbf{x}) \neq \emptyset$, then $T_{\mathbf{x}\mathbf{x}'} = 0$ for any $\mathbf{x}' \neq \mathbf{x}$, hence $T_{\mathbf{x}\mathbf{x}} = 1$ and $\mathbf{x}$ is absorbing.

Now we show that $\mathcal{A} \subseteq \mathcal{N}$.
By contradiction, suppose $\mathbf{x} \notin \mathcal{N}$. Then $U(\mathbf{x}) \neq \emptyset$. Consider a state $\mathbf{x}'$ where $x'_i = x_i$ if $i \in S(\mathbf{x})$, and $x'_i \neq x_i$ otherwise. We have that $\mathbf{x}' \neq \mathbf{x}$ and, by (P2), that $T_{\mathbf{x}\mathbf{x}'} > 0$, hence $T_{\mathbf{x}\mathbf{x}} < 1$ and $\mathbf{x}$ is not absorbing. □

PROPOSITION **2.** $\mathcal{A} = \mathcal{R}$.

Proof. The first inclusion $\mathcal{A} \subseteq \mathcal{R}$ follows from the definitions of $\mathcal{A}$ and $\mathcal{R}$.

Now we show that $\mathcal{R} \subseteq \mathcal{A}$.
We prove that every element $\mathbf{x}$ which is not in $\mathcal{A}$ is also not in $\mathcal{R}$. Suppose that $\mathbf{x} \notin \mathcal{A}$. We identify, by means of a recursive algorithm, a state $\hat{\mathbf{x}}$ such that $\hat{\mathbf{x}}$ is accessible from $\mathbf{x}$, but $\mathbf{x}$ is not accessible from $\hat{\mathbf{x}}$. This implies that $\mathbf{x} \notin \mathcal{R}$.

By proposition 1 we know that $\mathcal{A} = \mathcal{N}$. Then $\mathbf{x} \notin \mathcal{N}$ and we have that $U(\mathbf{x}) \neq \emptyset$. If $U^C(\mathbf{x}) \neq \emptyset$, we define $\mathbf{x}' \equiv \mathbf{x}$ and we go to Step 1, otherwise we jump to Step 2.

Step 1. We take $i \in U^C(\mathbf{x}')$ and we define state $\mathbf{x}''$ such that $x''_i \equiv 0 \neq x'_i = 1$ and $x''_j \equiv x'_j$ for all $j \neq i$.
Note that $||U^C(\mathbf{x}'')|| < ||U^C(\mathbf{x}')||$. This is because of two reasons: first of all, $i \in U^C(\mathbf{x}')$ and $i \notin U^C(\mathbf{x}'')$; the second is that $U^C(\mathbf{x}'') \subseteq U^C(\mathbf{x}')$, otherwise two neighbors contribute in $\mathbf{x}''$ and do not contribute in $U^C(\mathbf{x}')$, but that is not possible because $C(\mathbf{x}'') \subset C(\mathbf{x}')$. Moreover, by (P2) we have that $T_{\mathbf{x}'\mathbf{x}''} > 0$.
We redefine $\mathbf{x}' \equiv \mathbf{x}''$. Then, if $U^C(\mathbf{x}') = \emptyset$ we pass to Step 2, otherwise we repeat Step 1.

Step 2. We know that $U^C(\mathbf{x}') = \emptyset$. We take $i \in U^D(\hat{\mathbf{x}})$ and we define state $\mathbf{x}''$ such that $x''_i \equiv 1 \neq x'_i = 0$ and $x''_j \equiv x'_j$ for all $j \neq i$.
Note that $||U^D(\mathbf{x}'')|| < ||U^D(\mathbf{x}')||$. This is because of two reasons: first of all, $i \in U^D(\mathbf{x}')$ and $i \notin U^D(\mathbf{x}'')$; the second is that $U^D(\mathbf{x}'') \subseteq U^D(\mathbf{x}')$, otherwise two neighbors do not contribute in $\mathbf{x}''$ and do contribute in $U^C(\mathbf{x}')$, but that is not possible because $D(\mathbf{x}'') \subset D(\mathbf{x}')$.
We also note that still $U^C(\mathbf{x}'') = U^C(\mathbf{x}') = \emptyset$, since only i has become contributor and all i's neighbors are defectors.
By (P2) we have that $T_{\mathbf{x}'\mathbf{x}''} > 0$. Finally, if $U^D(\mathbf{x}'') \neq \emptyset$ we redefine $\mathbf{x}' \equiv \mathbf{x}''$ and repeat Step 2, otherwise it means that $\hat{\mathbf{x}} = \mathbf{x}''$ and we have reached the goal of the algorithm.

The sequence of states we have constructed shows that $\hat{\mathbf{x}}$ is accessible from $\mathbf{x}$.
Since $U(\mathbf{x}') = \emptyset$, we have that $T_{\hat{\mathbf{x}}\hat{\mathbf{x}}} = 1$ by (P2), and hence $\mathbf{x}$ is not accessible from $\mathbf{x}'$. $\square$

An immediate corollary of Propositions 1 and 2 is that $\mathcal{R} = \mathcal{A} = \mathcal{N}$.

We end this introductory section clarifying some points that could generate confusion as there is ambiguity between the terminologies in game theoretical network economics (surveyed in Jackson (2008)) and in the theory of stochastic stability (Young (1998)). The best shot game is a one–shot game defined on a fixed network. In the topology of this network we are interested only in characterizing neighborhoods. All the nodes of the network decide simultaneously between contributing or not and every strategy profile in pure strategies will de identified by a vector $\mathbf{x} \in X$.

Then we define a dynamical system in discrete time, based on the best shot game, as is done in the theory of stochastic stability. We call any element $\mathbf{x}$ as a state of the dynamical system, and the rule in (P2) defines a Markov chain between those states. This Markov chain can be represented as a directed network between the states, but should not be confused with the original undirected network on which the game is played.[5] We will use all the words with a dynamic implication (e.g. *accessible*, *path*. . .) only for the Markov chain.

The algorithm of the proof of Proposition 2 uses those transitions in T where only one unsatisfied player changes its action. However, it should be noted that (P2) allows with positive probability any change in which any subset of the unsatisfied nodes change.

