An Analysis of the German University Admissions System*

Preliminary and Incomplete - Please do not circulate

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March 19, 2010

Abstract

This paper analyzes the centralized assignment procedure by which places in medicine and related disciplines at public universities in Germany are allocated. The total capacity at each university is divided into three *quotas* reserved for specific applicant groups. Places are allocated sequentially using the well known *Boston* and *college optimal stable mechanisms.* Quotas are *floating* in the sense that capacity can be redistributed across quotas in response to insufficient demand. Assuming that universities are not strategic, I characterize *complete information equilibrium outcomes* of the revelation game induced by the assignment procedure and show that Pareto dominated equilibrium outcomes exist.

I introduce a theory of *affirmative action problems with floating quotas* and show that under two simple assumptions there exists a *strategyproof* and *stutdent optimal stable matching mechanism* (SOSM). It is shown that the German admissions problem is a special case of this class of matching problems and that the associated SOSM Pareto dominates *all* complete information equilibria of the currently employed procedure. Apart from this application, I show that the theory of affirmative action with floating quotas provides useful new ways for dealing with multi-dimensional affirmative action constraints in school choice.

JEL codes: C78; D02

Keywords: University Admissions; Matching; Stability; Strategyproofness; Floating Quotas

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[*] I owe special thanks for continuous encouragement and advice to Benny Moldovanu. Furthermore, comments and suggestions by Sebastian Braun, Peter Coles, Nadja Dwenger, Lars Ehlers, Paul Heidhues, Fuhito Kojima, Dorothea Kubler, Konrad Mierendorff, Al Roth, and a seminar audience in Berlin were very helpful and are greatly appreciated. Any remaining errors are, of course, my responsibility.

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1 Introduction

According to German legislation, every student who obtains the *Abitur* (i.e., successfully finishes secondary school) or some equivalent qualification is entitled to study any subject at any public university. In accordance with this principle of free choice university admission was a completely decentralized process prior to the 1960s: a student with the appropriate qualification could just enroll at the university of her choice. Problems emerged in the early 1960s when some universities had to reject a substantial number of applicants for medicine and dentistry. Rejections were usually based on some measure of the quality of the *Abitur*, mostly the average degree. This often led to a threshold for average grades, called *Numerus Clausus*, such that applicants with higher averages were not admitted.[1] The problem quickly spread to other disciplines and many universities had to establish local admission criteria. This resulted in a very complicated decentralized admission procedure that forced students to spend more time on maximizing their chances of admission than to figure out which university fitted their needs.[2] To solve these problems a centralized clearinghouse, the *Zentralstelle für die Vergabe von Studienplätzen* (ZVS), was established in 1973. Ever since its introduction the ZVS has been subject to immense public scrutiny and political debates. These debates led to gradual changes in the assignment procedure, with the last major revision in 2005.

In this paper I analyze the most recent version of the ZVS procedure that is used to allocate places for medicine and three related subjects (dentistry, pharmacy, and veterinary medicine). For the winter term 2009/2010, 12 684 places for freshmen in these four subjects were allocated among 52 472 applicants. The procedure consists of three steps that sequentially allocate parts of total capacity: In step one twenty percent of available places at each university can be allocated among applicants with exceptional average grades. This is implemented by first using average grades to *select* as many applicants as places can be allocated in step one and then assigning selected applicants (henceforth *top-grade applicants*) to universities using the *Boston mechanism.* In step two a completely analogous procedure is used to allocate up to twenty percent of available places at each university among applicants who have unsuccessfully participated in previous assignment procedures (henceforth *wait-time applicants*) on basis of average grades and social criteria. In the third step all remaining places - this includes in particular all places that could have been but were not allocated in the first two steps - are assigned

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[1] In Germany average grades range from 1.0 to 6.0, with 1.0 representing the best possible average grade. Hence, *high* average grades indicate a *bad* performance in secondary school.

[2] In a study of the admission procedures at 40 universities, Scheer (1999) finds that 70 different admission criteria were used.

among remaining applicants according to criteria chosen by the universities using the *college (university) proposing deferred acceptance algorithm* (CDA). In a sense, the ZVS procedure tries to have the *best of both worlds* by using the applicant proposing Boston mechanism for the priority based steps of the procedure (steps 1 and 2) and letting universities take an active role in the last step of the procedure, where they are able to evaluate applicants themselves.

With respect to reported preferences, the ZVS procedure can lead to very undesirable allocations. For example, an applicant assigned in the first step may prefer a university at which she could have been admitted in the third step. However, the procedure is highly manipulable so that reported need not correspond to true preferences. In particular, prospective students can submit one ranking for each step of the procedure which allows them to condition their reports on the different admission criteria and assignment mechanisms used in the three steps of the procedure. In general, applicants have to make a difficult trade-off between securing a match in an early step and staying in the procedure in hope of obtaining a better assignment in a later step. I argue that given the structure of the German university admissions system it is reasonable to assume that universities do not act strategically. Under this assumption we show that the set of (complete information) equilibrium outcomes coincides with the set of matchings that are *stable* with respect to the true preferences of applicants and the admissions environment. Here, the stability constraints take into account that each university has three types of places with differing admissions criteria and that vacant seats can be redistributed across quotas. I also briefly consider the case of incomplete information. Two simple examples point out the problems associated with (i) the sequential allocation of places, and (ii) allowing universities to use their position in applicants' preference rankings as an admission criterion for the last step of the procedure.

While stability is a reasonable allocative goal, the current assignment procedure forces applicants to participate in a complicated revelation game and I show that, even in the absence of information asymmetries, the procedure supports *Pareto dominated equilibrium outcomes.* These deficiencies motivate a redesign based around the ideas of (i) achieving a stable matching that is as favorable as possible to applicants, and (ii) ensuring that truthful preference revelation is a dominant strategy for applicants. What precludes a direct application of existing matching theory is that the German system combines rigid affirmative action constraints - in the form of reserving some parts of capacity for special student groups or *types* - with the possibility of redistributing unused capacity across type-specific quotas. This case has not been covered by the existing literature. I introduce an *affirmative action problem with floating quotas* where (i) students have (potentially multidimensional) types, (ii) each college has a fixed total capacity initially split up into several (rigid) *type specific quotas* that reflect the target type distribution the college is trying to achieve, and (iii) each college has a *choice protocol* that describes in which sequence types are to be considered (a type may be considered more than once) and how capacity is to be redistributed between type-specific quotas if there is insufficient demand from some student types. It is shown that as long as choice protocols satisfy two simple assumptions of *monotonicity* and *consistency*, an affirmative action problem with floating quotas gives rise to an associated college admissions problem for which a student optimal stable matching exists and the associated student optimal stable matching mechanism (SOSM) is (group) strategyproof for students. Before applying these results to the German system, I show how these results can be used to deal with multi-dimensional affirmative action constraints in school choice. An example shows that consistent and monotonic choice protocols allow schools or school districts to express a strict preference for a balanced distribution of sexes *and* ethnicities in a class while ensuring that demanded seats are never left empty if balancedness cannot be achieved. I then show that the German admissions problem can be interpreted as an affirmative action problem with floating quotas for which the above assumptions on choice protocols are satisfied. Furthermore, stability in the associated college admissions problem is shown to coincide with the stability notion that characterizes complete information equilibrium outcomes of the current German assignment procedure. This implies in particular that interpreting the German admissions problem as an affirmative action problem with floating quotas not only provides straightforward incentives to applicants, but also achieves an outcome that results in unambiguous welfare gains for students compared to the (equilibria of) the currently employed assignment system.

Related Literature

Braun, Dwenger, and Kübler (2008) study the German university admissions system from an empirical perspective. Using data for the winter term 2006/2007 they find considerable support for the hypothesis that applicants try to manipulate the ZVS procedure.[3] My paper, which was drafted independently of this empirical study, complements this research since it shows precisely how these findings can be explained by applicants' strategic incentives. A major benefit of the more theoretical approach is that I am not only able to design a promising alternative but can also compare it directly to the equilibrium outcomes of the current procedure.

Since the assignment procedure analyzed in this paper combines the Boston mechanism with the college optimal stable matching mechanism, the theoretical and applied literature

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[3] There have been some minor changes in the procedure since then, which I detail in an Appendix.

concerned with these two algorithms is also closely related.[4] There are three incidents of real-life matching procedures that were found to be equivalent to (one of the versions of) a deferred acceptance algorithm: Roth (1984a) showed that the matching algorithm used to match graduating medical students to their first professional position in the US from 1951 until the late 1990s was equivalent to the CDA. In a similar vein, Balinski and Sönmez (1999) showed that the mechanism used to assign Turkish high school graduates to public universities, the *multi-category serial dictatorship*, was also equivalent to this mechanism. More recently, Guillen and Kesten (2008) have shown that the mechanism used to allocate on campus housing among students of the Massachusetts Institute of Technology is equivalent to the SDA.

The Boston mechanism has been extensively studied in the matching literature since Abdulkadiroglu and Sönmez (2003)'s influential study of school choice systems. Ergin and Sönmez (2006) show that the set of pure strategy equilibrium outcomes coincides with the set of stable matchings that are stable with respect to the preferences of students.[5] Well known results from the theory of two-sided matching markets then imply that the SDA outcome weakly dominates any equilibrium outcome of the Boston mechanism with respect to the true preferences of students. In an empirical investigation of the Boston mechanism, Abdulkadiroglu, Pathak, Roth, and Sönmez (2006) find strong evidence that many students try to manipulate the mechanism. They argue that the strategic choices of some families hurt other families who strategize suboptimally.[6] In an experimental comparison of school choice mechanism, Chen and Sönmez (2006) found that the student optimal stable matching mechanism outperformed the Boston mechanism in terms of efficiency.[7] My study contributes to this literature by reporting another case of a real-life assignment procedure that uses the CDA. However, in the German system this mechanism is combined with the well known Boston mechanism which, to the best of my knowledge, is the first time a combined use of these two popular mechanisms has been observed and analyzed. I show that Ergin and Sönmez (2006)'s equilibrium characterization has a natural extension to the more complicated German admission system. Furthermore, I show that the SDA can accommodate the specific constraints of the German market without losing

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[4] An excellent summary of the earlier research in this area is Roth and Sotomayor (1991).

[5] Kojima (2008) shows that this result also holds for *generalized priority structures*, which are formally equivalent to substitutable preferences over subsets of students. These generalized priority structures can accommodate e.g. affirmative action constraints, which are often present in real-life applications of the school choice problem.

[6] For a theoretical argument in this vein see Pathak and Sönmez (2008).

