Compensations in the Shapley value and the compensation solutions for graph games*

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Abstract

We consider an alternative expression of the Shapley value that reveals a system of compensations: each player receives an equal share of the worth of each coalition he belongs to, and has to compensate an equal share of the worth of any coalition he does not belong to. We give an interpretation in terms of formation of the grand coalition according to an ordering of the players and define the corresponding compensation vector. Then, we generalize this idea to cooperative games with a communication graph. Firstly, we consider forest graphs and extend the compensation vector by considering all rooted spanning trees of the graph (Demange [3]) instead of orderings of the players. The associated allocation rule, called the compensation solution, is characterized by component efficiency and relative fairness. The latter axiom takes into account the relative position of a player with respect to his component. Secondly, we consider arbitrary graphs and construct rooted spanning trees by using the classical algorithms **DFS** and **BFS**. If the graph is complete, we show that the compensation solutions associated with **DFS** and **BFS** coincide with the Shapley value and the equal surplus division respectively.

Keywords: Shapley value, compensations, relative fairness, compensation solution, **DFS**, **BFS**, equal surplus division solution.
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1 Introduction

The Shapley value (Shapley [15]) is the most studied allocation rule for cooperative games with transferable utility (TU-games henceforth). One way to interpret the Shapley value consists in considering orderings of players. For each such ordering, the players enter a bargaining room one by one, and upon entering each player is paid his marginal contribution. This procedure yields a payoff vector, which is called the marginal vector, and the Shapley value is the average over all orderings of the players of the marginal vector.

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Another less-known interpretation of the Shapley value is due to Eisenman [4].[1] Eisenman envisages a situation in which the grand coalition is split in a "two-alliances", *i.e* two blocks bargaining against each other. This configuration is similar to the original idea with which von Neumann and Morgenstern [13] consider an n-person game as a game between a coalition and its complement. In the two-alliances, each player is paid an equal share of the worth of the coalition he belongs to and must contribute an equal share for the other coalition. The total of these payoffs is then zero. Eisenman considers the step-by-step model in which the first two-alliances is selected at random and then, inductively, one player of the smallest coalition is randomly chosen to grow the largest coalition until the grand coalition is formed. The resulting expected payoff vector is the Shapley value.

In this article we introduce another interpretation of the Shapley value, which is similar in spirit to the principle of compensation formulated by Eisenman [4] but preserves Shapley's idea of a one-by-one formation of the grand coalition. More specifically, consider any ordering of the players. The first player enters and can claim a payoff equal to the worth he produces, otherwise he can refuse to keep on forming the grand coalition. In order to prevent this failure, the remaining players should pay him this worth, and each of the remaining players should be charge an equal share of this compensation. Then, the second player enters and forms a coalition with the first player. As before this coalition can ask for a compensation equal to its worth in return for continuing to form the grand coalition. Each of the remaining players should pay an equal share of the compensation and the two entered players should split the compensation in two identical parts. Now repeat these stages until the last player enters the room. At this point the grand coalition eventually forms and each player gets an equal share of its worth. In all but the last stage, the total flow of payoffs is null: the players use a compensation scheme. The whole procedure induces a payoff vector which we call the compensation vector. In a first result we show that the Shapley value is the average of the compensation vectors associated with all orderings of the players.

The second and main part of this article consists in extending the principle of the compensation vector from TU-games to cooperative games with a communication graph, the so-called graph games. For the class of graph games, a crucial point is to study how the communication constraints influence the allocation rules. There are at least two ways to measure this impact. In a first approach, the communication constraints determine how a coalition is *evaluated*. There is no restriction on the formation of coalitions, but if a coalition is not connected through the communication graph, its worth is the sum of the worths of its connected parts. This approach is investigated by Myerson [12] who introduces the Myerson value in order to generalize the Shapley value from TU-games to graph games. The Myerson value is the Shapley value of a graph-restricted TU-game: all orderings of the players are considered, but the worth of unconnected coalitions is evaluated according to their connected components. In a second approach, the communication constraints determine how the coalitions are to be *formed*. Orderings of the players that induces unconnected coalitions are ruled out: the formation of the grand coalition requires a communication at any stage. In order to satisfy the communication constraints, Demange [3] proposes to represent the sequential formation of the grand coalition by a rooted spanning tree of the communication graph. Each such rooted spanning tree is a partial order of the player set such that any

[1] See also Kleinberg and Weiss [11] for another formulation of the Shapley value.

coalition formed by the arrival of a new player is connected. A rooted spanning tree singles out a unique player, called the root, and assigns to each other player a unique superior. Any player can communicate with the root by communicating iteratively with superiors. Demange [3] extends the marginal vector from orderings of the players to rooted spanning trees. This second approach is also studied by Herings *et al.* [7] who introduce the average tree solution for graph games in which the communication graph is a forest (the graph has no cycle). This allocation rule is the average of the marginal vectors associated with all rooted spanning trees of the forest communication graph. Herings *et al.* [8] show that an extension of the average tree solution to arbitrary graph games can be seen as another generalization of the Shapley value.

The principle of the compensation vector can be extended in both ways. The first approach leads immediately to the Myerson value since in the graph-restricted game the average of the compensation vectors associated with all orderings coincides with the Shapley value. In this article we therefore adopt the second approach: we generalize the compensation vector from orderings of the players to rooted spanning trees. As in Herings *et al.* [8] and Béal *et al.* [2], we distinguish two cases depending on the presence or absence of cycles in the communication graph.

Firstly, if the communication graph is a forest, each player is the root of exactly one rooted spanning tree. In such a case, we define the compensation solution as the average of the compensation vectors associated with all rooted spanning trees. For this class of graph games, the compensation solutions can be characterized by the axioms of component efficiency and relative fairness. The first axiom is standard and states that the payoffs of the members of a component of the communication graph add up to the worth of the component. The second axiom originates in the considerable empirical and experimental evidence that considerations of fairness affect economic behavior explains the spectacular development of the literature that incorporates the player's relative payoff position into economic models (see for instance Fehr and Schmidt [5] and Ho and Su [9]). The second axiom takes into account such effects: a player not only cares about the payoff allocated to him but also cares about the payoff allocated to relevant reference players. Depending on the situation, the relevant reference agents may be the colleagues in the firm with whom a person interacts most frequently, a person's relatives, the trading partners, or a person's neighbors. In a forest communication graph, it is natural to consider the set of players with whom a player communicates as the set of reference agents. The axiom of relative fairness examines the negotiation for the creation of a link connecting two players. The creation of the link will merge the two components of the graph of which the two negotiating players are members. The two players drive a look to their components to evaluate their payoff and judge whether they have been treated fairly. They care about their relative positions with respect to their component. The average payoff in their components can be used as a reference point or benchmark for these players to compare their well-being. Relative fairness says that the relative position of the two players with respect to the average payoff in their pre-existing components should be the same. We show that the compensation solution is the unique allocation rule on the class of forest graph games that satisfies component efficiency and relative fairness. The axiom of relative fairness can be regarded as a mixture between Myerson's fairness [12] and the axiom of component fairness introduced by Herings *et al.* [7]. Fairness says that deleting a link between two players yields for both players the same average change in payoff while component fairness says that deleting a link between two players yields for both resulting components the same average change in payoff, where the average is taken over the players in the component. The Myerson value is the unique allocation rule on the class of all graph games that satisfies component efficiency and fairness and the average tree solution is the unique allocation rule on the class of forest graph games that satisfies component efficiency and component fairness.