Example 4. Consider the network from Figure 2: both states (a) and (b) shown there are absorbing, as they are Nash equilibria. Consider now the new state (c) on the same network, shown in Figure 4: the satisfied nodes here are only the defectors Ann, Dan and Eve. Both states (a) and (b) are accessible from state (c), but through different paths. To reach (a) from (c), the unsatisfied contributor Cindy should turn to defection, so that Eve would become (the only) unsatisfied and would be forced to become a contributor. To reach (b) from (c), both the unsatisfied contributors Bob and Cindy should simultaneously turn to defection,[6] then all five nodes would be unsatisfied. If we now turn to contribution exactly Ann, Dan and Eve, we reach state (b). □

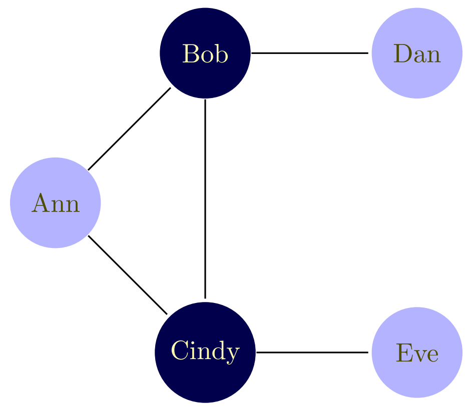
The diagram in Figure 5 sketches how the T dynamics works. The four nodes in the figure represent the same agent, with the letters within each node indicating whether such agent is a contributor (C) or a defector (D), and if she is satisfied (S) or unsatisfied (U). Arrows represent all and only the possible changes in terms of cooperation/defection and satisfaction/unstatisfaction that can occur to an agent when the T dynamics is applied.

4 Perturbed Dynamics

Given the multiplicity of Nash equilibria, we are uncertain about the final outcome of (X, T), that depends in part on the initial state and in part on the realizations of the probabilistic passage from states to states. In order to obtain a more sharp prediction, which is independent of the initial state, we introduce a small amount of perturbations and we use the techniques developed in economics by Foster and Young (1990), Young (1993), Kandori et al.

[5] Such a network of all possible configurations, in which directed links are driven by best reply functions, is in the same spirit of meta–networks in Jackson and Watts (2002).

[6] This is not in contradiction with the algorithm defined in the proof of Proposition 2, because there the starting state $\mathbf{x}$ is supposed to be a Nash equilibrium, while (c) is not.



(c)

Figure 4: A non-Nash (non-absorbing) state for the same network of Figure 2. Here Bob and Cindy are contributing, while Ann, Dan and Eve are not.

(1993). Since the way in which perturbations are modeled have in general important consequences on the outcome of the perturbed dynamics (see Bergin and Lipman, 1996), we consider three specific perturbation schemes, each of which has its own interpretation and may better fit a particular application.

We introduce perturbations by means of a *regular perturbed Markov chain* (Young, 1993), that is a triple $(X, T, (T^\epsilon)_{\epsilon\in(0,\bar{\epsilon})})$ where (X, T) is the unperturbed Markov chain and:

1. (X, T^ϵ) is an ergodic Markov chain, for all $\epsilon \in (0, \bar{\epsilon})$;

2. $\lim_{\epsilon\to 0} T^\epsilon = T$;

3. there exists a *resistance* function $r : X \times X \to \mathbb{R}^+ \cup \{\infty\}$ such that for all pairs of states $\mathbf{x}, \mathbf{x}' \in X$,

$$\begin{cases} \lim_{\epsilon\to 0} \dfrac{T^\epsilon_{\mathbf{x}\mathbf{x}'}}{\epsilon^{r(\mathbf{x},\mathbf{x}')}} \text{ exists and is strictly positive} & \text{if } r(\mathbf{x},\mathbf{x}') < \infty \;; \\ T^\epsilon_{\mathbf{x}\mathbf{x}'} = 0 \text{ for sufficiently small } \epsilon & \text{if } r(\mathbf{x},\mathbf{x}') = \infty \;. \end{cases}$$

The resistance $r(\mathbf{x}, \mathbf{x}')$ is meant to measure the amount of perturbations required to move the system from $\mathbf{x}$ directly to $\mathbf{x}'$. It also defines a directed network between the states in X, with weights that depends, through (iii), on those of the Markov chain. If $r(\mathbf{x}, \mathbf{x}') = 0$, then the system can move from state $\mathbf{x}$ directly to state $\mathbf{x}'$ in the unperturbed dynamics, that is

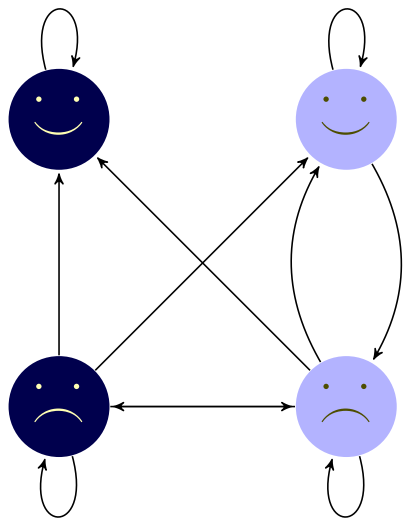


Figure 5: Single node dynamics in T. Dark blue stands for contribution, light blue stands for defection, $\because\!\smile$ stands for being satisfied, $\because\!\frown$ stands for being unsatisfied.

$T_{\mathbf{x}\mathbf{x}'} > 0$. If $r(\mathbf{x}, \mathbf{x}') = \infty$, then the system cannot move from $\mathbf{x}$ directly to $\mathbf{x}'$ even in the presence of perturbations, that is $T^{\epsilon}_{\mathbf{x}\mathbf{x}'} = 0$ for ϵ sufficiently small.

Even if T and r are defined on all the possible states of X, we can limit our analysis to the absorbing states only, which are all and only the recurrent ones (Proposition 2). This technical procedure is illustrated in Young (1998) and simplifies the complexity of the notation, without loss of generality. Given $\mathbf{x}, \mathbf{x}' \in \mathcal{A}$, we define $r^*(\mathbf{x}, \mathbf{x}')$ as the minimum sum of the resistances between absorbing states over any path starting in $\mathbf{x}$ and ending in $\mathbf{x}'$.

Given $\mathbf{x} \in \mathcal{A}$, an $\mathbf{x}$-tree on $\mathcal{A}$ is a subset of $\mathcal{A} \times \mathcal{A}$ that constitutes a tree rooted at $\mathbf{x}$.[7] We denote such $\mathbf{x}$-tree with $F_{\mathbf{x}}$ and the set of all $\mathbf{x}$-trees with $\mathcal{F}_{\mathbf{x}}$. The *resistance of an* $\mathbf{x}$*-tree*, denoted with $r^*(F_{\mathbf{x}})$, is defined to be the sum of the resistances of its edges, that is:

$$r^*(F_{\mathbf{x}}) \equiv \sum_{\mathbf{x}\mathbf{x}' \in F_{\mathbf{x}}} r^*(\mathbf{x}, \mathbf{x}') \ .$$

Finally, the *stochastic potential* of $\mathbf{x}$ is defined to be

$$\rho(\mathbf{x}) \equiv \min\{r^*(F_{\mathbf{x}}) : \ F_{\mathbf{x}} \in \mathcal{F}_{\mathbf{x}}\} \ .$$

[7] By *tree* we will refer only to this structure between absorbing states, and in no way to the topology of the underlying exogenous undirected network on which the best shot game is played.