[7] For a more positive perspective on the Boston mechanism see Miralles (2008), Featherstone and Niederle (2008), and Abdulkadiroglu, Yasuda, and Chen (2010).

its desirable incentive and allocative properties. This shows that at least the theoretical arguments in favor of deferred acceptance algorithms remain valid despite the complex constraints in Germany.

Organization of the paper

This remainder of this paper is structured as follows: In section 2, I introduce three known matching algorithms that are important for the analysis. Section 3 contains a description of the current admission procedure for German universities. In section 4, I analyze the revelation game induced by this procedure under the assumption that universities do not act strategically. Section 5 first introduces a general method of dealing with affirmative action constraints and then shows how this method can be applied to construct a Pareto improving alternative to the current German system. Section 6 concludes. Some proofs as well as details for and data on the current procedure are relegated to an Appendix.

2 Basic model and assignment algorithms

A *college admissions problem* consists of

- a finite set of students I
- a finite set of colleges C,
- a profile of student preferences $R_I = (R_i)_{i \in I}$, where R_i is a strict ranking of $C \cup \{i\}$,[8] and
- a profile of college preferences $R_C = (R_c)_{c \in C}$, where R_c is a strict ranking of 2^I.

Given a strict ranking R_i, we denote by cP_ic' that i strictly prefers college c over college c' and by cR_ic' that either cP_ic', or $c = c'$ (a similar remark applies to R_c). College c is acceptable to i if cP_ii, and set of students $J \subseteq I$ is acceptable to college c if $JP_c\emptyset$. A *matching* is a mapping μ from $I \cup C$ into itself such that (i) for all students i, $\mu(i) \in C \cup \{i\}$ for all $i \in I$, (ii) for all colleges c, $\mu(c) \subseteq I$, and (iii) $i \in \mu(c)$ if and only if $\mu(i) = c$. We assume throughout that agents only care about their own partner(s) in a matching so that their preferences over matchings are congruent with their preferences over potential partners. The sets of students and colleges are assumed to be fixed so that a college admissions problem is given by a profile of student and college preferences $R = (R_I, R_C)$.

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[8] More formally, for all students $i \in I$, R_i is a complete, reflexive, transitive, and antisymmetric binary relation on $C \cup \{i\}$. The same remark applies to college preferences.

We assume that college preferences are *substitutable* (Kelso and Crawford (1982)): If a student i is admitted by college c when students in $J \subseteq I$ applied then i is still admitted by c when only students in $\tilde{J} \subseteq J$ apply to it. More formally, given some $J \subseteq I$ let $Ch_c(J)$ denote the R_c-most preferred subset of J, c's *choice from* J (unique by the assumption of strict preferences). College c's preferences are substitutable if whenever $i \in Ch_c(J) \cap \tilde{J}$ for some $\tilde{J} \subset J \subset I$ then $i \in Ch_c(\tilde{J})$. An important subclass of substitutable preferences which will also play an important role in this paper is the class of *responsive preferences* introduced by Roth (1985): A college with responsive preferences has a strict ranking $R{*}_c$ of $S \cup \{i\}$ which is related to its preference relation over groups of students by $JP_c(J \setminus \{j\} \cup \{k\})$ for some $J \subseteq I$, $j \in J$, $k \in I \setminus J$, if and only if jP_c^*k. In many applications with responsive preferences one assumes that each college has a fixed upper bound q_c, it's *quota*, on the number of students it can admit. In this case $Ch_c(J)$ consists of the q_c most preferred acceptable students in J with respect to R_c^* (or all acceptable students in J if there are fewer than q_c such students in J).

The key allocative criterion in the literature is *(pairwise) stability* as introduced by Gale and Shapley (1962). Given a college admissions problem R, a matching μ is pairwise stable if (i) no student is matched to an unacceptable college, that is, $\mu(i)R_ii$ for all $i \in I$, (ii) no college prefers to reject some of its assigned students, that is, $Ch_c(\mu(c)) = \mu(c)$, and (iii) there is no student-college pair that blocks μ, that is, there is no pair (i, c) such that $cP_i\mu(i)$ and $i \in Ch_c(\mu(c) \cup \{i\})$.

In the following, I briefly describe three assignment procedures that play an important role in the literature and the remainder of this paper.

The Student Proposing Deferred Acceptance Algorithm

The *student proposing deferred acceptance algorithm* (SDA), due to Gale and Shapley (1962) (and extended to college admission problems with substitutable preferences by Roth and Sotomayor (1991)), will play an important role in my proposal for a redesign of the German admission system and proceeds as follows

> In the first round, every student applies to her favorite acceptable college. Each college c temporarily accepts its choice from the set of applicants in this round and rejects all other applicants.
>
> In the tth round, every student applies to her most preferred acceptable college (if any) among those that have not rejected her in any previous round of the algorithm.

> Each college c temporarily accepts its choice from the set of applicants in this round and rejects all other applicants.

It is known (Roth (1984b)) that the matching chosen by this algorithm is *student optimal* and *college pessimal* among all stable matchings, that is, it is the unanonimously most preferred stable matching for students and the unanonimously least preferred stable matching for colleges.

Another major benefit of this procedure is that it induces straightforward behavior by students under one additional assumption on college preferences introduced by Hatfield and Milgrom (2005): College preferences satisfy the *law of aggregate demand* (LAD), if $J \subseteq \tilde{J}$ implies $|Ch_c(J)| \leq |Ch_c(\tilde{J})|$. If college preferences satisfy substitutability and LAD then the SDA is *group strategy-proof for students* (Hatfield and Kojima (2009))[9] in the sense that no group of students (in particular no individual student) can ever obtain a strictly more preferred assignment for all of its members from the SDA by submitting a profile of preference rankings which differ from the true rankings of the group's members. More formally, let $f^I(R)$ denote the matching chosen by the SDA when the profile of submitted preferences is R and $f^I_i(R)$ denote student i's assignment under this matching. Then there is no group of students $J \subseteq I$ and preference profile R of students and colleges, such that for all $i \in J$, $f^I_i(\tilde{R}_J, R_{-J}) P_i f^I_i(R)$ for some (joint) manipulation $\tilde{R}_J = (\tilde{R}_i)_{i \in J}$ of students in J.

The college proposing deferred acceptance algorithm

The *college proposing deferred acceptance algorithm* (CDA) proceeds analogously with colleges taking an active role. This algorithm plays an important role in the current German admission procedure and proceeds as follows:

> In the first round, each college offers admission to its choice from the set of all students. Each student i temporarily holds on to her most preferred offer and rejects all other offers.
>
> In the tth round, each college offers admission to its choice from the set of all students that have not rejected one of her offers in previous round. Each student i temporarily holds on to her most preferred offer and rejects all other offers.

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[9] Hatfield and Milgrom (2005) showed that the SDA is strategyproof for individual students under these two assumptions.

The matching chosen by this algorithm has diametrically opposed properties to the matching chosen by the SDA (add citation): It is the unanonimously most preferred stable matching for colleges and the unanonimously least preferred stable matching for students. Even when college preferences satisfy LAD, the CDA does not in general induce straightforward behavior from either students or colleges. If we assume that colleges do not act strategically, the set of complete information Nash equilibria (of the revelation game between students induced by the CDA) corresponds to the set of stable matchings with respect to the true preferences of students and colleges (Sotomayor (2007)).[10]

The School Choice Problem and the Boston Mechanism

A *school choice problem* (Abdulkadiroglu and Sönmez (2003)) is a college admissions problem (with responsive preferences) in which each college c (we maintain the terminology of students and colleges throughout), is exogenously assigned a strict *priority ordering* $\succ_c$ of students and a capacity q_c which are treated as a primitive of the problem.[11] An important difference in interpretation is that $\succ_c$ does not (necessarily) represent college c's preferences but is imposed on it by laws and regulations. Under this interpretation, only students count for welfare considerations.

The *Boston mechanism* is a popular real-life assignment procedure which is based on the following algorithm.

> In the first round, every student applies to her top choice college. Each college c admits the q_c highest priority students who apply in this round (or all those students if there are fewer than q_c). All other students are rejected. Let q_c^2 denote the remaining capacity of college c.
>
> In the tth round, every remaining student applies to her tth most preferred acceptable college (if any). Each college c admits the q_c^t highest priority students who apply in this round (or all those students if there are fewer than q_c^t). All other students are rejected. Let q_c^{t+1} denote the remaining capacity of college c.

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[10] Haeringer and Klijn (2009) show that this result is not true for the SDA if students are allowed to rank more than one college. However, if lists are of unlimited length it can be argued that truth-telling is a focal point for students and any such equilibrium would involve some students playing (weakly) dominated strategies.

[11] See Erdil and Ergin (2008) and Abdulkadiroglu, Pathak, and Roth (2009) on how to deal with ties in the priority structure.

It is known that this algorithm produces a matching that is efficient with respect to the *reported* preferences of students. However, it is also known that a matching mechanism based on the above procedure is highly manipulable and there is empirical (Abdulkadiroglu, Pathak, Roth, and Sönmez (2006)) as well as experimental (Chen and Sönmez (2006)) evidence that students (or their parents) do act on these incentives and that their strategic behavior leads to welfare losses. Ergin and Sönmez (2006) show that the above procedure viewed as a direct matching mechanism induces stability in Nash equilibrium in the sense that the set of Nash equilibrium outcomes coincides with the set of stable matchings with respect to the true preferences of participants. Combined with the above, this shows that at least for complete information environments the SDA dominates the Boston mechanism with respect to the true preferences of students.

3 The German University Admissions System

The ZVS assigns places for medicine and related subjects. There is a separate assignment procedure for each subject and each applicant can take part in at most one of these assignment procedures. The assignment procedures are identical for all subjects and proceed in three sequential steps:

1. In the first step (up to) one fifth of total places at each university are allocated among applicants with an *exceptional* qualification, that is, an excellent, or very low, average grade in school leaving examinations.
2. In the second step, (up to) one fifth of total places at each university are allocated among applicants with an exceptionally long *waiting time*, that is, a long time since obtaining their high-school degree.[12]
3. In the third step, all remaining places, in particular those places not taken in the first two steps, are allocated among applicants not assigned in the first two steps on basis of universities' preferences.