Secondly, if the communication graph is arbitrary, we allow any nonempty set of rooted spanning trees in the definition of the compensation solution. The question of the creation of such a set of rooted spanning trees naturally arises. In order to answer this question, we use two algorithms well-known in computer science. More specifically, we consider the compensation solutions associated with the sets of rooted spanning trees created by the tree-growing algorithms called **DFS** (for Depth-First Search) and **BFS** (for Breadth-First Search) respectively. Such algorithms explore the communication graph so as to construct rooted spanning trees by growing a tree, one player and one link at a time. In **DFS**, players are explored out of the most recently visited player who still has unvisited neighbors. Thus, the rooted spanning trees constructed by **DFS** have the greatest height, *i.e.* the greatest distance between a player and the root. Algorithm **BFS** systematically explores the links of the graph in order to visit every unexplored player that is reachable from an initial player. Therefore, **BFS** constructs the rooted spanning trees with the smallest height. The rooted trees considered in Herings *et al.* [8] are constructed by **DFS** (a proof is given in Béal *et al.* [2]). We prove that the compensation solution associated with this set of rooted spanning trees coincides with the Shapley value when the communication graph is complete. A consequence of this result is that the compensation solution can be seen as a generalization of the Shapley value from TU-games to graph games. We also show that the compensation solution with respect to trees constructed by **BFS** yields the equal surplus division when the communication graph is complete. The same two results are obtained in [2] for the two average tree solutions obtained from **DFS** and **BFS**. These results make a comparison between the Shapley value and the equal surplus division in terms of differences in the formation of the grand coalition whereas van den Brink [16] study the two allocation rules by comparable axiomatic characterizations.

The rest of the article is organized as follows. In section 2, we introduce the compensation vector and prove that the Shapley value is the average over all orderings of the players of the compensation vectors. Section 3 is devoted to the generalization to graph games. Following the usual definitions, we split the results in two subsections. Subsection 3.1 contains the axiomatic characterization for the class of forest graph games. Subsection 3.2 introduces algorithms **DFS** and **BFS** and proves the connections between the compensation solutions and both the Shapley value and the equal surplus division. Section 4 concludes.

2 Reinterpreting the Shapley value

In order to interpret the formation of the grand coalition as the result of a bargaining process, let us consider any arbitrary 3-player TU-game (N, v) with player set $N = \{1, 2, 3\}$. Assume we have the order $(1, 2, 3)$: player 1 shows up first, then player 2, and finally player 3 completing the formation of the grand coalition N.

In the first stage, coalition $\{1\}$ is formed. This coalition can claim a compensation of $v(\{1\})$ in return for continuing to form N. This claim is a credible threat since $\{1\}$ produces a worth $v(\{1\})$. If the remaining players 1 and 2 accept to pay this compensation, then coalition $\{1\}$ cannot threaten any longer to stop forming N. Both players 2 and 3 should be charged an equal amount $v(\{1\})/2$ of the compensation.

Next player 2 enters the scene and forms coalition $\{1,2\}$ with player 1. As before, coalition $\{1,2\}$ can claim a compensation of $v(\{1,2\})$ to keep on forming the grand coalition. The remaining player 3 will have to pay $v(\{1,2\})$ in order to ensure that N will eventually form, and it seems reasonable to split the obtained compensation $v(\{1,2\})$ equally between players 1 and 2.

Finally, player 3 shows up and the grand coalition is formed. The bargaining process ends and each player obtains an equal share of $v(N)$. The following table summarizes the payoffs of the players.

	Stage 1	Stage 2	Stage 3	Total payoff
Player 1	gets $v(\{1\})$	gets $\frac{v(\{1,2\})}{2}$	gets $\frac{v(\{1,2,3\})}{3}$	$v(\{1\}) + \frac{v(\{1,2\})}{2} + \frac{v(\{1,2,3\})}{3}$
Player 2	pays $\frac{v(\{1\})}{2}$	gets $\frac{v(\{1,2\})}{2}$	gets $\frac{v(\{1,2,3\})}{3}$	$-\frac{v(\{1\})}{2} + \frac{v(\{1,2\})}{2} + \frac{v(\{1,2,3\})}{3}$
Player 3	pays $\frac{v(\{1\})}{2}$	pays $v(\{1,2\})$	gets $\frac{v(\{1,2,3\})}{3}$	$-\frac{v(\{1\})}{2} - v(\{1,2\}) + \frac{v(\{1,2,3\})}{3}$

Extending this procedure to the n-person case, we obtain a compensation vector: at each stage of the formation of N, each of the remaining players pays an equal share of the worth of the currently formed coalition, and the total amount is split equally between the entered players. Moreover, since no ordering is pre-determined for a TU-game, we will average over all possible orderings.

Formal definitions are provided below. Let $N = \{1, \ldots, n\}$ be a finite set of players. A *cooperative game with transferable utility* on N, or simply TU-game, is a *characteristic function* $v : 2^N \longrightarrow \mathbb{R}$ such that $v(\emptyset) = 0$. For each $S \subseteq N$, $v(S)$ is the *worth* of *coalition* S. The set of all TU-games on N is denoted by Γ^N. A *payoff vector* $x \in \mathbb{R}^n$ on N is an n-dimensional vector giving a payoff $x_i \in \mathbb{R}$ to each player $i \in N$. A *solution* on Γ^N is a function $f : \Gamma^N \longrightarrow \mathbb{R}^n$ that assigns a payoff vector $f(v) \in \mathbb{R}^n$ to each $v \in \Gamma^N$.

An *ordering* of N is a bijective function π on N, where $\pi(i)$ is the player at position $i \in \{1, \ldots, n\}$ in π. For any ordering π on N and any player $i \in N$, define the coalition containing player i and the set of its predecessors in π as $S_i^\pi = \{j \in N : \pi^{-1}(j) \leq \pi^{-1}(i)\}$. For any TU-game $v \in \Gamma^N$, we define the *compensation vector* $c^\pi(v) \in \mathbb{R}^n$ as:

$$\forall i \in N, \quad c_i^\pi(v) = \sum_{j \in N : i \in S_j^\pi} \frac{v(S_j^\pi)}{|S_j^\pi|} - \sum_{j \in N : i \in N \setminus S_j^\pi} \frac{v(S_j^\pi)}{n - |S_j^\pi|} \tag{1}$$

For any $v \in \Gamma^N$ consider the *marginal contribution* vector $m^\pi(v) \in \mathbb{R}^n$ defined as $m_i^\pi(v) = v(S_i^\pi) - v(S_i^\pi \setminus \{i\})$ for each $i \in N$. The *Shapley value* is the allocation rule Sh that assigns to each TU-game $v \in \Gamma^N$ the average of all $n!$ marginal contribution vectors $m^\pi(v)$:

$$\forall v \in \Gamma^N, \forall i \in N, \quad \mathrm{Sh}_i(v) = \frac{1}{n!} \sum_{\pi} m_i^\pi(v)$$

Equivalently, the Shapley value is the allocation rule such that:

$$\forall v \in \Gamma^N, \forall i \in N, \quad \mathrm{Sh}_i(v) = \sum_{S \subseteq N : i \in S} \frac{(n-s)!(s-1)!}{n!} [v(S) - v(S \backslash \{i\})] \tag{2}$$

where s stands for the cardinal of coalition S. In a first result, we show that the average, over the set of all possible orderings of N, of the compensation vector $c^\pi(v)$ is the Shapley value of the TU-game $v \in \Gamma^N$.