A state $\mathbf{x}$ is said *stochastically stable* (Foster and Young, 1990) if $\rho(\mathbf{x}) = \min\{\rho(\mathbf{x}) : \mathbf{x} \in \mathcal{A}\}$. Intuitively, stochastically stable states are those and only those states that the system can occupy after very long time has elapsed in the presence of very small perturbations.[8]

We consider three types of perturbations in Sections 4.1, 4.2, and 4.3. Since what matters for determining stochastically stable states is the resistance function, instead of working with a specific regular perturbed Markov chain we prefer to focus on properties of the resistance function. So doing we refer to the class of regular perturbed Markov chains whose resistance function satisfies the required properties. By means of examples, on which we base also the simulations commented in Section 5, we show that such classes are non-empty.

4.1 Perturbations affect all agents

When perturbations affect all agents it is intuitively always possible to move from a state directly to any other state. The resistance $r(\mathbf{x}, \mathbf{x}')$ is required to be equal to the number of agents playing an action in $\mathbf{x}'$ that is (i) different from the action in $\mathbf{x}$ and (ii) not obtainable as best reply response to $\mathbf{x}$. We can easily summarize this in an algebraic formula: for all $\mathbf{x} \in X$,

$$r(\mathbf{x}, \mathbf{x}') \equiv ||S(\mathbf{x}) \cap M(\mathbf{x}, \mathbf{x}')|| = \sum_{i \in I} (x'_i - x_i)(x'_i - b_i(\mathbf{x})) \tag{P3}$$

In other words, $r(\mathbf{x}, \mathbf{x}')$ is the number of agents that are required to change action making a mistake in order to pass from $\mathbf{x}$ to $\mathbf{x}'$. Note that there exists a unique resistance function that satisifes (P3), which can then be considered as a definition.

The diagram in Figure 6 helps understanding how the T^ϵ dynamics works when (P3) holds. This diagram is the analogous of that in figure 5 for the unperturbed dynamics.

Example 5. Consider the following family of perturbed transition matrices: for all $\epsilon \in (0, \bar{\epsilon})$,

$$T^\epsilon = \sum_{\mathbf{x}'' \in X} T_{\mathbf{x}\mathbf{x}''} \widehat{T}^\epsilon_{\mathbf{x}''\mathbf{x}'},$$

with

$$\widehat{T}^\epsilon_{\mathbf{x}''\mathbf{x}'} \equiv \epsilon^{d_{10}(\mathbf{x}'', \mathbf{x}')} (1 - \epsilon)^{n - d_{10}(\mathbf{x}'', \mathbf{x}')},$$
$$d_{10}(\mathbf{x}'', \mathbf{x}') \equiv ||M(\mathbf{x}'', \mathbf{x}')|| = \sum_{i \in I} (x'_i - x''_i)(x'_i - x''_i).$$

Here the unperturbed transition matrix is first applied, and then each agent independently switches action with probability ϵ. It is easy to check that such $(X, T, (T^\epsilon)_{\epsilon \in (0, \bar{\epsilon})})$ is indeed a regular perturbed Markov chain.

[8] For a formal statement see Young (1993).

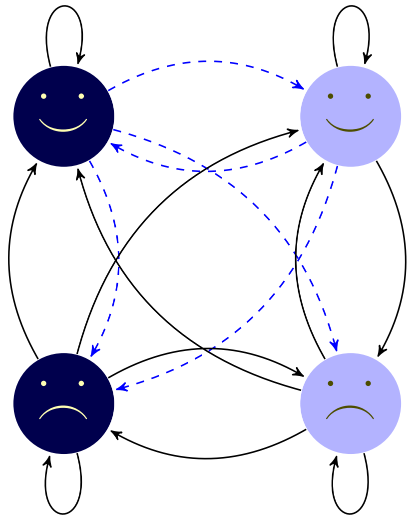


Figure 6: Single node dynamics in T^ϵ under (P3). Dark blue: contribution, light blue: defection, ◡: being satisfied, ⌢: being unsatisfied. Blue dashed lines stand for changes that are possible only by perturbation.

1. (X, T^ϵ) is ergodic for all positive ϵ; this can be seen applying the last sufficient condition for ergodicity in (Fudenberg and Levine, 1998, appendix in chap. 5), once we take into account that i) $\mathcal{A} = \mathcal{R}$ by Proposition 2, and ii) $r^*(\mathbf{x}\mathbf{x}') < \infty$ for all $\mathbf{x}, \mathbf{x}' \in \mathcal{A}$ by the following Lemma 1.

2. $\lim_{\epsilon\to 0} T^\epsilon = T$, since $\lim_{\epsilon\to 0} \widehat{T}^\epsilon$ is equal to the identity matrix.

3. Function r satisfying (P3) is a resistance function, since the agents in $U(\mathbf{x}) \cap M(\mathbf{x}, \mathbf{x}')$ can change when T is applied, while the agents in $S(\mathbf{x}) \cap M(\mathbf{x}, \mathbf{x}')$ can change only when $\widehat{T}^\epsilon$ is applied.

The last point above shows also that the resistance function of the regular perturbed Markov chain under consideration satisfies indeed (P3). □

The following simple remark states a lower bound for the resistance to move between Nash equilibria, and it is of help to establish which states have minimum stochastic potential.

Remark **1.** *When (P3) holds,* $r^*(\mathbf{x}, \mathbf{x}') \geq 1$ *for all* $\mathbf{x}, \mathbf{x}' \in \mathcal{N}$.

Next proposition provides a characterization of stochastically stable states under (P3).

Proposition 3. *When (P3) holds, a state* $\mathbf{x}$ *is stochastically stable in* $(X, T, (T^\epsilon)_{\epsilon\in[0,\bar{\epsilon})})$ *if and only if* $\mathbf{x} \in \mathcal{N}$.

Proof. It follows from Proposition 4 when we replace Remark 1 to Remark 2. □

4.2 Perturbations affect only the agents that are playing action 0

When perturbations affect only the agents that are playing action 0, some passages from a state to another one can be unfeasible, and the associated resistances will be infinite. We therefore set $r(\mathbf{x}, \mathbf{x}')$ whenever some satisfied contributor changes action from $\mathbf{x}$ to $\mathbf{x}'$. When instead no satisfied constributor changes action, then we set $r(\mathbf{x}, \mathbf{x}')$ equal to the number of satisfed defectors changing action from $\mathbf{x}$ to $\mathbf{x}'$. We can express this as follows: for all $\mathbf{x} \in X$,

$$r(\mathbf{x}, \mathbf{x}') = \begin{cases} ||S^C(\mathbf{x}) \cap M(\mathbf{x}, \mathbf{x}')|| & \text{if } ||S^D(\mathbf{x}) \cap M(\mathbf{x}, \mathbf{x}')|| = 0 \\ \infty & \text{otherwise} \end{cases} \tag{P4}$$

The diagram in Figure 7 shows how the T^ϵ dynamics works when (P4) holds. Again, blue dashed lines stand for changes that are possible only by perturbation. With respect to the diagram in Figure 6 we have two blue dashed lines missing, those going from a satisfied contributor to a satisfied defector and to an unsatisfied defector.