Since assignment procedures for different subjects are separate and applicants can participate in at most one of these procedures, I concentrate on one assignment procedure in the following. Let A be the set of applicants and U denote the set of universities. In order to participate in the centralized assignment procedure applicants have to submit an ordered (preference) list of

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[12] In this context, an applicant's waiting-time is defined as the number of half years (or semesters) that have passed since he or she graduated from secondary school net of the number of semesters that he or she was enrolled at some public university (in a different subject).

universities for each step of the procedure. There is no consistency requirement on the three lists and the list submitted for step $t \in \{1, 2, 3\}$ is used only to determine assignments in step t. All three preference lists are submitted simultaneously before any assignments are determined. For step 2, applicants can rank as many universities as they want. For steps 1 and 3 at most six universities can be ranked. Let $Q_a = (Q_a^1, Q_a^2, Q_a^3)$ denote the profile of preference lists submitted by applicant $a \in A$. An applicant *applies for a place in step* t, if she ranks at least one university for step t of the procedure. Let q_u denote the total number of places that university u has to offer. Let $q_u^1 = q_u^2 = \frac{1}{5}q_u$ and $q^1 = q^2 = \frac{1}{5}\sum_{u\in\mathcal{U}} q_u$ denote the number of places at university u and the total number of places available in steps 1 and 2, respectively. To avoid integer problems I assume that, for all $u \in U$, q_u is a multiple of five. With these preparations, the ZVS procedure can be described as follows.[13]

Step 1: Assignment for excellent applicants

(Selection) Select q^1 applicants from those who applied for a place in step 1. If there are more than q^1 such applicants, order applicants lexicographically according to (i) average grade, (ii) time since obtaining qualification, (iii) completion of military or civil service, (iv) lottery. Select the q^1 highest ranked applicants in this ordering.

(Assignment) Apply the Boston mechanism to determine assignments of selected applicants: University u can admit at most q_u^1 applicants, the preference relation of a selected applicant a is Q_a^1, and an applicant's priority for a university is determined lexicographically by (i) average grade, (ii) social criteria,[14] (iii) lottery. Denote the matching produced in step 1 by $f^{Z1}(Q)$, the set of admitted applicants by A_1 and the set of remaining applicants by $A^2 = A \setminus A_1$.

Step 2: Assignment for wait-time applicants

(Selection) Select q^2 applicants from those in A^2 who applied for a place in step 2. If there are more than q^2 such applicants, order applicants lexicographically according to (i) time

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[13] The main reference for this description are the *Vergabeverordnung ZVS, Stand: WS 2009/2010*, which can be found at www.zvs.de. The following is a simplified version of the actual assignment procedure and some omitted details can be found in Appendix C.

[14] In this category, applicants are ordered lexicographically according to the following criteria: 1. Being severely disabled. 2. Main residence with spouse or child in the district or a district-free city associated to the university. 3. Granted request for preferred consideration of top choice. 4. Main residence with parents in the area associated with the university. Note that, in contrast to the selection stage, an applicant's priority may thus differ across universities.

since obtaining qualification, (ii) average grade, (iii) completion of military or civil service, (iv) lottery. Select the q^2 highest ranked applicants in this ordering.

(Assignment) Apply the Boston mechanism to determine assignments of selected applicants: University u can admit at most q_u^2 applicants, the preference relation of a selected applicant a is Q_a^2, and an applicant's priority for a university is determined lexicographically by (i) social criteria, (ii) average grade, (iii) lottery. Denote the matching produced in step 2 by $f^{Z2}(Q)$, the set of admitted applicants by A_2 and the set of remaining applicants by $A^3 = A^2 \setminus A_2$.

Step 3: Assignment according to universities' preferences

For each university $u \in U$, all remaining places are allocated in this step. Let $q_u^3 = q_u - \left|f_u^{Z1}(Q)\right| - \left|f_u^{Z2}(Q)\right|$, that is, the total capacity of this university minus places assigned in the first two steps. Each university u submits a strict ranking R_u of applicants in A^3 and the option of leaving a place unfilled.

(**Assignment**) Apply the college proposing deferred acceptance algorithm to determine an assignment of applicants to universities: University u can admit at most q_u^3 applicants, the preference relation of an applicant $a \in A^3$ is given by Q_a^3, and the preference relation of university u over individual applicants is given by R_u. Denote the matching produced in this step of the procedure by $f^{Z3}(Q, R_U)$.

3.1 An example

In the following, I illustrate the ZVS procedure by calculating the chosen assignment in a simple example. This will also serve as a first step in the analysis of the procedure since the example already highlights some of the problems.

Suppose that $A = \{a_1, \ldots, a_9\}$ and $U = \{u_1, u_2, u_3\}$. For simplicity, assume that each university has three places to allocate among students and that one place at each university is available in all three steps of the ZVS procedure.[15] Applicants are indexed in increasing order of their average grades, so that a_i has the ith best average grade among $a_1, \ldots, a_9$ (I assume

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[15] It is unproblematic to let each universities' capacity be some multiple of five if one includes more applicants to take the additional places. Larger examples do not facilitate the understanding of the mechanism and all the points made below apply equally well to larger, more realistic settings. This point applies to all examples considered in this paper.

that there are no ties in average grades). The applicants with the longest waiting time are a_7, a_8, a_9. Applicants' (true) preferences are summarized in the following table

R_A	R_{a_1}	R_{a_2}	R_{a_3}	R_{a_4}	R_{a_5}	R_{a_6}	R_{a_7}	R_{a_8}	R_{a_9}
	u_1	u_1	u_3	u_2	u_2	u_3	u_2	u_2	u_1
	u_2	u_3	u_2	u_1	u_3	u_2	u_1	u_1	u_2
	u_3	u_2	u_1	u_3	u_1	u_1	u_3	u_3	u_3

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This notation means that according to R_{a_1} applicant a_1 strictly prefers u_1 over u_2 over u_3. The other columns have analogous interpretations. I now calculate the assignment chosen by the ZVS procedure under the assumption that all applicants submit their preferences truthfully for each step of the procedure. This assumption is made for illustrative purposes and is not meant to indicate that applicants have straightforward incentives under the ZVS procedure.

In step 1, applicants a_1, a_2, a_3 are selected since they have the best average grades. The Boston mechanism produces the following assignment

$$f^{Z1}(R_A) = \begin{array}{ccc} u_1 & u_2 & u_3 \\ a_1 & a_2 & a_3 \end{array}.$$

Since the Boston mechanism is used to determine assignments this matching is efficient with respect to the preferences of a_1, a_2, a_3. Note that a_2 has an incentive to overreport her preference for u_3, since she would have been assigned to (the more preferred) university u_3 if only she had ranked it first.

Next, we calculate the assignment in step 2. Given the above description, a_7, a_8, and a_9 are eligible for a place in this step. To pin down assignments, assume that the priority ordering in the assignment stage of step 2 is a_8, a_7, a_9 at university u_1 (applicants are listed in decreasing order of priority), a_9, a_7, a_8 at u_2, and a_7, a_8, a_9 at u_3. In this case, the Boston mechanism produces the following assignment

$$f^{Z2}(R_A) = \begin{array}{ccc} u_1 & u_2 & u_3 \\ a_9 & a_7 & a_8 \end{array}.$$

Similar to the assignment chosen for step 1, the assignment is efficient with respect to reported preferences and a_8 would have been better off claiming that her most preferred university is u_1. In addition, a_8 and a_2 would both benefit if they were allowed to trade their places. Hence, the matchings chosen in the first two steps of the ZVS procedure are not necessarily efficient with respect to the reported preferences of all applicants assigned in steps 1 and 2.

Finally, we calculate the assignment in step 3. To pin down assignments, assume that $R_{u_1} : a_4, a_5, a_6$, $R_{u_2} : a_6, a_5, a_4$, and $R_{u_3} : a_4, a_5, a_6$. The college proposing deferred acceptance algorithm in step 3 produces the following assignment

$$f^{Z3}(R_A, R_U) = \begin{array}{ccc} u_1 & u_2 & u_3 \\ a_4 & a_6 & a_5 \end{array} .$$

Note that this is the university optimal- and applicant pessimal stable matching in a college admissions problem with participants $\{a_4, a_5, a_6, u_1, u_2, u_3\}$ and preferences as given above if universities' preferences over groups of students are responsive to R_U. Furthermore, note that a_5 would have been able to secure a place at her first choice university u_2 if she had declared all other universities as unacceptable for step 3.

4 Analysis of the assignment procedure: Strategic Incentives of Applicants

The example in the last section showed that applicants sometimes have an incentive to manipulate the ZVS procedure by submitting a ranking of universities that does not correspond to their true preferences. Strategic behavior is encouraged by the ability to submit different preference lists for the three steps of the procedure since study conditions are the same no matter in which step an applicant receives a place at a given university. In its official information brochures the ZVS makes it very clear to applicants that they should choose submitted preference lists "carefully" in order to maximize their chances of admission. These materials, available at www.zvs.de, even contain examples where profitable manipulations are explicitly calculated. Braun, Dwenger, and Kübler (2008) provide empirical evidence showing that applicants do act upon the incentives to manipulate the assignment procedure. In order to evaluate the performance of the university admissions system it is thus important to analyze the strategic incentives induced by the ZVS mechanism.

For this analysis I assume that universities do not act strategically. This assumption is reasonable since prior to the application deadline for applicants universities have to announce a set of criteria according to which they evaluate applicants in the third step of the ZVS procedure. Upon having announced it's criteria, a university can strategize only if it employs "subjective" criteria such as performance in interviews. The assumption of non-strategic universities is therefore not without loss of generality, but (i) only a limited number of universities use subjective criteria,[16] and (ii) there have not been reports about strategic behavior by universities. For these reasons the assumption of non-strategic universities is a useful approximation and I will concentrate on the strategic incentives of applicants in the following. This is not to say that universities do not act strategically at all: Rather, the game induced by the ZVS procedure can be viewed as a two-stage game where in the first stage universities (strategically) announce their evaluation criteria and in the second stage applicants submit their rankings of universities. In this paper I focus on the game between applicants.

Before proceeding to the analysis, I introduce some additional terminology. First, let $\tilde{A}^1 \subseteq A$ be the set of applicants who would be selected in step 1 if all applicants applied for a place in step 1. For $u \in U$, define the *top-grade priority ordering* $\succ_u^1$ of $A \cup \{u\}$ by $a \succ_u^1 b$ if and only if a ranks higher than b with respect to the criteria of the assignment stage in step 1, and $a \succ_u^1 u$ if and only if $a \in \tilde{A}^1$. The *wait-time priority ordering* $\succ_u^2$ is defined analogously. Applicant $a \in A$ is a *top-grade applicant* (*wait-time applicant*) if $a \succ_u^1 u$ ($a \succ_u^2 u$) for all $u \in U$. I assume that both of the just defined orderings are strict so that no two distinct students can have equal top-grade or wait-time priority (see the discussion in section 5 on how to deal with indifferences in the priority structure). The *university admission environment* summarizes all factors that are taken to be fixed throughout and is given by the five-tuple $(A, U, q, \succ^1, \succ^2)$.