Lemma 1 *For any $v \in \Gamma^N$ and any $i \in N$, it holds that*

$$Sh_i(v) = \frac{1}{n!} \sum_\pi c_i^\pi(v).$$

Proof. For any $v \in \Gamma^N$ and any $i \in N$, we have

$$\begin{aligned}
\frac{1}{n!} \sum_\pi c_i^\pi(v) &= \frac{1}{n!} \sum_\pi \left(\sum_{j \in N : i \in S_j^\pi} \frac{v(S_j^\pi)}{|S_j^\pi|} - \sum_{j \in N : i \in N \backslash S_j^\pi} \frac{v(S_j^\pi)}{n - |S_j^\pi|} \right) \\
&= \frac{1}{n!} \sum_{S \subseteq N} \left(\sum_{\pi : S_j^\pi = S, i \in S} \frac{v(S)}{s} - \sum_{\pi : S_j^\pi = S, i \in N \backslash S} \frac{v(S)}{n-s} \right) \\
&= \frac{1}{n!} \left(\sum_{S \subseteq N : i \in S} (n-s)!s! \times \frac{v(S)}{s} - \sum_{S \subseteq N : i \in N \backslash S} (n-s)!s! \times \frac{v(S)}{n-s} \right) \\
&= \sum_{S \subseteq N : i \in S} \frac{(n-s)!(s-1)!}{n!} \times v(S) - \sum_{S \subseteq N : i \in N \backslash S} \frac{(n-(s+1))!s!}{n!} \times v(S) \\
&= \sum_{S \subseteq N : i \in S} \frac{(n-s)!(s-1)!}{n!} [v(S) - v(S \backslash \{i\})] \\
&= \mathrm{Sh}_i(v),
\end{aligned}$$

which proves the result. ∎

As a consequence, another expression of the Shapley value is given by:

$$\forall v \in \Gamma^N, \forall i \in N, \quad \mathrm{Sh}_i(v) = \sum_{S \subseteq N : i \in S} \frac{(n-s)!s!}{n!} \times \frac{v(S)}{s} - \sum_{S \subseteq N : i \in N \backslash S} \frac{(n-s)!s!}{n!} \times \frac{v(S)}{n-s}.$$

The interpretation of this expression is similar to the usual probabilistic interpretation of the Shapley value. For a given coalition size $s = 1, \ldots, n$, draw a set from the urn containing the subsets of N of size s, where each set has the same probability $(n-s)!s!/n!$ to be drawn. If S is drawn with $|S| = s$, then either player i belongs to S and obtains a share $1/s$ of $v(S)$ or player i does not belong to S and pays a share $1/(n-s)$ of $v(S)$. The expected payoff for player i in this random procedure is the Shapley value of i in game $v \in \Gamma_N$.

We conclude this section by the definition of another allocation rule on Γ^N that will be used in section 3.2. It is the *equal surplus division* ESD, which first assigns to each player $i \in N$ his stand-alone payoff $v(\{i\})$ and then distributes the remainder of $v(N)$ equally among all players in N:

$$\forall v \in \Gamma^N, \forall i \in N, \quad \mathrm{ESD}_i(v) = v(\{i\}) + \frac{v(N) - \sum_{j \in N} v(\{j\})}{n} \tag{3}$$

3 Generalization to graph games

Consider a 4-person cooperative game (N, v) with $N = \{1, 2, 3, 4\}$ and such that the bilateral relationships between the players are represented by an undirected graph (N, L) on N where L contains the links $\{4, 3\}$, $\{4, 2\}$ and $\{3, 1\}$. This graph is drawn on the left side of Figure 1 and reflects the communication constraints faced by the players. In order to describe the formation of the grand coalition as the result of a bargaining process similar to the one in section 2, we have to take these constraints into account. In particular, imagine that player 4 is the last player who shows up. Because player 2 does not directly communicate with players 1 and 3, the formation of N cannot be described by a total order. However, several partial orders are compatible with the limited communication possibilities. As an example, let us suppose that there are two bargaining rooms that simultaneously used for the formation the grand coalition. In the first room, player 1 shows up first and then player 3. In the second room, player 2 shows up. Finally, player 4 shows up and completes the formation of N by connecting the coalitions $\{1, 3\}$ and $\{2\}$ formed in the two rooms. This partial order is represented by a rooted spanning tree of (N, L) drawn on the right side of Figure 1, where a directed link from i to j means that i shows up after player j.

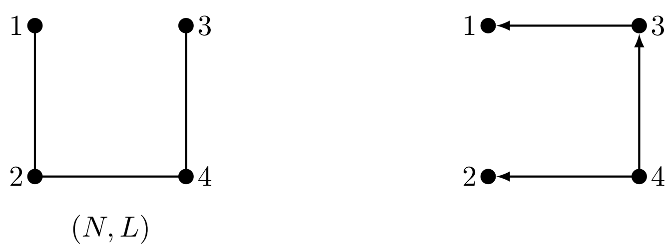

Figure 1: A communication graph and a rooted spanning tree

During this process, players and coalitions of players that form can threaten to refuse to continue the formation of the grand coalition. In the first bargaining room coalition $\{1\}$ forms in the first step and can credibly claim a worth $v(\{1\})$. As in the first section, the other players 2, 3 and 4 should bear an equal share of the compensation $v(\{1\})$ requested by player 1 so as to ensure that the process will go on. Note that the determination of these intermediary payoffs does note rely on whether coalition $\{2\}$ has already been formed in the second bargaining room since player 2's current situation has no influence on his need of the agreement of player 1 to continue to form N. In the second step of the first bargaining room player 3 shows up and forms coalition $\{1, 3\}$. Players 2 and 4 will have to pay each a compensation $v(\{1,3\})/2$ to guarantee that this coalition will pursue the formation of the grand coalition, and players 1 and 3 will get each an amount of $v(\{1,3\})/2$. Continuing in this fashion, we obtain the following payoffs:

	$\{1\}$	$\{1,3\}$	$\{2\}$	N	Total payoff
Player 1	$v(\{1\})$	$\frac{v(\{1,3\})}{2}$	$-\frac{v(\{2\})}{3}$	$\frac{v(N)}{4}$	$v(\{1\})+\frac{v(\{1,3\})}{2}-\frac{v(\{2\})}{3}+\frac{v(N)}{4}$
Player 2	$-\frac{v(\{1\})}{3}$	$-\frac{v(\{1,3\})}{2}$	$v(\{2\})$	$\frac{v(N)}{4}$	$-\frac{v(\{1\})}{3}-\frac{v(\{1,3\})}{2}+v(\{2\})+\frac{v(N)}{4}$
Player 3	$-\frac{v(\{1\})}{3}$	$\frac{v(\{1,3\})}{2}$	$-\frac{v(\{2\})}{3}$	$\frac{v(N)}{4}$	$-\frac{v(\{1\})}{3}+\frac{v(\{1,3\})}{2}-\frac{v(\{2\})}{3}+\frac{v(N)}{4}$
Player 4	$-\frac{v(\{1\})}{3}$	$-\frac{v(\{1,3\})}{2}$	$-\frac{v(\{2\})}{3}$	$\frac{v(N)}{4}$	$-\frac{v(\{1\})}{3}-\frac{v(\{1,3\})}{2}-\frac{v(\{2\})}{3}+\frac{v(N)}{4}$