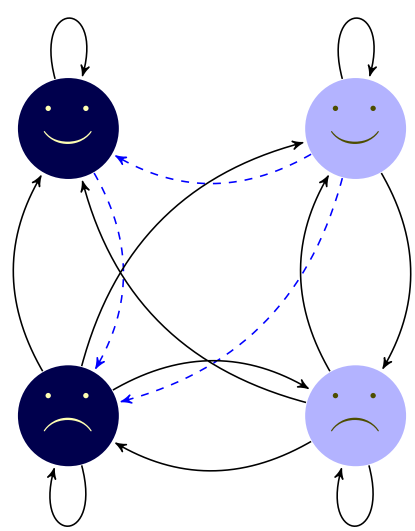


Figure 7: Single node dynamics in T^ϵ under (P4). Dark blue: contribution, light blue: defection, ◡: being satisfied, ⌒: being unsatisfied, blue dashed lines: changes that are possible only by perturbation.

Example 6. The following family of perturbed transition matrices is an example of a regular perturbed Markov chain that satisfies (P4): for all $\epsilon \in [0, \bar{\epsilon})$,

$$T^\epsilon = \sum_{\mathbf{x}'' \in X} T_{\mathbf{x}\mathbf{x}''} \widehat{T}^\epsilon_{\mathbf{x}''\mathbf{x}'},$$

with

$$\widehat{T}^\epsilon_{\mathbf{x}''\mathbf{x}'} \equiv \epsilon^{d_0(\mathbf{x}'',\mathbf{x}')}(1-\epsilon)^{n-||C(\mathbf{x}'')||-d_0(\mathbf{x}'',\mathbf{x}')},$$
$$d_0(\mathbf{x}'', \mathbf{x}') \equiv ||M(\mathbf{x}'', \mathbf{x}') \cap D(\mathbf{x}'')|| = \sum_{i \in I} (x'_i - x''_i)(x'_i - x''_i)(1 - x''_i).$$

After the unperturbed transition matrix is applied, each defector independently switches action with probability ϵ, while contributors keep on contributing. We check that the example is indeed a regular perturbed Markov chain whose resistance function satisfies (P4).

1. The same of point 1 in Example 1.

2. The same of point 2 in Example 1.

3. Function r satisfying (P4) is a resistance function, since i) $T^\epsilon_{\mathbf{x}\mathbf{x}'} = 0$ if $r(\mathbf{x}, \mathbf{x}') = \infty$ (in such case $S^C(\mathbf{x}) \cap M(\mathbf{x}, \mathbf{x}') \neq \emptyset$ and there is no way to go from $\mathbf{x}$ to $\mathbf{x}'$), and ii) when $r(\mathbf{x}, \mathbf{x}') < \infty$ (so $S^C(\mathbf{x}) \cap M(\mathbf{x}, \mathbf{x}') = \emptyset$) the agents in $U(\mathbf{x}) \cap M(\mathbf{x}, \mathbf{x}')$ can change when T is applied, while the agents in $S(\mathbf{x}) \cap M(\mathbf{x}, \mathbf{x}')$ can change only when $\widehat{T}^\epsilon$ is applied.

□

In the next remark we provide a lower bound for the resistance to move between Nash equilibria under this perturbation scheme, and we then use such remark in Proposition 4.

Remark **2.** *When (P4) holds,* $r^*(\mathbf{x}, \mathbf{x}') \geq 1$ *for all* $\mathbf{x}, \mathbf{x}' \in \mathcal{N}$.

The following lemma, which is of help in the proof of Proposition 4, shows that under this perturbation scheme any two absorbing states are connected through a sequence of absorbing states, with each step in sequence having resistance 1.

Lemma **1.** *When (P4) holds, for all* $\mathbf{x}, \mathbf{x}' \in \mathcal{A}$, $\mathbf{x} \neq \mathbf{x}'$, *there exists a sequence* $\mathbf{x}^\mathbf{0}, \ldots, \mathbf{x}^\mathbf{s}, \ldots, \mathbf{x}^\mathbf{k}$, *with* $\mathbf{x}^\mathbf{s} \in \mathcal{A}$ *for* $0 \leq s \leq k$, $\mathbf{x}^\mathbf{0} = \mathbf{x}$ *and* $\mathbf{x}^\mathbf{k} = \mathbf{x}'$, *such that* $r^*(\mathbf{x}^\mathbf{s}, \mathbf{x}^\mathbf{s+1}) = 1$ *for* $0 \leq s < k$.

Proof. We denote with k the cardinality of the set of contributors in $\mathbf{x}'$ that are defectors in $\mathbf{x}$, namely $k = ||C(\mathbf{x}') \cap D(\mathbf{x})||$. Since $vecx \neq \mathbf{x}'$, we have that $k \geq 1$. We set $\mathbf{x}^\mathbf{0} = \mathbf{x}$.
For $0 \leq s < k$:

Take $i \in C(\mathbf{x'}) \cap D(\mathbf{x^s})$. We define state $\tilde{\mathbf{x}}$ such that $\tilde{x}_i \equiv 1 \neq x_i^s = 0$ and $\tilde{x}_j \equiv x_j^i$ for all $j \neq i$. Note that $r(\mathbf{x^i}, \tilde{\mathbf{x}}) = 1$. We now define state $\mathbf{x^{i+1}}$ such that $x_j^{i+1} \equiv 0$ for all $j \in N_i$, and $x_j^{i+1} \equiv \tilde{x}_j$ for all $j \notin N_i$. Note that $r(\tilde{\mathbf{x}}, \mathbf{x^{i+1}}) = 0$. By Lemma 2 in Dall'Asta et al. (2009a) we have that $\mathbf{x^{i+1}} \in \mathcal{N}$, and hence $\mathbf{x^{i+1}} \in \mathcal{A}$ by Lemma 1. Therefore, $r^*(\mathbf{x^i}, \mathbf{x^{i+1}}) = 1$.

Note that, since $i \notin D(\mathbf{x^{i+1}})$ and $(j \in N_i \Rightarrow j \in D(\mathbf{x'}))$, then $||C(\mathbf{x'}) \cap D(\mathbf{x^{i+1}})|| = ||C(\mathbf{x'}) \cap D(\mathbf{x^i})|| - 1$. Therefore, $||C(\mathbf{x'}) \cap D(\mathbf{x^k})|| = 0$, and this means that $\mathbf{x^k} = \mathbf{x'}$. □

Next proposition provides a characterization of stochastically stable states under (P4).