Next, I define feasible assignments for the university admissions environment. Since the total capacity is divided into three parts, a matching has to specify not only to which university an applicant is matched, but also which of the three types of places she receives. More formally, a *matching for the university admission environment* is a three-tuple of matchings $\mu = (\mu^1, \mu^2, \mu^3)$ such that

(i) for all $t \in \{1, 2, 3\}$ and all $a \in A$, $\mu^t(a) \in U \cup \{a\}$

(ii) $|(\mu^1(a) \cup \mu^2(a) \cup \mu^3(a)) \cap U| \leq 1$ for all $a \in A$,

(iii) for all $t \in \{1, 2\}$ and all $u \in U$, $|\mu^t(u)| \leq q_u^t$,

(iv) for all $u \in U$, $|\mu^3(u)| \leq q_u^3(\mu^1, \mu^2) = q_u - |\mu^1(u)| - |\mu^2(u)|$, and

(v) for all $t \in \{1, 2, 3\}$, $a \in \mu^t(u)$ if and only if $\mu^t(a) = u$.

I usually suppress the dependency of q_u^3 on μ^1 and μ^2 in what follows. Applicant a's *assignment under matching* μ is u if, for some $t \in \{1, 2, 3\}$, $\mu^t(a) = u$, and it is a if for all $t \in \{1, 2, 3\}$, $\mu^t(a) = a$. I denote a's assignment under μ by $\mu(a)$ and a is *unassigned* if $\mu(a) = a$.

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[16] In Pharmacy for example, only 2 out of 22 universities employ subjective criteria. See Appendix B.

4.1 Complete Information

In this section I assume that the admissions environment, applicants' true preferences, and the rankings of universities to be used for step 3 are all common knowledge among applicants. While the assumption of complete information is strong, applicants can often rely on the outcomes of past assignment procedures to estimate their chances of admission. In order to simplify the analysis and to be able to characterize complete information equilibrium outcomes I make four additional assumptions.

(A1) University u's criteria for step 3 induce a strict ranking $\succ_u^3$ of $A \cup \{u\}$.

(A2) All applicants always rank at least one university for each part of the procedure.

(A3) No applicant who is selected in step 1 can also be selected in step 2.

(A4) If an applicant could be matched to the same university in step 1 (2) and step 3, she prefers to be assigned in 1 (2).

In general, universities only evaluate applicants for step 3 who have not been admitted in the first two steps. However, applicants are evaluated according to criteria that are announced prior to the application deadline and $\succ_u^3$ is taken to be the ranking that would result if all applicants were evaluated by u according to its chosen criteria. If a university uses subjective criteria such as performance in interviews, the first assumption is of course quite strong since entails that applicants can perfectly assess their performance *before* the interview has taken place. See also the discussion after Proposition 1 below. The second assumption ensures that the set of applicants selected for steps 1 and 2 does *not* depend on the profile of submitted preferences. The empirical evidence in Braun, Dwenger, and Kübler (2008) offers strong support in favor of the assumption. The assumption is restrictive, since it can be shown that it may in rare cases be in an applicant's best interest to manipulate the set of students eligible for a place in steps 1 or 2 by strategically abstaining from the admission procedure in these steps (by listing no university). An example demonstrating this point can be found in Appendix A. The third assumption ensures that the set of applicants selected in step 2 does not depend on the assignment in step 1. The assumption is reasonable since top-grade applicants are typically assigned a place of study before they could become eligible for step 2 since average grade is usually an important criterion in the evaluation procedures of universities for the third step (data can be found on www.zvs.de). Finally, the fourth assumption is reasonable since assignments in steps 1 and 2 are determined more than one month before step 3 is conducted (to give universities enough time to evaluate applicants). Since study conditions do not depend on the particular step of the procedure an applicant is admitted through, the additional time to search for an apartment, prepare to move, and so on, motivates strict preference for early assignment.

I now describe the revelation game between applicants induced by the ZVS mechanism. Given that universities are assumed to be non-strategic, I set $\succ = (\succ^1, \succ^2, \succ^3)$ and refer to $(A, U, q, \succ)$ as an *extended university admissions environment.* I assume that each applicant a has a strict ranking R_a of $U \cup \{a\}$ that describes her true ranking of universities and the option of remaining unmatched. In light of A4, this induces a preference ranking $\tilde{R}_a$ over matchings for the admissions environment such that $(\mu^1, \mu^2, \mu^3) = \mu \tilde{P}_a \tilde{\mu} = (\tilde{\mu}^1, \tilde{\mu}^2, \tilde{\mu}^3)$, if and only if either $\mu(a) P_a \tilde{\mu}(a)$, or for some $t \in \{1, 2\}$ and $u \in U$, $\mu^t(a) = \tilde{\mu}^3(a) = u$. For each applicant a, the set of admissible *strategies* consists of three-tuples of rankings $Q_a = (Q_a^1, Q_a^2, Q_a^3)$, such that, for $t \in \{1, 3\}$, Q_a^t is an ordered list containing at least one and at most six universities in order of decreasing preference, and Q_a^2 is an ordered list containing at least one university. Let $\mathcal{Q}$ denote the set of admissible strategies and let $\mathcal{Q}^{|A|}$ be the set of all strategy profiles. Given a profile of applicant reports $Q \in \mathcal{Q}^{|A|}$, the extended admissions environment $(A, U, q, \succ)$, let $f^Z(Q)$ be the three-tuple of matchings produced by the ZVS procedure. Given a profile of applicant preferences R, the *game induced by the ZVS mechanism* is denoted by $\Gamma^Z(R) = (A, \mathcal{Q}^{|A|}, f^Z, \tilde{R})$. A strategy profile Q is a *Nash equilibrium* of $\Gamma^Z(R)$ if there is no profitable unilateral deviation, that is, for all $a \in A$, $f_a^Z(Q) \tilde{R}_a f_a^{ZVS}(Q', Q_{-a})$, for all $Q' \in \mathcal{Q}$. I now define a notion of stability for matchings that is adapted to the specific nature of the German admission environment and that will turn out to be crucial in characterizing equilibrium outcomes. First of all, a *university admissions problem* is given by an extended admissions environment and a profile of applicants' preferences. Since everything else is assumed to be fixed, an admissions problem is given by a profile of student preferences R.

Definition 1. *A matching $\mu = (\mu^1, \mu^2, \mu^3)$ is* ***stable for the university admissions problem*** *R if*

(i) for all $t \in \{1, 2, 3\}$, $u \in U$, $a \in \mu^t(u)$, $a \succ_u^t u$.

(ii) for all $a \in A$, $\mu(a) R_a a$,

(iii) if $\mu^3(a) = u$ for some university u then there is no $t \in \{1, 2\}$ such that either ($a \succ_u^t u$ and $|\mu^t(u)| < q_u^t$), or ($a \succ_u^t \tilde{a}$ for some applicant $\tilde{a} \in \mu^t(u)$).

(iv) there is no applicant-university pair (a, u) such that $u P_a \mu(a)$ and, for some $t \in \{1, 2, 3\}$, either ($a \succ_u^t u$ and $|\mu^t(u)| < q_u^t$) or ($a \succ_u^t \tilde{a}$ for some $\tilde{a} \in \mu^t(u)$).

This definition of stability takes into account that different criteria are used to regulate admission in the three different steps of the assignment procedure. Part (iii) of this definition ensures that in case of multiple possibilities of admission at a university, an applicant taks the place that was intended for her. Finally, note that this definition of stability takes into account that places reserved for, but not taken by top-grade and wait-time applicants can be allocated according to universities' criteria. The following is the main result of this section.

Proposition 1. *Let R be a profile of strict applicant preferences. The set of pure strategy Nash equilibrium outcomes of $\Gamma^Z(R)$ coincides with the set of stable matchings for the university admissions problem R.*

The result is closely related to the equilibrium characterizations of the Boston mechanism by Ergin and Sönmez (2006) and of the COSM by Sotomayor (2007). The non-standard part is the stability across the three quotas and the possibility of capacity redistribution. To get some intuition (the formal proof follows below) for the result, suppose a top-grade student a is matched in the first step of the ZVS procedure but could be admitted at some strictly preferred university u in step 3 (given the set of applicants admitted in that step). In this case a could profitably deviate by ranking only u for each step of the procedure: Since a was matched in the first step under her original report, only a subset of applicants have to wait for step 3 when she deviates in the just described way. But this implies that all universities make more offers in the CDA of step 3 and in particular a must receive an offer by u. Before proceeding to the formal proof of Proposition 1, it is illustrative to calculate the set of stable matchings for a simple example.

Example 1. *Consider again the example of section 3.1. To calculate the set of stable matchings, we first need to complete the preferences of universities for step U to a ranking of all applicants. I assume that*

$$\succ^3_{u_1}\colon\ a_1,\ a_4,\ a_5,\ a_2,\ a_6,\ a_3,\ a_7,\ a_8,\ a_9$$
$$\succ^3_{u_2}\colon\ a_1,\ a_6,\ a_2,\ a_3,\ a_5,\ a_4,\ a_7,\ a_8,\ a_9$$
$$\succ^3_{u_3}\colon\ a_1,\ a_4,\ a_5,\ a_2,\ a_6,\ a_3,\ a_7,\ a_8,\ a_9$$

Note that restricted to a_4, a_5, a_6 these orderings coincide with the preferences of universities given in section 3.1. Preferences of applicants and the priority structures for steps 1 and 2 are as given in section 3.1. For this specification, there are exactly two stable matchings.

$$\mu_1^1 = \begin{array}{ccc} u_1 & u_2 & u_3 \\ a_1 & \emptyset & a_3 \end{array}, \qquad \mu_1^2 = \begin{array}{ccc} u_1 & u_2 & u_3 \\ a_8 & a_9 & a_7 \end{array}, \qquad \mu_1^3 = \begin{array}{ccc} u_1 & u_2 & u_3 \\ a_2 & \{a_4, a_5\} & a_6 \end{array}, \text{ and}$$

$$\mu_2^1 = \begin{array}{ccc} u_1 & u_2 & u_3 \\ & & \\ a_1 & a_3 & a_2 \end{array}, \qquad \mu_2^2 = \begin{array}{ccc} u_1 & u_2 & u_3 \\ & & \\ a_8 & a_9 & a_7 \end{array}, \qquad \mu_2^3 = \begin{array}{ccc} u_1 & u_2 & u_3 \\ & & \\ a_4 & a_6 & a_5 \end{array}.$$