This procedure can be extended to any n-person cooperative game with a communication graph and to any of its rooted spanning tree. We get a payoff vector: at each stage of the (partially ordered) formation of N, each of the players outside of the considered coalition pays an equal share of the worth of this coalition, and the total amount is split equally between the players of the coalition. Moreover, since no particular rooted spanning tree is pre-determined for a TU-game with a communication graph, we will average over all possible rooted spanning trees. The resulting allocation rule will be called the compensation solution. The above procedure also applies to any connected component of a non connected communication graph.

Now let us give formal definitions and notations. An *undirected graph* is a pair (N,L) where N is a set of nodes and L is a collection of *links*, *i.e.* $L \subseteq L^N$ where $L^N = \{\{i,j\} : i,j \in N, i \neq j\}$. For ease of notation we write ij instead of $\{i,j\}$ and L_{-ij} instead of $L \backslash \{\{i,j\}\}$. For each $S \subseteq N$, $L(S) = \{ij \in L : ij \subseteq S\}$ is the set of links between nodes of S. The graph $(S, L(S))$ is the subgraph of (N,L) induced by S. A sequence of distinct nodes $(i_1,\ldots,i_k)$ is a *path* in (N,L) if $i_q i_{q+1} \in L$ for each $q = 1,\ldots,k-1$. Two nodes i and j are *connected* in (N,L) if $i = j$ or there exists a path $(i_1,\ldots,i_k)$ with $i_1 = i$ and $i_k = j$. A graph (N,L) is *connected* if any two nodes $i,j \in N$ are connected. A *tree* is a connected graph (N,L) such that for each link $ij \in L$, the graph (N, L_{-ij}) is not connected. A subset S of N is connected in (N,L) if $(S,L(S))$ is a connected graph. The empty subset $\emptyset$ is trivially connected. A subset $C \subseteq N$ is a *component* of (N,L) if $(C,L(C))$ is maximally connected, *i.e.* if $(C,L(C))$ is connected and for each $i \in N\backslash C$, $(C \cup \{i\}, L(C \cup \{i\}))$ is not connected. The collection of components of (N,L), denoted by N/L, forms a partition of N. A graph (N,L) is a *forest* if for each component $C \in N/L$, $(C,L(C))$ is a tree.

The combination of a TU-game and of a communication graph is a so-called *graph game*, given by a triple (N,v,L) where N is the set of players, v is the characteristic function on N and L the set of links on N. In this article we consider graph games on N such that (N,L) is a connected graph. Let $\mathcal{G}_N$ denote the set of all such graph games. Also, let $\mathcal{G}^*_N$ be the set of all graph games on N such that (N,L) is a forest and $\mathcal{G}^{**}_N$ be the set of all graph games on N such that (N,L) is arbitrary. Denote by K_N the complete graph on N and by $\mathcal{G}_{K_N}$ the class of all graph games on N with a complete communication graph. As for TU-games we omit N in our notation. Let $\mathcal{G}$ be any class of graph games on N. An *allocation rule* on $\mathcal{G}$ is a function f that assigns to each $(v,L) \in \mathcal{G}$ a payoff vector $f(v,L) \in \mathbb{R}^n$.

For each component C of a graph (N, L), a *spanning tree* on C is a tree on C. A *rooted spanning tree* on C is a directed graph that arises from this spanning tree by selecting a player $r \in C$, called the *root*, and directing all links away from the root. For a given spanning tree on C, each player $r \in C$ is the root of exactly one rooted spanning tree denoted by t_r. For each t_r and each $j \in C\backslash\{r\}$, there is exactly one directed link (i, j). Player i is the unique *predecessor* of j and j is a *successor* of i in t_r. Denote by s_i^r the possibly empty set of successors of player $i \in C$ in t_r. A player j is a *subordinate* of i in t_r if there is a directed path from i to j, *i.e.* if there is a sequence of distinct players $(i_1, \ldots, i_k)$ such that $i_1 = i$, $i_k = j$, and, for each $q = 1, \ldots, k-1$, $i_{q+1} \in s_{i_q}^r$. The set S_i^r denotes the union of all subordinates of i in t_r and $\{i\}$.

For each graph game $(v, L) \in \mathcal{G}_N^{**}$, each component $C \in N/L$ and each rooted spanning tree t_r on C, Demange [3] defines the marginal vector

$$\forall i \in C, \quad m_i^r(v, L) = v(S_i^r) - \sum_{j \in s_i^r} v(S_j^r).$$

Now we are ready to adapt the compensation vector for TU-games in the context of graph games. For each graph game (v, L), each component $C \in N/L$ and each rooted spanning tree t_r on C, we define the compensation vector as:

$$\forall i \in C, \quad c_i^r(v, L) = \sum_{j \in C: i \in S_j^r} \frac{v(S_j^r)}{|S_j^r|} - \sum_{j \in C: i \in C\backslash S_j^r} \frac{v(S_j^r)}{|C\backslash S_j^r|} \tag{4}$$

Firstly, the contribution of player $i \in C$ in t_r consists in sharing equally the worth $v(C)$ with the other members of component C. Then, for each coalition S_j^r, $j \in C\backslash\{r\}$, formed according to the partial order t_r, player i receives a share $v(S_j^r)/|S_j^r|$ if he belongs to this coalition or pays $v(S_j^r)/|C\backslash S_j^r|$ otherwise. Then we are ready to give the definition of the compensation solutions.

For the class $\mathcal{G}_N^{**}$ of all graph games on N, we assigns to each possible graph a nonempty set of rooted spanning trees. Define a function $\mathscr{T}$ that assigns to each connected graph (N, L) and to each component $C \in N/L$ a nonempty set $\mathscr{T}(L, C)$ of rooted spanning trees on N. The *compensation solution* $\mathrm{CS}^{\mathscr{T}}(v, L)$ with respect to $\mathscr{T}$ on $\mathcal{G}_N^{**}$ is defined as:

$$\forall (v, L) \in \mathcal{G}_N^{**}, \forall i \in N, \quad \mathrm{CS}_i^{\mathscr{T}}(v, L) = \frac{1}{|\mathscr{T}(L, C)|} \sum_{t_r \in \mathscr{T}(L,C)} c_i^r(v, L) \tag{5}$$

For the class of forest graph games on N, the *compensation solution* CS on $\mathcal{G}_N^*$ is defined as the average over all rooted spanning trees of the contribution vector (4). Formally:

$$\forall (v, L) \in \mathcal{G}_N^*, \forall C \in N/L, \forall i \in C \quad \mathrm{CS}_i(v, L) = \frac{1}{|C|} \sum_{r \in C} c_i^r(v, L). \tag{6}$$