PROPOSITION 4. *When (P4) holds, a state* $\mathbf{x}$ *is stochastically stable in* $(X, T, (T^\epsilon)_{\epsilon \in [0,\bar{\epsilon})})$ *if and only if* $\mathbf{x} \in \mathcal{N}$.

Proof. We first show that $\mathbf{x} \in \mathcal{N}$ implies $\mathbf{x}$ stochastically stable. Theorem 2 in Samuelson (1994) implies that if $\mathbf{x'}$ is stochastically stable, $\mathbf{x} \in \mathcal{A}$, $r^*(\mathbf{x'}, \mathbf{x})$ is equal to the minimum resistance between recurrent states, then $\mathbf{x}$ is stochastically stable. Since at least one recurrent state must be stochastically stable, Proposition 2 implies that there must exist an absorbing state $\mathbf{x'}$ that is stochastically stable. For any $\mathbf{x} \in \mathcal{N}$, if $\mathbf{x} = \mathbf{x'}$ we are doone. If $\mathbf{x} \neq \mathbf{x'}$, then we by Proposition 1 we can use Lemma 1 to say that there exists a sequence of absorbing states from $\mathbf{x'}$ to $\mathbf{x}$. Remark 2, together with Propositions 2 and 1, implies that 1 is the minimum resistance between recurrent states. A repeated application of Theorem 2 in Samuelson (1994) shows that each state in the sequence is stochastically stable, and in particular the final state $\mathbf{x}$.

It is trivial to show that $\mathbf{x}$ stochastically stable implies $\mathbf{x} \in \mathcal{N}$. By contradiction, suppose $\mathbf{x} \notin \mathcal{N}$. Then, by Propositions 2 and 1, $\mathbf{x} \notin \mathcal{R}$, and hence cannot be stochastically stable. □

4.3 Perturbations affect only the agents that are playing action 1

When perturbations affect only the agents that are playing action 1, again some resistances can be infinite. In particular, there will be infinite the resistances of passages that require some mistake from 0 to 1. The following definition makes the precise requirement: for all $\mathbf{x} \in X$,

$$r(\mathbf{x}, \mathbf{x'}) = \begin{cases} ||S^D(\mathbf{x}) \cap M(\mathbf{x}, \mathbf{x'})|| & \text{if } ||S^C(\mathbf{x}) \cap M(\mathbf{x}, \mathbf{x'})|| = 0 \\ \infty & \text{otherwise} \end{cases} \tag{P5}$$

The diagram in Figure 8 shows how the T^ϵ dynamics works when (P5) holds. With respect to the diagram in figure 6 we have three blue dashed lines missing, the two going from a satisfied defector to a satisfied contributor and to an unsatisfied contributor, and the one (the most surprising) going from a satisfied contributor to a satisfied defector.

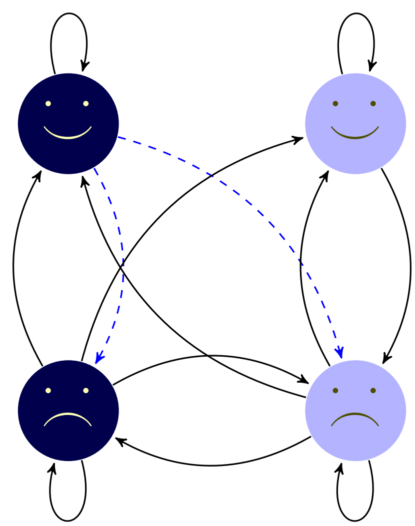


Figure 8: Single node dynamics in T^ϵ under (P5). Dark blue: contribution, light blue: defection, ◡̈: being satisfied, ⁔̈: being unsatisfied, blue dashed lines: changes that are possible only by perturbation.

Example 7. We provide the following family of perturbed transition matrices as an example of a regular pertubed Markov chain that satisfies (P5): for all $\epsilon \in (0, \bar{\epsilon})$,

$$T^\epsilon = \sum_{\mathbf{x}'' \in X} T_{\mathbf{x}\mathbf{x}''} \widehat{T}^\epsilon_{\mathbf{x}''\mathbf{x}'},$$

with

$$\begin{aligned} &\widehat{T}^\epsilon_{\mathbf{x}''\mathbf{x}'} \equiv \epsilon^{d_1(\mathbf{x}'',\mathbf{x}')}(1-\epsilon)^{n-||D(\mathbf{x}'')||-d_1(\mathbf{x}'',\mathbf{x}')}, \\ &d_1(\mathbf{x}'',\mathbf{x}') \equiv ||M(\mathbf{x}'',\mathbf{x}') \cap C(\mathbf{x}'')|| = \sum_{i \in I}(x'_i - x''_i)(x'_i - x''_i)x''_i. \end{aligned}$$

After the unperturbed transition matrix is applied, each contributor independently switches action with probability ϵ, while defector keep on defecting. We check that the provided example is indeed a regular perturbed Markov chain satisfying (P5).

1. The same of point 1 in Example 1, once we replace Lemma 2 to Lemma 1.

2. The same of point 2 in Example 1.

3. Function r satisfying (P5) is a resistance function, since i) $T^\epsilon_{\mathbf{x}\mathbf{x}'} = 0$ if $r(\mathbf{x}, \mathbf{x}') = \infty$ (in such case $S^D(\mathbf{x}) \cap M(\mathbf{x}, \mathbf{x}') \neq \emptyset$ and there is no way to go from $\mathbf{x}$ to $\mathbf{x}'$), and ii) when

$r(\mathbf{x}, \mathbf{x}') < \infty$ (so $S^D(\mathbf{x}) \cap M(\mathbf{x}, \mathbf{x}') = \emptyset$) the agents in $U(\mathbf{x}) \cap M(\mathbf{x}, \mathbf{x}')$ can change when T is applied, while the agents in $S(\mathbf{x}) \cap M(\mathbf{x}, \mathbf{x}')$ can change only when $\widehat{T}^{\epsilon}$ is applied.

□

This remark plays the same role of Remarks 1 and 2.

Remark **3.** *When (P4) holds,* $r^*(\mathbf{x}, \mathbf{x}') \geq 1$ *for all* $\mathbf{x}, \mathbf{x}' \in \mathcal{N}$.

The following Lemma shows that the resistance between any two absorbing states is equal to the number of contributors that that must change to defection. This result is less trivial than it might appear: it shows that there is no possibility that by changing only some of the contributors to defectors, the remaining ones are induced to change by the unperturbed dynamics.

Lemma **2.** *When (P5) holds, for all* $\mathbf{x}, \mathbf{x}' \in \mathcal{A}$, $r^*(\mathbf{x}, \mathbf{x}') = ||C(\mathbf{x}) \cap D(\mathbf{x}')||$.