It is easy to see that these are the only stable matchings. By Proposition 1, there are thus two pure strategy equilibrium outcomes of the revelation game among applicants. Note that all applicants weakly prefer μ_1 over μ_2. One strategy profile that implements the first matching is the following: All top-grade applicants rank only their most preferred university for step 1 and submit their true ranking for step 3. Wait-time applicants rank only their assignment under the stable matching μ_1 for step 2. The remaining three applicants rank only their most preferred university for step 3. For a_2 this means that she truncates *her true preferences so that she will stay in the procedure until step 3, where she can be assigned a place at her most preferred university u_1 given the reports of the others. Wait-time applicants on the other hand, overreport their preferences in fear of falling through the cracks in the Boston mechanism of step 2 and knowing that their chances of obtaining a more preferred university in step 3 are slim. The empirical analysis of Braun, Dwenger, and Kübler (2008) suggests that such strategies are indeed used since (i) about 28 percent of the places available to top-grade applicants are not filled in step 1 because selected applicants submit very short preference lists, and that (ii) there are almost no places reserved for, but not filled in, step 2.*

However, note that a_2 is guaranteed to obtain a place at u_3 in step 1 if she ranks this university first - irrespective of the reports of the other applicants. On the other hand, a_2 has to rely on others to follow the right equilibrium strategy in order to reach the Pareto dominant equilibrium. In particular, she might receive a place at her third choice u_2 if others strategize suboptimally.[17] *In this sense the Pareto dominant equilibrium is more risky for a_2 so that she might be inclined to use the safe strategy of overreporting her preference for u_3 in step 1.*

Proof of Proposition 1:

We show first that for any profile of strict applicant preferences R, if Q is a pure strategy Nash equilibrium of $\Gamma^Z(R)$ then $f^Z(Q)$ must be a stable matching for the university admissions problem R. The only non-trivial part of the proof is to show that $f^Z(Q)$ satisfies (iv). For economy of notation let $f^{ZVS}(Q) = (\mu^1, \mu^2, \mu^3)$ and q^3 be the corresponding capacity vector for step 3.

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[17] This would happen, if e.g. a_6 ranks u_2 higher than u_3. In contrast to standard college admission problems, this is not necessarily a dominated strategy for a_6 here since u_2 may consider only students who ranked it first. Such *ranking constraints* are popular with German universities. As discussed at the end of this section, the presence of such constraints does not change the equilibrium characterization.

Suppose to the contrary there is an applicant-university pair (a, u) such that (iv) is violated for some $t \in \{1, 2, 3\}$. Let $\tilde{Q}_a$ be an alternative report for applicant a that lists only u for each step of the procedure. Let $\tilde{Q} = (\tilde{Q}_a, Q_{-a})$, $f^Z(\tilde{Q}) = (\tilde{\mu}^1, \tilde{\mu}^2, \tilde{\mu}^3)$, and $\tilde{q}^3$ be the corresponding capacity vector for step 3. It is clear that unless $\tilde{\mu}(a) = a$, $\tilde{Q}_a$ is a profitable deviation for a. I now show that $\tilde{\mu}(a) = a$ is impossible. If $t \in \{1, 2\}$, we must clearly have $\tilde{\mu}^t(a) = u$ since by (iv) at most $q^t_u - 1$ eligible applicants with higher $\succ^t_u$-priority than a could have listed u as their first choice. So suppose that $t = 3$. Let A^3 and $\tilde{A}^3$ denote the sets of applicants apart from a who remain in the procedure by the beginning of step 3 under Q and $\tilde{Q}$, respectively. Suppose to the contrary that $\tilde{\mu}(a) = a$. Note that this implies $\tilde{A}^3 \subseteq A^3$, $\tilde{q}^3_u = q^3$, $\tilde{q}^3_v \geq q^3_v$ for all $v \in U \setminus \{u\}$, and, since (iv) is violated w.r.t. $t = 3$, $|\tilde{\mu}^3(u)| = \tilde{q}^3$. This implies that if a does not receive an offer by u in the course of the CDA under $\tilde{Q}$, all applicants in $\tilde{A}^3$ receive a superset of the set of offers they got when the profile of submitted preferences was Q. In particular, any applicant in $\tilde{A}^3$ who received and declined an offer by u in the CDA under Q will also decline an offer by u in the CDA under $\tilde{Q}$. This implies $|\mu^3(u)| = q^3_u$. Since (iv) is violated, there has to be an applicant $\tilde{a} \in \mu^3(u)$ such that $a \succ^3_u \tilde{a}$. Given the above, for any applicant $\hat{a}$ such that $\hat{a} \succ^3_u \tilde{a}$ and $\hat{a} \notin \mu^3(u)$ we must also have $\hat{a} \notin \mu^3(u)$. But then $\tilde{\mu}^3(u)$ has to contain at least one applicant with strictly lower $\succ^3_u$-priority than a so that u would have made an offer to a in the CDA under $\tilde{Q}$, a contradiction.

Now let $\mu = (\mu^1, \mu^2, \mu^3)$ be a stable matching in the university admissions problem R. If $\mu(a) = a$, let a rank her six most preferred acceptable universities according to R_a steps 1 and 3, and all acceptable universities for step 2. If $\mu(a) = u$, let a rank only u for all parts of the procedure. Let Q be the resulting strategy profile.

We show first that $f^Z(Q) = \mu$. Let $f^Z(Q) = (\tilde{\mu}^1, \tilde{\mu}^2, \tilde{\mu}^3)$. It is easy to see that (iii) and (iv) imply $\tilde{\mu}^1 = \mu^1$ and $\tilde{\mu}^2 = \mu^2$. Given this, any applicant a with $\mu^3(a) \in U$ will not be assigned in steps 1 or 2 of the ZVS procedure under Q. Since all of these applicants rank only their assigned university under μ^3 for step 3 while all other unassigned applicants rank their six most preferred acceptable universities (w.r.t. R), (iv) would be violated if one of the unassigned applicants received a place in step 3 of the ZVS procedure under Q.

Next, we show that Q is a Nash equilibrium profile. Let $\tilde{Q}_a$ be an alternative report for applicant a, $\tilde{Q} = (\tilde{Q}_a, Q_{-a})$, and $f^Z(\tilde{Q}) = (\tilde{\mu}^1, \tilde{\mu}^2, \tilde{\mu}^3)$. Note that the sets of top-grade and wait-time applicants are the same as under Q since, for $t \in \{1, 2\}$, $\tilde{Q}^t_a$ contains at least one university by A1. Now suppose to the contrary that $\tilde{\mu}(a) \tilde{P}_a \mu(a)$. It cannot be the case that $\tilde{\mu}^t(a) = u = \mu^3(a)$ for some $t \in \{1, 2\}$ since all applicants in $\mu^t(u)$ apply to u in the first round of the Boston mechanism under $\tilde{Q}$ so that μ could not satisfy (iii) or (iv) otherwise. A similar argument shows that a cannot obtain a university strictly R_a-preferred to $\mu(a)$ in steps 1 or 2. It remains to be shown that a cannot strictly prefer $\tilde{\mu}^3(a)$ over $\mu(a)$. Consider first an applicant a such that $\mu^1(a) = \mu^2(a) = a$. Given that μ satisfies (iii) and (iv) (and the construction of Q), no alternative report $\tilde{Q}_a$ that leads to different assignments in steps 1 or 2 can be profitable for a. We can hence assume w.l.o.g. that $\tilde{Q}_a^1 = Q_a^1$ and $\tilde{Q}_a^2 = Q_a^2$. But then if $\tilde{\mu}^3(a) P_a \mu^3(a)$ we obtain an immediate contradiction to (iv) for $t = 3$ since all applicants in $\mu^3(u)$ are available in step 3 of the ZVS procedure under $\tilde{Q}$ and rank only u. Now consider an applicant a such that for some $u \in U$, $\mu^1(a) = u$ (a completely analogous argument takes care of the case where an applicant is matched in the second step). By A3, a can never obtain a place at some university in step 2. By (iv) and the construction of Q, there is no alternative report for a such that she obtains a strictly R_a-preferred university in step 1. Thus the only way that a could potentially improve upon her assignment under μ is that $\tilde{\mu}^1(a) = a (= \tilde{\mu}^2(a))$. But the only applicants who could take the leftover seat at u in step 1 are those who are either unassigned under μ or who are matched to u under μ^3. In particular, for all universities $v \neq u$, all applicants in $\mu^3(v)$ remain in the procedure by the beginning of step 3 under $\tilde{Q}$ and $\tilde{q}_v^3 = q_v^3$. If $\tilde{\mu}^3(a) P_a u$, we must thus obtain a contradiction to (iv). This completes the proof.

□

Proposition 1 shows that the potential instabilities of ZVS procedure we saw in the example of section 2.1 are "corrected" by the strategic behavior of applicants. We will later see that the set of stable matchings for the university admissions problem coincides with the set of stable matchings for a related college admissions problem with substitutable preferences. Since for such problems there exists a student optimal stable matching that all students (weakly) prefer to any other stable matching, this implies that the ZVS procedure supports Pareto dominated outcomes. This will prove to be a useful benchmark for a comparison between my proposed redesign and the current procedure in section 5.

We close this section with a discussion of the influence of universities' evaluation criteria on the equilibrium characterization of Proposition 1. In the actual ZVS procedure universities are allowed to use their rank in the preference lists of applicants submitted for step 3. For example, a university may decide to consider only applicants who ranked it first and order these applicants according to their average grade. Assumption 1 above could be interpreted as saying that the other criteria a university u uses apart from such *ranking constraints* induces the strict ranking $\succ_u^3$. In the just mentioned example $\succ_u^3$ would simply list applicants in decreasing order of average grade. The ranking $\succ_u^3$ can thus be understood as the ranking that would result if all applicants had ranked u sufficiently high to satisfy its ranking constraint. While I have abstracted from ranking constraints in the above, Proposition 1 remains valid without this restriction. To see this note that in the above any stable matching can be achieved by a strategy profile in which matched applicants only rank their assigned university and the deviations used to show that any strategy profile must be stable similarly use a preference list of length one. Thus, the ranking constraints set by universities never come into play in a complete information equilibrium. An interesting corollary of the above result is therefore that neither ranking constraints nor the constraint that applicants can rank at most six universities for steps 1 and 2, have any effect on the set of matchings that are attainable as equilibrium outcomes. The type of strategy used in the proof is of course very risky since it entails a potentially high probability of being left unassigned by the end of the procedure if applicants are only slightly mistaken about the preferences of universities and other applicants. The discussion is meant to point out that *if* applicants had a very reliable estimate of their chances of admission at each university *then* constraints would be irrelevant.