To further illustrate the compensation solution, we consider the following three-person graph games. Consider $(v, L) \in \mathcal{G}_N^*$ where $N = \{1, 2, 3\}$, $L = \{12, 23\}$ and such that v is given by:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	30	0	0	0	0	30	60

Observe that v can be interpreted as the composition of two games: one game on $\{1\}$ and one game on $\{2,3\}$. The formation of the grand coalition does not create any extra worth compared to the partition $\{\{1\},\{2,3\}\}$. Therefore, the presence of link 12 in the communication graph does not really matter in terms of worth (the same conclusion would hold if link 13 were to replace 12). All in all, it seems natural that each coalition of the above-mentioned partition gets the worth it produces, *i.e.* player 1 should obtain 30 and players 2 and 3 should share 30. Moreover, players 2 and 3 are symmetric in v so that they should split equally the payoff 30. The induced vector $(30,15,15)$ is precisely the compensation solution of this graph game, whereas the average tree solution is $(30,10,20)$ and the Myerson value is $(25,10,25)$. Note that the core of v^L contains both the compensation solution and the average tree solution (actually the compensation solution is the center of gravity of this core) but not the Myerson value.

3.1 The compensation solution for forest graph games

In this section, we provide a characterization of the compensation solution for forest graph games. First we need some definitions. For a component $C \in N/L$ of a forest (N,L) and a link $ij \in L(C)$, let C_k be the component in (N, L_{-ij}) containing k, where $k = i, j$. For each component C, we denote by Δ_L^C the set of coalitions that contains, for each $ij \in L(C)$, the two components C_i and C_j. We consider four axioms. The first two well-known axioms are enunciated for the class of all graph games.

Component efficiency. For each $(v,L) \in \mathcal{G}_N^{**}$ and each $C \in N/L$, it holds that

$$f_C(v,L) = v(C).$$

Fairness. For each $(v,L) \in \mathcal{G}_N^{**}$ and each $ij \in L$, it holds that

$$f_i(v,L) - f_i(v,L_{-ij}) = f_j(v,L) - f_j(v,L_{-ij}).$$

Fairness says that deleting a link between two players yields for both players the same change in payoff. The unique allocation rule on $\mathcal{G}_N^{**}$ that satisfies component efficiency and fairness is the Myerson value (Myerson [12]). It is the allocation rule on $\mathcal{G}_N^{**}$ that assigns to each graph game $(v,L) \in \mathcal{G}_N$ the Shapley value of the graph restricted game v^L defined as

$$\forall S \subseteq N, \quad v^L(S) = \sum_{T \in S/L(S)} v(T).$$

Note that both axioms are still valid on the class of forest graph games. The last two axioms are enunciated for the class of forest graph games.

Component fairness. For each $(v,L) \in \mathcal{G}_N^{*}$, each $C \in N/L$ and each $ij \in L(C)$, it holds that

$$\frac{1}{|C_i|}\Big(f_{C_i}(v,L) - f_{C_i}(v,L_{-ij})\Big) = \frac{1}{|C_j|}\Big(f_{C_j}(v,L) - f_{C_j}(v,L_{-ij})\Big).$$

Component fairness says that deleting a link between two players yields for both resulting components the same average change in payoff, where the average is taken over the players in the component. The unique allocation rule on $\mathcal{G}_N^*$ that satisfies component efficiency and component fairness is the average tree solution (Herings *et al.* [7]). It is the allocation rule on $\mathcal{G}_N^*$ that assigns to each graph game in this class the average over all rooted spanning trees of the Demange's marginal vector.

Relative fairness. For each $(v, L) \in \mathcal{G}_N^*$, each $C \in N/L$ and each $ij \in L(C)$, it holds that

$$f_i(v, L) - \frac{1}{|C_i|} f_{C_i}(v, L_{-ij}) = f_j(v, L) - \frac{1}{|C_j|} f_{C_j}(v, L_{-ij}).$$

Relative fairness has the following interpretation. Players i and j are negotiating the creation of link ij. These players are members of the two components C_i and C_j that are about to merge. Rather than focusing solely on their allocation changes as in the axiom of fairness, the two players i and j drive a look to their components to evaluate their payoff and judge whether they have been treated fairly. They care about their relative positions with respect to their components. The average payoff in their components can be used as a reference point for these players to compare their well-being. Relative fairness says that the relative position of players i and j with respect to average payoff in their pre-existing components C_i and C_j should be the same. As such, relative fairness shares with the axiom of fairness the feature that the negotiating players are those involved in the considered link. The axiom of relative fairness also shares with the axiom of component fairness the feature that the payoffs of two involved components are relevant for the creation of a link.

The next two results show that the compensation solution given by (6) is the unique allocation rule on $\mathcal{G}_N^*$ that satisfies component efficiency and relative fairness.

Theorem 1 *On the class $\mathcal{G}_N^*$, there is a unique allocation rule that satisfies component efficiency and relative fairness.*

Proof. Suppose that f satisfies the two axioms on $\mathcal{G}_N^*$. Pick any $(v, L) \in \mathcal{G}_N^*$, any $C \in N/L$ and any $ij \in L(C)$. Note that $C_i \in N/L_{-ij}$ and $C_j \in N/L_{-ij}$. Thus component efficiency of f yields

$$f_{C_i}(v, L_{-ij}) = v(C_i) \text{ and } f_{C_j}(v, L_{-ij}) = v(C_j) \tag{7}$$

so that relative fairness becomes

$$f_i(v, L) - f_j(v, L) = \frac{1}{|C_i|} v(C_i) - \frac{1}{|C_j|} v(C_j), \tag{8}$$

with the convention that $i < j$. Therefore, we obtain a system of $|L|$ equations of the form (8) and $|N/L| = |N| - |L|$ equations of the form (7). Let us show that this system of equations has a unique solution. Consider the matrix M of coefficients given by these n equations. Specifically, for each $C \in N/L$, let m^C be the row of M corresponding to the axiom of component efficiency for component C. For each link $ij \in L$, let $m^{(i,j)}$ be the row of M corresponding to equation (8) associated with link ij, $i < j$, i.e. $m_i^{(i,j)} = 1$, $m_j^{(i,j)} = -1$ and $m_k^{(i,j)} = 0$ for each $k \in N \backslash \{i, j\}$. We will use elementary operations on these rows in order to transform M into the identity matrix of size n.