Proof. We first show that $r^*(\mathbf{x}, \mathbf{x}') \geq ||C(\mathbf{x}) \cap D(\mathbf{x}')||$. By contradiction, suppose $r^*(\mathbf{x}, \mathbf{x}') < ||C(\mathbf{x}) \cap D(\mathbf{x}')||$. Then, some $i \in C(\mathbf{x}) \cap D(\mathbf{x}')$ must switch from contribution to defection along a path from $\mathbf{x}$ to $\mathbf{x}'$ by best reply to the previous state. This requires that in the previous state there must exist some $j \in N_i$ that contributes. However, $j \in D(\mathbf{x})$ and j can never change to contribution as long as i is a contributor, neither by best reply nor by perturbation when (P5) holds.

We now show that $r^*(\mathbf{x}, \mathbf{x}') \leq ||C(\mathbf{x}) \cap D(\mathbf{x}')||$. Define state $\tilde{\mathbf{x}}$ such that $\tilde{x}_i \equiv 0 \neq x_i = 1$ for all $i \in C(\mathbf{x}) \cap D(\mathbf{x}')$, and $\tilde{x}_i \equiv x_i$ otherwise. Note that $r(\mathbf{x}, \tilde{\mathbf{x}}) = ||C(\mathbf{x}) \cap D(\mathbf{x}')||$. Note also that $b_i(\tilde{\mathbf{x}}) = 1$ for all $i \in C(\mathbf{x}') \cap D(\tilde{\mathbf{x}})$. This means that $r(\tilde{\mathbf{x}}, \mathbf{x}') = 0$, and therefore $r^*(\mathbf{x}, \mathbf{x}') \leq r(\mathbf{x}, \tilde{\mathbf{x}}) + r(\tilde{\mathbf{x}}, \mathbf{x}') = ||C(\mathbf{x}) \cap D(\mathbf{x}')||$. □

We now use the above Lemma 2 to relate algebraically the resistance move away from $\mathbf{x}$ to $\mathbf{x}'$ to the resistance to come back $\mathbf{x}'$ to $\mathbf{x}$.

Lemma **3.** *When (P5) holds, for all* $\mathbf{x}, \mathbf{x}' \in \mathcal{A}$, $r^*(\mathbf{x}, \mathbf{x}') = r^*(\mathbf{x}', \mathbf{x}) + ||C(\mathbf{x})|| - ||C(\mathbf{x}')||$.

Proof. From Lemma 2 we know that $r^*(\mathbf{x}, \mathbf{x}') = ||C(\mathbf{x}) \cap D(\mathbf{x}')||$. Note that $||C(\mathbf{x}) \cap D(\mathbf{x}')|| = ||C(\mathbf{x})|| - ||C(\mathbf{x}) \cap C(\mathbf{x}')||$. Always from Lemma 2 we also know that $r^*(\mathbf{x}, \mathbf{x}') = ||C(\mathbf{x}') \cap D(\mathbf{x})|| = ||C(\mathbf{x}')|| - ||C(\mathbf{x}) \cap C(\mathbf{x}')||$, from which $||C(\mathbf{x}) \cap C(\mathbf{x}')|| = C(\mathbf{x}') - r^*(\mathbf{x}', \mathbf{x})$, which substituted in the former equality gives the desired result. □

Lemma 3 allows us to provide in next Proposition a characterization of stochastically stable states under (P5).

Proposition 5. *When (P5) holds, a state* $\mathbf{x}$ *is stochastically stable in* $(X, T, (T^\epsilon)_{\epsilon\in[0,\bar{\epsilon})})$ *if and only if* $\mathbf{x} \in \arg\max_{\mathbf{x'}\in\mathcal{N}} ||C(\mathbf{x'})||$.

Proof. We first prove that only a state in $\arg\max_{\mathbf{x'}\in\mathcal{N}} ||C(\mathbf{x'})||$ may be stochastically stable. Ad absurdum, suppose $||C(\mathbf{x})|| \notin \arg\max_{\mathbf{x'}\in\mathcal{N}} ||C(\mathbf{x'})||$ and $\mathbf{x}$ is stochastically stable. There must exist $\mathbf{x'}$ such that $||C(\mathbf{x'})|| > ||C(\mathbf{x})||$. Take an $\mathbf{x}$-tree $F_\mathbf{x}$. Consider the path in $F_\mathbf{x}$ going from $\mathbf{x'}$ to $\mathbf{x}$, that is the unique $\{(\mathbf{x^0}, \mathbf{x^1}), \ldots, (\mathbf{x^{k-1}}, \mathbf{x^k})\}$ such that $\mathbf{x^0} = \mathbf{x'}$, $\mathbf{x^k} = \mathbf{x}$, and $(\mathbf{x^i}, \mathbf{x^{i+1}}) \in F_\mathbf{x}$ for all $i \in \{0, k-1\}$. We now modify $F_\mathbf{x}$ by reverting the path from $\mathbf{x'}$ to $\mathbf{x}$, so we define $F_{\mathbf{x'}} = (F_\mathbf{x} \setminus \{(\mathbf{x^i}, \mathbf{x^{i+1}}) : \ i \in \{0, k-1\}\}) \cup \{(\mathbf{x^{i+1}}, \mathbf{x^i}) : \ i \in \{0, k-1\}\}$, which is indeed an $\mathbf{x'}$-tree. It is straightforward that $r^*(F_{\mathbf{x'}}) = r^*(F_\mathbf{x}) - \sum_{i=0}^{k-1} r^*(\mathbf{x^i}, \mathbf{x^{i+1}}) + \sum_{i=0}^{k-1} r^*(\mathbf{x^{i+1}}, \mathbf{x^i})$. Applying Lemma 3 we obtain $r^*(F_{\mathbf{x'}}) = r^*(F_\mathbf{x}) + \sum_{i=0}^{k-1}(||C(\mathbf{x^{i+1}})|| - ||C(\mathbf{x^i})||)$, that simplifies to $r^*(F_{\mathbf{x'}}) = r^*(F_\mathbf{x}) + ||C(\mathbf{x})|| - ||C(\mathbf{x'})||)$. Since $||C(\mathbf{x'})|| > ||C(\mathbf{x})||$, then $r^* F_{\mathbf{x'}} < r^* F_\mathbf{x}$. In terms of stochastic potentials, this implies that $\rho(\mathbf{x'}) < \rho(\mathbf{x})$, against the hypothesis that $\mathbf{x}$ is stochastically stable.