4.2 Incomplete Information

The last section considered the case of complete information. This informational environment is a reasonable approximation if universities mostly rely on objective measures such as average grades to evaluate applicants. However, calculating chances of admission is significantly harder for universities using subjective criteria such as interviews: First, there is usually very little to no data available on past admission decisions (for step 3) in this case. Secondly, applicants may find it hard to estimate their potential performance in an interview. Assuming complete information is thus more appropriate when relatively few universities use subjective criteria. In particular, the applicability of the results in the last section may depend on the subject under consideration: In the assignment procedure for pharmacy, only 2 out of 22 universities used subjective criteria to assign at least part of their capacity. On the other hand, 10 out of 34 universities in the assignment procedure for medicine gathered additional information about applicants by conducting interviews (see Appendix B). A full answer to the question of which outcomes can be expected from the ZVS procedure may thus require modeling applicant expectations about the interviewing process, about other applicants' preferences, and so on. Given the complexities of the ZVS procedure the problem of characterizing incomplete information equilibria can quickly become very difficult or even intractable. Instead of aiming for general results, I use two very simple examples to point out the possible complications caused by (i) the sequential allocation of places, and (ii) the possibility that universities use ranking constraints for step 3. These features will be abolished in my proposal for a redesign in the next section.

As before, I assume that applicants know the preferences and average grades of their peers. Applicant a's average grade is denoted by $g(a)$. In contrast to the last section, however, applicants are uncertain about their performance in interviews and thus their chances of admission at a university in step 3: Applicant a's performance in interviews is the same across universities and is summarized by a one-dimensional random variable θ_a with realizations in some finite set $\Theta \subset \mathbb{R}$. Interview performance is identically and independently distributed across applicants. All universities rank applicants on basis of the sum of average grade and performance in the interview: Applicant a ranks higher in u's preferences than applicant a' if and only if $g(a) + \tilde{\theta}_a < g(a') + \tilde{\theta}_{a'}$. Note that high realized values of θ are thus associated with bad performance in an interview. The first example outlines the problems of the sequential nature of the ZVS procedure.

Example 2 (The Problems of Sequential Allocation). *Suppose there are seven applicants* $a_1, \ldots, a_7$ *and three universities* u_1, u_2, u_3. *Applicants are ordered in increasing order of their average grades, that is,* $g(a_i) < g(a_j)$ *for* $i < j$. *For simplicity I concentrate on the assignment procedures for steps 1 and 3 in this example and assume that each university has only one seat to allocate in each part of the procedure. Hence,* a_1, a_2, a_3 *are the top-grade applicants. For this example, we assume that* $\Theta = \{-x, 0, x\}$, *where* $x > g(a_5) - g(a_2)$.[18] *Let* $p := Prob(\theta_a = -x)$, $q := Prob(\theta_a = 0)$, *and* $r := Prob(\theta_a = x)$. *Ordinal preferences of applicants are given by (only acceptable universities are listed)*

R	R_{a_1}	R_{a_2}	R_{a_3}	R_{a_4}	R_{a_5}	R_{a_6}	R_{a_7}
	u_1	u_1	u_2	u_1	u_2	u_3	u_3 .
		u_2					

For all applicants except a_2 *it is optimal to rank their only acceptable university for each step of the procedure.*[19] *Applicant* a_2*'s utilities for obtaining a place at* u_1 *and* u_2 *are given by* $\overline{V}$ *and* $\underline{V}$ *with* $\overline{V} > \underline{V} > 0$, *respectively. The utility of not being assigned a university is normalized to zero and the utility of being assigned to* u_3 *is negative. It is easy to see that* a_2 *should either try to secure a match in step 1 by ranking* u_2 *as her first choice for step 1, or*

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[18] This is not an unrealistic assumption when $g(a_5) - g(a_2)$ is not too large. In the ZVS procedure universities often use weighted averages of average grades and performance in interviews. It is not uncommon that a grade difference of, say, 0.3 points is reversed by performances in interviews.

[19] The other applicants can also be interpreted as representing a_2's beliefs about her competitors.

should rank only u_1 *as acceptable for step 1 and then submit her true ranking for step 3. The last statement follows since interview performance is the same across universities so that for any realization* u_1 *and* u_2 *have the same preferences over applicants.*

Suppose first that a_2 *decides to wait for step 3 and ranks only* u_1 *for step 1. Then the probability of obtaining a place at* u_1 *is* $p + q(1-p) + r^2$ *and the probability of obtaining a place at* u_2*, conditional on not obtaining a place at* u_1*, is* $qp(1-p) + r^2(1-r)$ *(remember that interview performance was assumed to be the same across universities). Hence,* a_2*'s expected value of waiting for step 3 is* $[p+q(1-p)+r^2]\overline{V} + [qp(1-p)+r^2(1-r)]\underline{V}$. *On the other hand, she will be matched to* u_2 *for sure if she ranks it as her first choice for step 1. Hence,* a_2 *will wait for step 3 if and only if* $[p + q(1-p) + r^2]\overline{V} + [qp(1-p) + r^2(1-r)]\underline{V} > \underline{V}$. *This can be rearranged to yield*

$$\frac{\overline{V}}{\underline{V}} > \frac{1 - [qp(1-p) + r^2(1-r)]}{p + q(1-p) + r^2}. \tag{1}$$

Note first that if the utility difference between obtaining first and second choice university is big, a_2 *is willing to accept a non-negligible risk of being left unassigned in step 3. For example, if* $\overline{V} = 3\underline{V}$ *and* $p = q = r$, a_2 *prefers to wait for assignment in step 3 even though there is a probability of* $qp^2 + r(1-r)^2 \approx 0.2$ *that she ends up unassigned. Thus, the sequential assignment procedure undermines the idea that top-grade applicants should not suffer too much from having a bad interview since they sometimes have to accept risky gambles in order to maximize their expected payoff from participating in the ZVS procedure.*[20]

Secondly, if the utility difference between obtaining first and second choice university is small, a_2 *will often prefer the safe option of taking a place at* u_2 *in step 1. Note that if* a_2 *refrains from taking part in step 3,* $p + q(1-p) + r^2$ *is the probability that* a_2 *would have been* u_1*'s top candidate had she not been assigned in step 1. This can be interpreted as the probability that* u_1 *ends up with the "wrong" applicant in step 3. Now suppose that* $\overline{V}/\underline{V} = 5/4$, $p = 0.5$, *and* $r = q = 0.25$. *From (1.1) we see that* a_2 *(weakly) prefers the safe option even though there is a chance of almost* 70 *percent that she receives a place at her most preferred university in step 3. Put differently, the probability that* u_1 *ends up with the wrong applicant in step 3 is close to* 70 *percent! This shows that from an* ex-ante *perspective the current ZVS procedure*

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[20] Braun, Dwenger, and Kübler (2008) report the case of an applicant with an average grade of 1.1 who did not receive a place in step 1 and who was subsequently rejected by all four universities he listed for step 3. An applicant with an average grade of 1.1 is usually among the top 2 percent of high school graduates. Thus, even for top-grade applicants there is a non-negligible risk of being left unassigned by the procedure if they fail to secure a place early in the procedure.

may produce outcomes that are very undesirable for universities.

Note that the problems outlined in example 2 are not peculiar to the precise form of the assignment procedure used in the sequential steps of the ZVS procedure. For example, the same problems would occur if instead the SDA procedure would be used in each step. In this sense, the above problems are direct consequences of the decision to allocate places in the three different quotas sequentially. Furthermore, note that the example does not even take into account that applicants might have a preference for being assigned in an early step of the procedure. The next example illustrates the problems associated with allowing universities to use *ranking constraints* which force applicants to rank a university sufficiently high if they want to be considered.

Example 3 (The problems of Ranking Constraints). *There are three applicants* a_1, a_2, a_3 *and two universities* u_1 *and* u_2*. As above, applicants are ordered in increasing order of average grades. For simplicity I consider only step 3 and assume that both universities have only one place to assign among applicants. We assume that* $\Theta = \{-y, 0, y\}$*, where* $y > g(a_3) - g(a_1)$*. Let* $p := Prob(\theta = -y)$*,* $q := Prob(\theta = 0)$*, and* $r = Prob(\theta = y)$*. Ordinal preferences of applicants are given by (as above only acceptable universities are listed)*

$$\begin{array}{c|ccc} R & R_{a_1} & R_{a_2} & R_{a_3} \\ \hline & u_1 & u_1 & u_2 \\ & u_2 & & \end{array} .$$

Applicant a_1*'s utilities for* u_1 *and* u_2 *are* $\overline{V}$ *and* $\underline{V}$ *with* $\overline{V} > \underline{V} > 0$*. Note that if no university uses ranking constraints then it is optimal for* a_1 *to submit her true ordinal ranking of universities for step 3. As above this follows from the assumption that interview performance is the same at all universities. This implies that if no university uses its rank in applications the outcome of step 3 must be stable with respect to the true preferences of participants. In this case the probability that* u_2 *is assigned its most preferred applicant (among the two applicants* a_1 *and* a_2 *it interviews) is* $q(1-p)p + r^2(1-r)$*, while* u_1 *always obtains its most preferred applicant (among* a_1 *and* a_3*). To see this note that (i) if* $\theta_{a_1} = -y$ *then* a_1 *is the top candidate of both universities with probability one so that* u_2 *will not get its most preferred applicant, (ii) if* $\theta_{a_1} = 0$ *then* a_1 *is the top candidate of* u_2 *if* $\theta_{a_3} \geq 0$ *but* u_2 *is assigned* a_1 *only if* $\theta_{a_2} = -y$*, and (iii) if* $\theta_{a_1} = y$ *then* a_1 *is the top candidate of* u_2 *if* $\theta_{a_3} = y$ *but* u_2 *is assigned* a_1 *only if* $\theta_{a_2} \leq 0$*. Note that the expected utility of applicant* a_1 *is given by* $[p + q(1-p) + r^2]\overline{V} + [qp(1-p) + r^2(1-r)]\underline{V}$ *as in Example 2.*

Now suppose u_2 *declares that it only considers applicants who ranked it first. In this case* a_2 *has to decide whether to report her first choice truthfully and thus forsake chances of admission at* u_2*, or to rank* u_2 *as her first choice university. Note that* a_2*'s expected utility of (truthfully) ranking* u_1 *first is now* $[p+q(1-p)+r^2]\overline{V}$*, since she would not be considered by* u_2 *if she fails to secure a place at* u_1*. On the other hand her expected utility for misrepresenting her first choice and ranking* u_2 *first is* $[p+q(1-p)+r^2]\underline{V}+[qp(1-p)+r^2(1-r)]\overline{V}$*. She will thus misrepresent her first choice if and only if* $[p+q(1-p)+r^2]\overline{V} < [p+q(1-p)+r^2]\underline{V}+[qp(1-p)+r^2(1-r)]\overline{V}$*. This is satisfied for example if* $p = q = r$ *and* $\overline{V}/\underline{V} = 5/4$*. For this parameter constellation,* u_2 *would increase the probability that it receives its most preferred applicant from* $4/27$ *to one by forcing* a_1 *to rank it first. Hence,* u_2 *has a strict incentive to employ this tool if we assume that* u_2 *has a strictly higher utility for obtaining its most preferred applicant. Note that for the above parameter constellation this means that* ex-post *there is a blocking pair (*a_1 *and* u_2*) for the ZVS outcome with probability* $p+q(1-p)^2+r^3 \approx 0.5$.