First, for each $ij \in L$ such that $i < j$ define the n-dimensional vector $m^{(j,i)}$ as $m^{(j,i)} = -m^{(i,j)}$. Recall that for each $C \in N/L$ and each $i, j \in N$ there is a unique directed path from i to j, which we denote by $(i_1, \ldots, i_k)$. Now consider any component $C \in N/L$ and pick a player $i \in C$. For each $j \in C\backslash\{i\}$, create the n-dimensional vector $m^{(i,j)}$ as

$$m^{(i,j)} = \sum_{q=1}^{k-1} m^{(i_q, i_{q+1})}.$$

Thus $m^{(i,j)}_i = 1$, $m^{(i,j)}_j = -1$ and $m^{(i,j)}_k = 0$ for each $k \in N\backslash\{i,j\}$. Next construct the n-dimensional vector b^i as

$$b^i = \frac{1}{|C|}\left(\sum_{j\in C\backslash\{i\}} m^{(i,j)} + m^C\right).$$

It follows that $b^i_i = 1$ and $b^i_j = 0$ for each $j \in N\backslash\{i\}$. We can repeat the preceding steps for each $i \in C$ and each $C \in N/L$ so as to obtain n vectors b^i, $i \in N$. For each $C \in N/L$ and each $i \in C$, note that b^i is the result of a linear combination of the $|C|$ rows of M associated with component C with no null coefficients. As a consequence, b^i can replace any of these rows. To accomplish such a replacement, for each $C \in N/L$, consider any rooted spanning tree t_r on C. Replace row m^C by b^r and for each $ij \in L(C)$, replace $m^{(i,j)}$ by b^i if (j,i) belongs to t_r or by b^j if (i,j) belongs to t_r. Therefore matrix M can be transformed into a matrix that consists in the n line vectors b^i, $i \in N$. This matrix is the identity matrix of size n, which implies that f is uniquely determined. ■

Theorem 2 *On the class $\mathcal{G}^*_N$, the compensation solution given by (6) satisfies component efficiency and relative fairness.*

Proof. Consider any $(v, L) \in \mathcal{G}^*_N$ and any $C \in N/L$. For a given rooted spanning tree t_r on C, we have

$$\begin{aligned}
\sum_{i\in C} c^r_i(v,L) &= \sum_{i\in C}\left(\sum_{j\in C: i\in S^r_j}\frac{v(S^r_j)}{|S^r_j|} - \sum_{j\in C: i\in C\backslash S^r_j}\frac{v(S^r_j)}{|C\backslash S^r_j|}\right)\\
&= \sum_{j\in C}\left(\sum_{i\in S^r_j}\frac{v(S^r_j)}{|S^r_j|} - \sum_{i\in C\backslash S^r_j}\frac{v(S^r_j)}{|C\backslash S^r_j|}\right)\\
&= v(S^r_r) + \sum_{j\in C\backslash\{r\}}\left(|S^r_j|\frac{v(S^r_j)}{|S^r_j|} - |C\backslash S^r_j|\frac{v(S^r_j)}{|C\backslash S^r_j|}\right)\\
&= v(S^r_r)\\
&= v(C).
\end{aligned}$$

Therefore

$$\sum_{i\in C}\mathrm{CS}_i(v,L) = \frac{1}{|C|}\sum_{i\in C}\sum_{r\in C} c^r_i(v,L) = \frac{1}{|C|}\sum_{r\in C}\sum_{i\in C} c^r_i(v,L) = \frac{1}{|C|}\sum_{r\in C} v(C) = v(C),$$

which proves that CS verifies component efficiency. In order to show that CS satisfies relative fairness, we first rewrite definition (6). For each $i \in C$,

$$\begin{aligned}
\mathrm{CS}_i(v, L) &= \frac{1}{|C|}\sum_{r\in C}\left(\sum_{j\in C: i\in S_j^r}\frac{v(S_j^r)}{|S_j^r|} - \sum_{j\in C: i\in C\setminus S_j^r}\frac{v(S_j^r)}{|C\setminus S_j^r|}\right)\\
&= \frac{1}{|C|}\left(v(C) + \sum_{S\in\Delta_L^C: i\in S}\sum_{r\in C\setminus S}\frac{v(S)}{|S|} - \sum_{S\in\Delta_L^C: i\in C\setminus S}\sum_{r\in C\setminus S}\frac{v(S)}{|C\setminus S|}\right)\\
&= \frac{1}{|C|}\left(v(C) + \sum_{S\in\Delta_L^C: i\in S}|C\setminus S|\frac{v(S)}{|S|} - \sum_{S\in\Delta_L^C: i\in S}\sum_{r\in S}\frac{v(C\setminus S)}{|S|}\right)\\
&= \frac{1}{|C|}\left(v(C) + \sum_{S\in\Delta_L^C: i\in S}\left(|C\setminus S|\frac{v(S)}{|S|} - v(C\setminus S)\right)\right)\\
&= \frac{1}{|C|}\left(v(C) + \sum_{S\in\Delta_L^C: i\in S}\frac{1}{|S|}\Big(|C\setminus S|v(S) - |S|v(C\setminus S)\Big)\right)
\end{aligned}$$

Using the former expression of CS, we obtain that for each $ij \in L$, the difference $\mathrm{CS}_i(v, L) - \mathrm{CS}_j(v, L)$ is equal to

$$\begin{aligned}
&= \frac{1}{|C|}\left(\sum_{S\in\Delta_L^C: i\in S}\frac{1}{|S|}\Big(|C\setminus S|v(S) - |S|v(C\setminus S)\Big) - \sum_{S\in\Delta_L^C: j\in S}\frac{1}{|S|}\Big(|C\setminus S|v(S) - |S|v(C\setminus S)\Big)\right)\\
&= \frac{1}{|C|}\left(\sum_{\substack{S\in\Delta_L^C:\\ i\in S, j\in C\setminus S}}\frac{1}{|S|}\Big(|C\setminus S|v(S) - |S|v(C\setminus S)\Big) - \sum_{\substack{S\in\Delta_L^C:\\ j\in S, i\in C\setminus S}}\frac{1}{|S|}\Big(|C\setminus S|v(S) - |S|v(C\setminus S)\Big)\right)
\end{aligned}$$

Observe that the two components C_i and C_j obtained from C after deleting link $ij \in L(C)$ are the unique elements in Δ_L^C that contain player i but not player j and player j but not player i respectively. As a consequence the previous expression can be rewritten as:

$$\begin{aligned}
&\frac{1}{|C|}\left(\frac{1}{|C_i|}\Big(|C_j|v(C_i) - |C_i|v(C_j)\Big) - \frac{1}{|C_j|}\Big(|C_i|v(C_j) - |C_j|v(C_i)\Big)\right)\\
&= \frac{1}{|C|}\left(\frac{1}{|C_i|} + \frac{1}{|C_j|}\right)\Big(|C_j|v(C_i) - |C_i|v(C_j)\Big)\\
&= \frac{|C_j|v(C_i) - |C_i|v(C_j)}{|C_i||C_j|}\\
&= \frac{v(C_i)}{|C_i|} - \frac{v(C_j)}{|C_j|}\\
&= \frac{1}{|C_i|}\mathrm{CS}_{C_i}(v, L_{-ij}) - \frac{1}{|C_j|}\mathrm{CS}_{C_j}(v, L_{-ij})
\end{aligned}$$

where the last equality follows from the fact that the compensation solution satisfies component efficiency. ■