We now prove that any state in $\arg\max_{\mathbf{x'}\in\mathcal{N}} ||C(\mathbf{x'})||$ is stochastically stable. Since at least one stochastically stable state must exist, from the above argument we conclude that there exists $\mathbf{x} \in \arg\max_{\mathbf{x'}\in\mathcal{N}} ||C(\mathbf{x'})||$ that is stochastically stable. Take any other $\mathbf{x'} \in \arg\max_{\mathbf{x'}\in\mathcal{N}} ||C(\mathbf{x'})||$. Following exactly the same reasoning as above we obtain that $\rho(\mathbf{x'}) = \rho(\mathbf{x})$. Since $\rho(\mathbf{x})$ is minimum, $\rho(\mathbf{x})$ is minimum too, and $\mathbf{x'}$ is hence stochastically stable. □

5 Simulation Results

In the previous section we have shown that under (P5)-type of perturbations, at the limit of vanishing perturbations, the regular perturbed Markov chain tends to Nash equilibria with maximum number of contributors, while under (P4)-type and (P3)-type of perturbations, at the same limit, every Nash equibrium is visited with positive probability in the long run. These results refer to the time-asymptotyc behavior when the amount of perturbations appproaches zero. We are here concerned to understand what happens in terms of average contribution in the presence of small but finite amount of perturbations, after very long time has elapsed for time–asymptotic behavior to be relevant. This kind of investigation is difficult to be done by mathematical deduction for general classes of networks, and therefore we rely on computer–based simulations.

We start, as a benchmark case, from a completely controlled environment: the regular random network shown in Example 3. For this network we know exactly that the maximum Nash equilibrium has a percentage of 40% of contributors (8 out of 20), but the percentage of contributors in Nash equilibria ranges down to 20%. Figure 9 shows the average number of contributors for the regular perturbed Markov chains from Examples 6 (the (P4)-type of perturbations) and 7 (the (P5)-type of perturbations). We have considered the average percentage of contributors over a large span of 10^6 time steps, for different values of the ϵ i.i.d. probability of a single node making an error, as illustrated in Examples 6 and 7. This has been iterated for 10 different time spans for each value of ϵ on a discrete grid of small values, for the two regular perturbed Markov chains.[9] By so doing we perform a comparative static between steady states, which is analogous as approaching first the limit on the time dimension, and then the limit on the probability ϵ of an error (see Kulkarni (1999) for the importance of the order in which limits are taken).

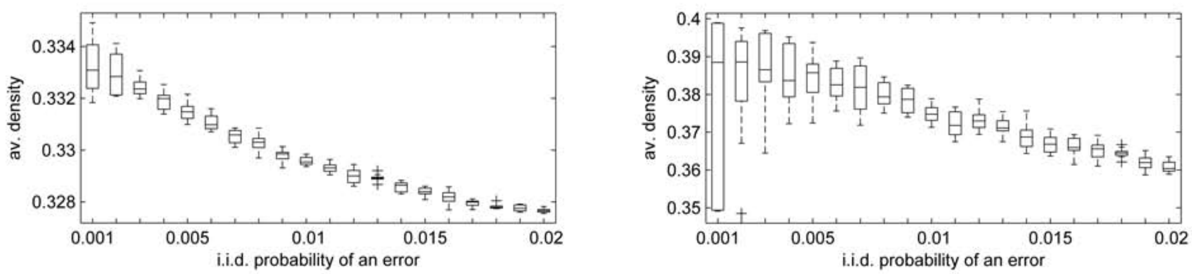


Figure 9: Box plots over 10 run of stochastic processes over the same regular random network of Figure 3. In both panels x–axis is the i.i.d. probability of an error and y–axis is the percentage of contributors averaged over 10^6 time steps (box plots over 10 different time–spans). Left panel: (P5)–type stochastic process. Right panel: (P4)–type stochastic process.

It is clear from a comparison of the two panels in Figure 9 that the average contribution in the (P4)-type of perturbations lies always below the (P5)-type. This is important since it allows to extend the conclusion that we obtain comparing either Proposition 4 or Proposition 3 and Proposition 5 to non-vanishing amounts of perturbations: (P5)-type of perturbations favors contribution. Moreover, we note that the slope of the relation between average contribution and the amount of perturbations is higher for (P5)-type of perturbations. This is likely to be related to the fact that in such a case there is the tendency for the system to move to equilibria with maximum number of contributors as the amount of perturbations approaches zero, while such tendency is absent for the other two types of perturbations (Propositions 3, 4 and 5).

[9] We have checked for this set of simulations and for the following ones that the (P3)-type of perturbations lies in–between.

In the case of the (P5)-type, moreover, the limit seems to converge linearly to the threshold value that we know to be the maximum possible in a Nash equilibrium, as is shown for the limit by Proposition 5. The higher variance is due to the fact that, as ϵ converges to 0, perturbations on the network are much more rare on the same time–span, and the regular perturbed Markov chain changes sites less frequently. Finally, a remark concerns the downward slope of the relation between the amount ϵ of perturbations and the average contribution, also for the (P4)-type of perturbations in the left panel. Since only agents that are playing action 0 can make a mistake, one would first think that the relation should be upward sloping. However, as a consequence of an agent's switch from 0 to 1 we have that other agents can switch their action, possibly leading to an equilibrium with an overall lower number of contributors. That the latter occurrence is indeed what is more likely to happen is a matter of fact for which we do not have a convincing argument.

The nice monotonicity of previous simulations could be the consequence of a very particular network. We compare then two well–known classes of random networks: regular random networks and scale–free networks.[10] For the two types of networks, and for the (P4)-type and (P5)-type of stochastic processes, we run 20 simulations, one for each different realization of the two classes of random networks.[11] The size of networks is always 50 nodes, that is a trade-off between computational tractability and variability.

Figure 10 reports the results for the regular perturbed Markov chain from Example 6 (the (P4)-type of perturbations). Figure 11 reports instead the results for the regular perturbed Markov chain from Example 7 (the (P5)-type of perturbations). In the graphs on the top of the figure we used networks that are realizations of a regular random network (with a time span of 10^6 time–steps), and in the graphs on the bottom of the figure we repeated the simulations using realizations of a scale-free network (for which a smaller time span of 10^5 time–steps was sufficient to get a reasonable stationarity). Here we reported also, in the left panels, another average quantity on the time–spans, which is the percentage unsatisfied agents across the regular perturbed Markov chain, that visits also non–equilibrium states.

[10] Newman (2003) and Jackson and Rogers (2007) show that regular random networks and scale–free networks miss some important statistical properties of real–world networks, such as *clustering*, as the number of nodes becomes large. For relatively small networks as the ones we are considering, this is not true and the two types of networks may be thought of as representatives of homogeneous and heterogeneous degree distributions.

[11] For regular random networks we adopted the procedure from McKay and Wormald (1990), while for scale-free networks we followed the preferential attachment model by Albert and Barabasi (1999). We did not use the same realization for the underlying network in order not to have results depending on a fortuitous realization.