There is one major benefit of allowing universities to use ranking constraints: If a university only considers applicants who ranked it first, its evaluation efforts are never wasted since any offer it makes must be accepted. Without ranking constraints, universities would potentially have to evaluate many more candidates to fill their places. This is problematic if the marginal cost of evaluating an additional candidate is not negligible as in the case of rankings that are based on interviews. Nevertheless, ranking constraints can hardly be seen as a satisfactory solution since they force applicants to forsake valuable chances of admission. For example, eight of the universities offering medicine only consider applicants who ranked them first (see Appendix B). Thus, an applicant interested in only these universities effectively has to decide on *one* university to rank for step 3 of the ZVS procedure. Furthermore, note that there were no explicit interviewing costs in the above example and yet u_2 had an incentive to restrict attention to applicants who ranked it first. This shows that it is not necessarily a cost saving motive that drives universities' incentives to employ ranking constraints. There is empirical evidence that universities make excessive use of such constraints: Many universities use *mechanical* evaluation procedures where the ranking of applicants can be easily computed from characteristics such as average grade. The marginal cost of "evaluating" an additional candidate is thus negligible. Still, many of the universities with mechanical evaluation procedures use ranking constraints. In medicine, for example, 16 out of the 34 universities have a (mostly) mechanical evaluation procedure and accept only applicants who ranked them "sufficiently" high (see Appendix B). Here, the ability to use ranking constraints lets universities take unduly advantage of the centralized procedure in the sense that they are able to elicit binding commitments from applicants that would not be possible in a decentralized procedure.

5 Towards a New Design

The main goal of this section is to develop a redesign of the German university admissions system. My approach is to keep the extended university admissions environment as defined in section 4.1 fixed and to construct an alternative mechanism within this environment. More specifically, I maintain the following features of the current assignment procedure:

(i) Each university's capacity is divided into three parts: One for top-grade applicants, one for wait-time applicants, and one for which universities can evaluate candidates.

(ii) Places not taken by top-grade and wait-time applicants can be allocated on basis of criteria chosen by universities among all applicants.

(iii) Places for top-grade and wait-time applicants are allocated on basis of $\succ^1$ and $\succ^2$.

In my redesign I focus on the welfare and incentives of applicants and continue to assume that universities do not act strategically. For conreteness, and in order to allow a comparison with the currently employed procedure, I assume that universities' preferences (used for those seats that they can allocate according to their own criteria) can be expressed by the same profile of rankings $\succ^3$ that was used in the analysis of the current procedure in section 4. The goal will be to design a stable, in the sense of Definition 1, and strategy-proof procedure that is as favorable as possible to applicants. Stability ensures that top-grade and wait-time applicants never lose their priority for a place in their quota, and that universities' preferences (as measured by $\succ^3$) are respected. The requirement that applicants should be matched as early as possible can be understood as the desire to reach a 1:1:3 distribution of students admitted through the top-grade, wait-time, and the university quota, respectively.

Given the goal of a stable admissions procedure, it is clear that strategyproofness demands that all places be allocated simultaneously. Furthermore, strategyproofness requires that applicants submit a single ranking of universities, are allowed to rank as many universities as they want,[21] and that universities are not allowed to use ranking constraints. This implies that the evaluation procedures of universities have to be conducted before any assignments are determined and that universities are not allowed to condition their invitations for interviews on their rank in applications. While this may lead to some wasted investment in the evaluation process

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[21] For an analysis of the strategic incentives induced by the SDA algorithm when the length of submitted preference lists is limited see Romero-Medina (1998) and Haeringer and Klijn (2009).

(since students that are evaluated by a university may now decline an offer if made), this is a sacrifice one has to make in order to guarantee dominant strategy incentives. Furthermore, such costs would (and do) also occur in a decentralized admission system so that an attempt to base interview invitations on applicants' rankings lets universities take unduly advantage of a centralized admission system.

The design problem at hand then shares important characteristics with the *affirmative action problem* that has been analyzed by e.g. Abdulkadiroglu and Sönmez (2003) in the context of school choice/college admissions. In this problem schools not only have a contstraint on the total number of students they can admit, but in addition are imposed *type-specific quotas*. These quotas can be *rigid*, in the sense that places in a type-specific quota are absolutely reserved for students with the corresponding type (which could be sex, ethnicity, and so on), or *flexible*, in the sense that they place an upper bound on the number of students of a certain type.[22] Abdulkadiroglu and Sönmez (2003) and Abdulkadiroglu (2005) show that the student proposing deferred acceptance algorithm can easily accommodate both flexible and rigid controlled choice constraints. In the German university admissions system a student can be either a top-grade, or a wait-time type (or can belong to neither of these groups). Type-specific constraints are rigid in the sense that places in the top-grade (wait-time) quota are absolutely reserved for top-grade (wait-time) students. However, the system allows capacity to be shifted from both type-specific quotas to the quota for which all applicants are potentially eligible (and places are allocated according to $\succ^3$). In the following I refer to the possibility of shifting places from one (rigid) type-specific quota to another as *floating quotas*. To the best of my knowledge the case of rigid but floating quotas has not been analyzed in the existing literature for the case where there is also a total capacity constraint. In the next section I therefore develop a theory of affirmative action with floating quotas and then show how this theory can be applied to design an appealing assignment procedure for the German university admissions problem.

5.1 Affirmative Action with Floating Quotas

A *school choice problem with affirmative action and floating quotas* (*affirmative action with floating quotas problem* in the following) is a version of the school choice problem in which students have (potentially) multi-dimensional types and schools have type specific quotas that are absolutely reserved for qualifying students. In contrast to the standard affirmative action

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[22] For a model in which each school has a lower and an upper bound for each type see Abdulkadiroglu and Ehlers (2007).

problem, types are considered in some pre-specified order and, more importantly, capacity can be redistributed among type-specific quotas in response to insufficient demand by some student types. More formally, an affirmative action with floating quotas problem is given by

- a finite set of schools C and a finite set of students I,
- a finite set of possible student types Θ,
- a correspondence $\tau : I \to \Theta$, where $\tau(i) \subseteq \Theta$ is the set of types that $i \in I$ inherits,
- for each college c a *choice protocol* consisting of
  (i) a sequence $\theta_c = (\theta_{(c,t)})_{t=1}^{T_c}$ in Θ,
  (ii) a vector $\succ_c = (\succ_{(c,\theta)})_{\theta\in\Theta}$, where $\succ_{(c,\theta)}$ is a strict ordering of $I \cup \{c\}$ such that $c \succ_{(c,\theta)} i$ for all i such that $\theta \notin \tau(i)$,
  (iii) a sequence of capacities $q_c = (q_{(c,t)})_{t=1}^{T_c}$, where $q_{(c,1)} \in \{0, \ldots, |I|\}$ and $q_{(c,t)} : \{0, \ldots, |I|\}^{t-1} \to \{0, \ldots, |I|\}$ for all $t \geq 2$,
- a profile of strict student preferences $R = (R_i)_{i\in I}$.

This formulation allows schools to control for the distribution of entering classes in multiple dimensions. In such applications, a student's type could be composed of sex, ethnicity, ability levels (as measured by some standardized test), and so on. In case of a firm(school) consisting of a set of heterogenous jobs, a worker(student)'s type may describe for which of the jobs he or she is qualified.[23] The priority orderings $\succ_{(c,\theta)}$ may or may not be the same for all student types: In applications to school choice, students are often ordered according to social criteria and the same criteria apply for all student types. In other applications, $\succ_{(c,\theta)}$ may rank applicants according to their suitability for position θ, so that an applicant's priority for a given school varies across quotas. Note that a student i with $\theta \in \tau(i)$ may still be unacceptable to c with respect to $\succ_{(c,\theta)}$. For example, a school that has reserved some of its seats for male students might evaluate according to the outcome of some standardized test and only consider applicants whose scores exceed some threshold. The capacity sequence starts with a fixed number of seats $q_{(c,1)}$ reserved for the first student type to be considered $\theta_{(c,1)}$, the number of seats $q_{(c,1)}(r_1)$ for the second student type to be considered $\theta_{(c,2)}$ depends on the number of seats r_1 that were not allocated due to insufficient demand by $\theta_{(c,1)}$ students, and so on.

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[23] For this and other applications it is clearly conceivable that students/workers are not indifferent to which type of seat/job they obtain at a given school/firm. We leave this interesting extension for future work.

A choice protocol (uniquely) defines schools' admission procedures that we now introduce formally. In the following, given a set of students $J \subseteq I$ we denote by $J(\theta)$ the set of all students $i \in J$ such that $\theta \in \tau(i)$. Given a set of applying students $J \subseteq I$ the *admission process* of a school c *induced by the choice protocol* $(\theta_c, \succ_c, q_c)$ works as follows (we omit the index for school c to simplify notation in the following):

> In the first round admit the q_1 highest ranking acceptable students in $J(\theta_1)$ according to $\succ_{\theta_1}$ (or all acceptable students in $J(\theta_1)$ if there are fewer than q_1). Denote the set of admitted students by $J_1(J)$,[24] the number of unused seats by $r_1(J) = q_1 - |J_1(J)|$, and let $J^2(J) = J \setminus J_1(J)$.
>
> $$\vdots$$
>
> In the tth round admit the $q_t(r_1(J), \ldots, r_{t-1}(J))\}$ highest ranking acceptable students in $J^t(\theta_t)$ according to $\succ_{\theta_t}$ (or all acceptable students in $J^t(\theta_t)$ if there are fewer than $q_t(r_1(J), \ldots, r_{t-1}(J))\}$). Denote the set of admitted students by $J_t(J)$, the number of unsed seats by $r_t(J) = q_t(r_1(J), \ldots, r_{t-1}(J)) - |J_t(J)|$, and let $J^{t+1}(J) = J^t(J) \setminus J_t(J)$.
>
> $$\vdots$$

The set of students admitted by c according to $(q_c, \succ_c, \theta_c)$ when the set of applicants is J is then $J_1 \cup \ldots \cup J_{T_c}$. Note that a given student type may be considered multiple times by a choice protocol: A student who is rejected in some round of the admission process since all seats in some type-specific quota were allocated to higher ranking students may benefit from capacity shifts induced by insufficient demand from other student types.