We conclude this section by a comparison between the compensation solution, the Myerson value and the average tree solution. Consider the glove graph game $(v, L) \in \mathcal{G}_N^*$ where $L = \{\{1i\}_{i \in N \backslash \{1\}}\}$ and such that player 1 has a one left-hand glove while players $2, \ldots, n$ have one right-hand glove each. A (left-right) pair is worth 1€. The corresponding function v is determined by the values: $v(S) = 1$ if $1 \in S$ and $|S| \geq 2$ and $v(S) = 0$ otherwise. The compensation solution assigns payoffs

$$\mathrm{CS}_1(v, L) = \frac{2}{n}$$

and

$$\mathrm{CS}_i(v, L) = \frac{n-2}{n(n-1)}$$

to each $i \in N \backslash \{1\}$. The Myerson value assigns payoffs $(n-1)/n$ to player 1 and $1/(n(n-1))$ to each $i \in N \backslash \{1\}$. The average tree solution assigns payoffs 1 to player 1 and 0 to each $i \in N \backslash \{1\}$. On the one hand the Myerson value converges to the average tree solution as n tends to infinity. The interpretation is that if n is very large, then the owner of unique left-hand glove obtains the entire worth of the grand coalition. On the other hand, payoffs assigned by the compensation solution tend to 0 when n tends to infinity. The interpretation is that if n is very large, sharing a single unit of worth implies that the payoffs of all players are very small. Nevertheless it should be noted that the payoff of the owner of unique left-hand glove still remains at least twice as large as the payoff of each other player: this emphasizes the idea used in the interpretation of the axiom of relative fairness that the players care about their relative position. In this example and in the three-person example studied before the beginning of section 3.1, the compensation solution seems be to a little bit more egalitarian than the Myerson value and the average tree solution. We do not know whether this observation is valid for a large class of situations.

3.2 The compensation solution for arbitrary graph games

In this section we study the compensation solutions for arbitrary graph games. The definition of the compensations solutions rely on the creation of nonempty sets of rooted spanning trees on a communication graph. A general algorithm, called **Tree-Growing**, is given for constructing spanning trees of a given graph. It is borrowed from computer science (see chapter 10 in Gross and Yellen [6]) and consists in growing a subtree, one link and one player at a time. Then, two particular instances of this algorithms will be considered. They create two sets of rooted spanning trees that can have a meaningful interpretation (see Béal *et al.* [2]). When the communication graph is complete, we prove that the compensation solutions associated with these sets of rooted spanning trees coincide with the Shapley value and the equal surplus division respectively.

Because the compensation vector (4) can be decomposed by the components of a graph, there is no loss of generality to focus on the class $\mathcal{G}_N$ of all graph games with a connected communication graph. So consider a connected communication graph (N, L). The algorithm introduced in this section can be easily applied to the connected components of a non-connected graph. A pair (S, L_S) with $S \in 2^N \backslash \{\emptyset\}$ and $L_S \subseteq L(S)$ is a *subtree* of (N, L) if (S, L_S) is a tree on S. Denote by G any such subtree. For any given subtree G of a graph (N, L), the links and players of G are called *tree links* and *tree players*, and the links and players in (N, L) that are not in G are called *non-tree links* and *non-tree players*. A *frontier link* for G is a non-tree link with one endpoint in G, called its *tree endpoint*, and one endpoint not in G, its *non-tree endpoint*. By definition, the graph resulting from adding any frontier link of G and its associated non-tree endpoint to the subtree G is still a subtree of (N, L).

An essential component of algorithm **Tree-Growing** is the rule `nextLink` which selects a frontier link to add to the current subtree. For any subtree G of a graph (N, L), let F denote the set of frontier links for G. Then the function `nextLink`$((N, L), F)$ chooses and returns as its value a frontier link in F that is to be added to subtree G. Then, the selected frontier link and its non-tree endpoint are added to the subtree G. Note that the rule `nextLink` may not be deterministic, depending on how it has been specified to select a frontier link in F. After a frontier link is added to the current subtree, the function `updateFrontier`$((N, L), F)$ removes from F those links that are no longer frontier links and adds to F those links that have become frontier links. The pseudocode of **Tree-Growing** is given by Algorithm 1.

Algorithm 1 – Tree-Growing

Input: a finite connected graph (N, L) and a starting player $r \in N$.
Output: a spanning tree G of (N, L).
Initial conditions: $G = (\{r\}, \emptyset)$, $F = \{ri \in L : i \in N\}$.

While $F \neq \emptyset$
    $e \longleftarrow$ `nextLink`$((N, L), F)$
    Let i be the non-tree endpoint of e
    Add link e and player i to G.
    `updateFrontier`$((N, L), F)$
Return tree G.

Each different specification of rule `nextLink` creates a different instance of **Tree-Growing**. In the remaining part of this section, we describe two well-known instances of **Tree-Growing** called Depth-First Search (**DFS**) and Breadth-First Search (**BFS**). Both algorithms rely on the discovery order. For each subtree G of (N, L) induced by **Tree-Growing**, the *discovery order* is a listing of players in N in the order in which they are added as subtree G is grown. Once the spanning tree G has been returned by **Tree-Growing**, one can easily consider its oriented version t_r, where the root is the starting player r specified as input in **Tree-Growing**. Henceforth, we will refer to t_r as the output of algorithm **Tree-Growing**. For any output t_r of **Tree-Growing**, the position of player i in the discovery order, starting with 0 for player r, is called the *discovery number* of i in t_r.

In algorithm **DFS**, `nextLink` selects a frontier link in F whose tree endpoint has the largest discovery number. In other words, **DFS** chooses a frontier link incident to the most recently discovered player. If such a link fails to exist, then **DFS** "backtracks" to the second most recently discovered player and tries again, and so on. Therefore, **DFS** discovers players "deeper" in the graph whenever possible. In this way, **DFS** creates spanning trees containing maximal directed paths starting at the root r. Let $\mathbf{DFS}(L)$ denote the set of all rooted spanning trees of graph (N, L) that **DFS** creates. For any $t_r \in \mathbf{DFS}(L)$, the discovery number of a player $i \in N$ is denoted $\mathtt{dfnumber}(t_r, i)$. Since `nextLink` is not necessarily a deterministic function, several different executions of **DFS** on a graph (N, L) can create the same rooted spanning tree t_r. In such a situation, $\mathtt{dfnumber}(t_r, i)$ can take different values depending on which region of the graph (N, L) is first explored.

In algorithm **BFS**, `nextLink` selects a frontier link in F whose tree endpoint has the smallest discovery number. In other words, algorithm **BFS** chooses a frontier link incident to the less recently discovered player. If such an link fails to exist, then **BFS** considers the second less recently discovered player and tries again, and so on. Therefore, **BFS** explores the graph by selecting frontier links incident to players as close to the root as possible. In this way, **BFS** creates shortest directed paths from the root to any other player (see Proposition 4.2.4 in Gross and Yellen [6]). Let $\mathbf{BFS}(L)$ denote the set of all rooted spanning trees of graph (N, L) that **BFS** creates.

The result of this section is a study of the compensation solutions with respect to the set of spanning trees created by **DFS** and **BFS** respectively. When the communication graph is complete the resulting CS solutions are shown to coincide with the Shapley value and the equal surplus division on Γ^N respectively.

Theorem 3 *(i) If (N, L) is a tree, then, for each $(v, L) \in \mathcal{G}_N$, the compensation solution defined with respect to $\mathbf{DFS}(L)$ and given by (5) is the average of n compensation vectors and coincides with (6).*

(ii) If (N, L) is the complete graph K_N, then, for $(v, L^N) \in \mathcal{G}_{K_N}$, the compensation solution defined with respect to $\mathbf{DFS}(L^N)$ and given by (5) is the average of $n!$ compensation vectors and coincides with the Shapley value given by (2).