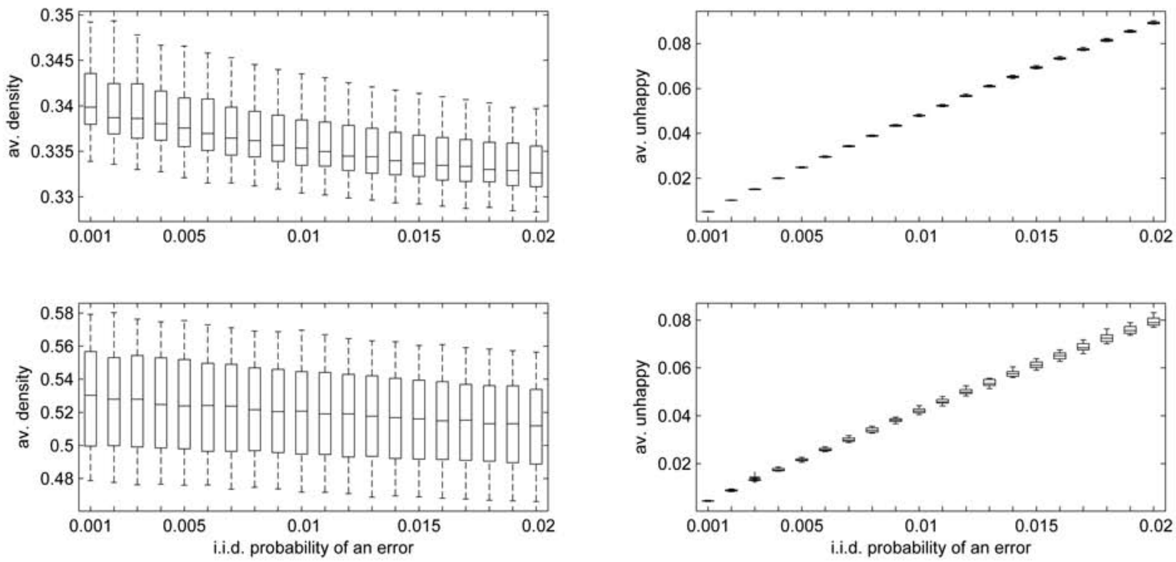


Figure 10: Average quantities over large time spans of the (P4)–type stochastic process. In all four panels x–axis is the i.i.d. probability of an error. Top–left panel: y–axis is the percentage of contributors averaged over 10^6 time steps (box plots over 20 different regular random networks of 50 nodes and degree 4). Top–right panel: y–axis is the percentage of unsatisfied nodes averaged over 10^6 time steps (box plots over the same sample). Lower panels differ from the above as the quantities are averaged over only 10^5 time steps, and box plots are over 20 different scale–free networks of 50 nodes and average degree 4.

Let us describe and compare Figures 10 and 11. We note that the downward and linear slope of the relation between the amount of perturbations and the average contribution is still present, as discussed for Figure 9. We do not know if the left panels of Figure 11 are actually approaching the exact maximum number of contributors in a Nash equilibrium, as we can claim for sure for the left panel of Figure 9. However, the right–top panel of Figure 11 approaches well the analytical predictions of Dall'Asta et al. (2009b).

In both figures the average contribution and its variability are always higher for scale-free networks. This is essentially due to the more asymmetric structure that scale-free networks have compared to regular random networks. In the former there are equilibria where few highly connected agents contribute, and others where many scarcely connected agents contribute. In the latter, instead, all equilibria are less diverse since they involve agents that all have the same number of connections.

We make some further remarks concerning the relation between the proportion of unsat-

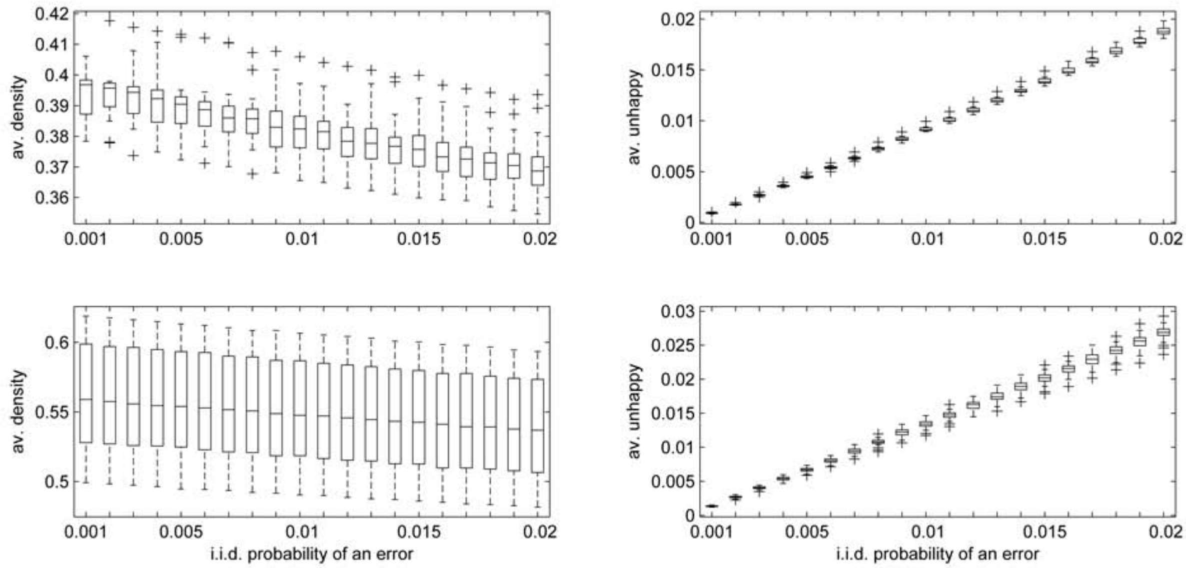


Figure 11: Average quantities over large time spans of the (P5)–type stochastic process. All four panels have respectively the same legend as in Figure 10.

isfied agents and the amount of perturbations. Its upward slope is rather trivial: starting from equilibria, where by definition no agent is unsatisfied, an increase in the amount of perturbations clearly raises the proportion of unsatisfied agents (those who made the mistake and possibly some of their neighbors). What is more interesting is the different slope for the two types of perturbations. Part of the explanation is related to the different proportions of agents that are susceptible of making mistakes. This leads, for instance, to conclude (as we checked) that with (P3)-type of perturbations the slope is the highest. Another reason that probably works in favor of a lower slope for (P5)-type of perturbations with respect to (P4)-type of perturbations is that a mistake of the former type leads to changes that are limited to the neighborhood of order 1 of the original node, while a mistake of the latter type leads to changes that are limited to the neighborhood of order 2 of the original node (see Lemma 2 in Dall'Asta et al., 2009a). Therefore we may roughly say that (P4)-type of perturbations has a larger impact than (P5)-type of perturbations, and that is probably responsible for part of the difference in slope.

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