Proof. *(i)* The proof is obvious and is omitted.

(ii) Note that $\mathbf{DFS}(L^N)$ contains only directed lines, since **DFS** can always grow the current tree by selecting a frontier link incident to the most recently discovered player. There are $n!$ such directed lines, one for each ordering of the players. In order to see this, fix a starting player r. Then, because the graph is complete, the tree can be grown by visiting any of the $(n-1)$ other players. In the next step, there are still $(n-2)$ unvisited players who can be reached from the most recently visited player. The tree can be grown by any of these players. Continuing in this fashion, it follows that for a given starting player, there are $(n-1)!$ different executions of **DFS** and each execution constructs a directed line. Since any of the n players can be chosen as starting player, we obtain the $n!$ directed lines. As a consequence, we get

$$
\begin{aligned}
\mathrm{CS}_i(v, L^N) &= \frac{1}{n!} \sum_{t_r \in \mathbf{DFS}(L^N)} c_i^r(v, L) \\
&= \frac{1}{n!} \sum_{t_r \in \mathbf{DFS}(L^N)} \left(\sum_{j \in N : i \in S_j^r} \frac{v(S_j^r)}{|S_j^r|} - \sum_{j \in N : i \in N \setminus S_j^r} \frac{v(S_j^r)}{|N \setminus S_j^r|} \right).
\end{aligned}
$$

Consider any such directed line t_r. Then for each $s \in \{1, \ldots, n\}$, there is a unique player $i \in N$ such that $|S_i^r| = s$. In addition, for a given $s \in \{1, \ldots, n\}$, there are exactly $n!/((n-s)!s!)$ different coalitions of size s. Therefore, there are precisely $(n-s)!s!$ of the $n!$ directed lines in which a coalition $S \subseteq N$ will be the unique set of subordinates of size s. Rewriting $\mathrm{CS}_i(v, L)$ by summing over the coalitions containing player i and over the coalitions not containing i, we obtain

$$\begin{aligned}
\mathrm{CS}_i^{\mathbf{DFS}}(v, L^N) &= \frac{1}{n!}\left(\sum_{S\subseteq N: i\in S} s!(n-s)!\frac{v(S)}{s} - \sum_{S\subseteq N: i\in N\setminus S} s!(n-s)!\frac{v(S)}{n-s}\right) \\
&= \frac{1}{n!}\left(\sum_{S\subseteq N: i\in S} s!(n-s)!\frac{v(S)}{s} - \sum_{S\subseteq N: i\in S} (s-1)!(n-s+1)!\frac{v(S\setminus\{i\})}{n-s+1}\right) \\
&= \frac{1}{n!}\left(\sum_{S\subseteq N: i\in S} (s-1)!(n-s)!v(S) - \sum_{S\subseteq N: i\in S} (s-1)!(n-s)!v(S\setminus\{i\})\right) \\
&= \mathrm{Sh}_i(v),
\end{aligned}$$

which completes the proof. ■

Theorem 4 *(i) If (N, L) is a tree, then, for each $(v, L) \in \mathcal{G}_N$, the compensation solution defined with respect to $\mathbf{BFS}(L)$ and given by (5) is the average of n compensation vectors and coincides with (6).*

(ii) If (N, L) is the complete graph K_N, then, for $(v, L^N) \in \mathcal{G}_{K_N}$, the compensation solution defined with respect to $\mathbf{BFS}(L^N)$ and given by (5) is the average of n compensation vectors and coincides with the equal surplus division given by (3).

Proof. *(i)* The proof is obvious and is omitted.

(ii) Note that for each $r \in N$, any player $i \in N\setminus\{r\}$ is at distance 1 of r since K_N is the complete graph. Hence, for any $r \in N$, the execution of **BFS** on K_N starting at r yields a unique spanning tree t_r in which r is the predecessor of all other players. The set $\mathbf{BFS}(L^N)$ contains n such rooted spanning trees, one for each $r \in N$. Therefore, for each $i \in N$, we have

$$\mathrm{CS}_i^{\mathbf{BFS}}(v, L^N) = \frac{1}{n}\sum_{r\in N} c_i^r(v, L^N),$$

where the vector of marginal contributions in t_r is then given by

$$c_r^r(v, L^N) = \frac{v(N)}{n} - \sum_{j\in N\setminus\{r\}} \frac{v(\{j\})}{n-1}$$

and

$$c_i^r(v, L^N) = \frac{v(N)}{n} + v(\{i\}) - \sum_{j\in N\setminus\{i,r\}} \frac{v(\{j\})}{n-1}$$

for each $i \in N\backslash\{r\}$. Replacing the compensations $c_i^r(v, L^N)$, $r \in N$, by their above expressions, $\text{CS}_i^{\mathbf{BFS}}(v, L^N)$ becomes

$$
\begin{aligned}
& \frac{1}{n}\left(\frac{v(N)}{n} - \sum_{j\in N\backslash\{i\}} \frac{v(\{j\})}{n-1} + \sum_{r\in N\backslash\{i\}} \left(\frac{v(N)}{n} + v(\{i\}) - \sum_{j\in N\backslash\{i,r\}} \frac{v(\{j\})}{n-1}\right)\right) \\
= \; & \frac{1}{n}\left(v(N) - \sum_{j\in N\backslash\{i\}} \frac{v(\{j\})}{n-1} + (n-1)v(\{i\}) - \sum_{r\in N\backslash\{i\}} \sum_{j\in N\backslash\{i,r\}} \frac{v(\{j\})}{n-1}\right) \\
= \; & \frac{1}{n}\left(v(N) - \sum_{j\in N\backslash\{i\}} \frac{v(\{j\})}{n-1} + (n-1)v(\{i\}) - (n-2) \sum_{j\in N\backslash\{i\}} \frac{v(\{j\})}{n-1}\right) \\
= \; & \frac{1}{n}\left(v(N) + (n-1)v(\{i\}) - \sum_{j\in N\backslash\{i\}} v(\{j\})\right) \\
= \; & v(\{i\}) + \frac{v(N) - \sum_{j\in N} v(\{j\})}{n} \\
= \; & \text{ESD}_i(v),
\end{aligned}
$$

which gives the result. ■

4 Conclusion

Various extensions of the average tree solutions have been proposed. For instance, Béal *et al.* [1] study the marginalist tree solutions defined as a linear combination of the marginal contribution vectors, and the random tree solutions defined as a probability distribution over the set of all marginal contribution vectors. It could be interesting to study similar extensions for the compensation solution. In addition, van den Brink *et al.* [17] and Khmelnitskaya [10] provide axiomatic characterizations of the Demange's marginal vector for line-graph games and forest graph games respectively. These articles also contain economic applications. So far there does not exist similar results on the compensation vector (4).

Moreover, a recent article by Ryuo *et al.* [14] explores another way to connect the axioms of fairness and component fairness. While fairness only compare the two players incident on the deleted link, component fairness compare the allocation change of all players in the resulting components, but the importance of each player in the component is the same. The authors consider the axiom of ε-parameterized fairness which incorporates the two preceding axioms. More specifically, parameter ε takes into account the fact that the importance of players should decrease with their distance to the deleted link.

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