In many applications, a school c will have a fixed upper bound $\overline{q}_c$ on the total number of students and some initial capacity distribution $(q_{(c,\theta)})_\theta$ with $\sum_{\theta \in \Theta} q_{(c,\theta)} = \overline{q}_c$ which represents c's intended type distribution. In this case a choice protocol $(\theta_c, \succ_c, q_c)$ is *adapted to the intended type distribution* if $q_{(c,t)}(0, \ldots, 0) = q_{(c,\theta_t)}$ if $\theta_{(c,s)} \neq \theta_{(c,t)}$ for all $s \leq t$. Apart from this, a choice protocol specifies in particular how capacity moves between type-specific quotas in case the intended type-distribution cannot be achieved. This is important since it may not be desirable (or even possible in the case of public schools) to reject students for the sake of affirmative action constraints even though not all places were filled. However, a school may prefer some deviations from the intended type distribution over others and choice protocols allow schools to express such preferences. We now illustrate this and the concepts introduce above with a simple application.

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[24] We supress the dependency of $J_1(J)$ on the choice protocol for economy of notation.

Application 1 (Two-dimensional affirmative action constraints)**.** *Consider a school interested in achieving an even distribution of ethnicities* <u>*and*</u> *sexes and, if this goal cannot be met, desires an even distribution of sexes.*

Suppose for simplicity that students can be either black (B) or white (W) and that there is just one school with four seats in total.[25] *The type space can then be described by* $\Theta = \{BM, BF, WM, WF, M, F\}$ *and each student's type* $\theta(i)$ *has the property that* $|\theta(i)| = 2$.

The initial type specific quota vector is defined by $q_{BM} = q_{BF} = q_{WM} = q_{WF} = 1$ *and* $q_M = q_W = 0$. *Suppose the order in which types are to be considered is* $(\theta^1, \ldots, \theta^6) = (BF, WF, BM, BM, W, M)$. *In order to specify the sequence of capacities it is sufficient to define* q^5 *given the goal described above. The above policy can be implemented by setting* $q^5(r^1, \ldots, r^4) = (0, 0, 0, 0, k, l)$ *with* $k = r^1 + r^2$ *and* $l = r^3 + r^4$. *To see this note that if, say,* $r^1 = 0$ *and* $r^2 = 1$ *then one female slot has been taken so that, if the four-seat school desires an equal distribution of sexes, then one more place can be allocated to a female.*

We now introduce three important conditions on choice protocols.

Definition 2. *(i) If* c *is a school with total capacity* $\overline{q}_c$*, then* $(\theta_c, \succ_c, q_c)$ ***respects capacity constraints*** *if for all* t *and all sequences* $(r_s)_{s=1}^{t-1}$, $\sum_{\theta \in \Theta} q_{(c,t)}(r_1, \ldots, r_{t-1}) \leq \overline{q}_c - \sum_{s=1}^{t-1} (q_{(c,s)}(r_1, \ldots, r_{s-1}) - r_s)$.

(ii) A choice protocol $(\theta_c, \succ_c, q_c)$ *is* ***monotonic****, if for all* t *and all pairs of sequences* $(r_s, \tilde{r}_s)_{s=1}^{t-1}$ *such that* $r_s \geq \tilde{r}_s$ *for all* $s \leq t-1$, $q_{(c,t)}(r_1, \ldots, r_{t-1}) \geq q_{(c,t)}(\tilde{r}_1, \ldots, \tilde{r}_{t-1})$.

(iii) A choice protocol $(\theta_c, \succ_c, q_c)$ *is* ***consistent****, if for all* t *and all pairs of sequences* $(r_s, \tilde{r}_s)_{s=1}^{t-1}$ *such that for* $s \leq t-1$, $r_s = \tilde{r}_s + k_s$ *for some* $k_s \geq 0$,

$$q_{(c,t)}(r_1, \ldots, r_{t-1}) - q_{(c,t)}(\tilde{r}_1, \ldots, \tilde{r}_{t-1}) \leq \sum_{s=1}^{t-1} [k_s - (q_{(c,s)}(r_1, \ldots, r_{s-1}) - q_{(c,s)}(\tilde{r}_1, \ldots, \tilde{r}_{s-1}))].$$

The first requirement simply says that a school should never distribute more capacity than it has. To see this note that $q_{(c,s)}(r_1, \ldots, r_{s-1}) - r_s$ is the number of students who receive a place at c in round s of its admission procedure. The second requirement postulates that the number of places available in some round t of the admission process should be weakly increasing in the number of unassigned seats in rounds 1 through $t-1$. The third requirement says that in response to greater demand in rounds 1 through $t-1$, students should not suffer too much in the sense that the total capacity reduction in rounds 1 up to and including t does not exceed the increase in demand $\sum_{s=1}^{t-1} k_s$. As I will now show, the main thrust of

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[25] The formulation can easily be generalized to the case of n ethnicities.

these properties is that they allow the affirmative action with floating quotas problem to be transformed into an *associated college admissions problem* for which mechanisms with desirable allocative and incentive properties exist. Given a subset $J \subseteq I$, let $Ch_c(J) \equiv Ch(J|\theta_c, \succ_c, q_c)$ be the set of students admitted by school c according to the choice protocol, that is, $Ch_c(J) = J_1(J) \cup \ldots \cup J_{T_c}(J)$, where $J_t(J)$ is the set of students admitted in round t of c's admission process. We have the following.

Proposition 2. *(i) If (θ^c, q^c) is monotone, $Ch_c(\cdot)$ is substitutable.*

(ii) If (θ^c, q^c) is also consistent, $Ch_c(\cdot)$ also satisfies the law of aggregate demand.

Proof:

(i) Since we consider a fixed school c in this proof, we again omit the school index to economize on notation. Suppose $\tilde{J} \subseteq J \subseteq I$. Let $(\tilde{J}_t, \tilde{r}_t, \tilde{J}^{t+1})$ and (J_t, r_t, J^{t+1}) denote the associated sequences of admitted students, remaining capacities, and remaining students for each round of the choice protocol. We show by induction that $(J_t \cap \tilde{J}^t) \subseteq \tilde{J}_t$, which proves the statement.

The statement is trivial for $t = 1$. Furthermore, it is easy to see that we must have $r_1 \leq \tilde{r}_1$ and $\tilde{J}^2 \subseteq J^2$. So suppose that for some $t \geq 1$, $(J_s \cap \tilde{J}^s) \subseteq \tilde{J}_s$, $r_s \leq \tilde{r}_s$, and $\tilde{J}^{s+1} \subseteq J^{s+1}$ for all $s \leq t$. We show that the same statements hold for $t + 1$ as well.

Note first that by the inductive assumption and monotonicity we obtain that $q_{t+1} := q_{t+1}(r_1, \ldots, r_t) \leq q_{t+1}(\tilde{r}_1, \ldots, \tilde{r}_t) =: \tilde{q}_{t+1}$. This already yields $(J_{t+1} \cap \tilde{J}^{t+1}) \subseteq \tilde{J}_{t+1}$. To see this note that the inductive assumption of $\tilde{J}^{t+1} \subseteq J^{t+1}$ together with $\tilde{q}_{t+1} \geq q_{t+1}$ implies that the $\tilde{q}_{t+1}$th lowest ranking applicant in $\tilde{J}_{t+1}$ with respect $\succ_{\theta_{t+1}}$ must rank weakly lower than the q_{t+1}th lowest ranking applicant in J_{t+1}.

It remains to be shown that $r_{t+1} \leq \tilde{r}_{t+1}$ and $\tilde{J}^{t+2} \subseteq J^{t+2}$. By definition $\tilde{r}_{t+1} - r_{t+1} = \tilde{q}_{t+1} - q_{t+1} + |J_{t+1}| - |\tilde{J}_{t+1}|$. If $|J_{t+1}| < q_{t+1}$, note that $\tilde{J}^{t+1}(\theta_{t+1}) \subseteq J^{t+1}(\theta_{t+1})$ implies $|J_{t+1}| = |J^{t+1}(\theta_{t+1})| \geq |\tilde{J}^{t+1}(\theta_{t+1})|$. Since $\tilde{q}_{t+1} \geq q_{t+1}$ this implies $|J_{t+1}| \geq |\tilde{J}^{t+1}(\theta_{t+1})| = |\tilde{J}_{t+1}|$ and we obtain $\tilde{r}_{t+1} - r_{t+1} \geq \tilde{q}_{t+1} - q_{t+1} \geq 0$. If $|J_{t+1}| = q_{t+1}$, we obtain $\tilde{r}_{t+1} - r_{t+1} = \tilde{q}_{t+1} - |\tilde{J}_{t+1}| \geq 0$. Finally, suppose to the contrary that there is some student $i \in \tilde{J}^{t+2} \setminus J^{t+2}$. Since $\tilde{J} \subseteq J$, this implies $i \in (J_1 \cup \ldots \cup J_{t+1}) \cap \tilde{J}$. But we have already shown that $J_s \cup \tilde{J}^s \subseteq \tilde{J}_s$ for all $s \leq t + 1$ so that $i \in \tilde{J}_1 \cup \ldots \cup \tilde{J}_{t+1}$.

(ii) As shown in the proof of statement (i), for all $t \geq 1$, we must have $\tilde{r}_t = r_t + k_t$ for some $k_t \geq 0$, if $\tilde{J} \subseteq J$. Now note that $|J_s| = q_s - \tilde{q}_s + |\tilde{J}_s| + k_s$ by the definition of $\tilde{r}_s$ and r_s.

This implies

$$\begin{aligned}\sum_{s=1}^{t}|J_s| &= \sum_{s=1}^{t}|\tilde{J}_s| + \sum_{s=1}^{t}[k_s - (\tilde{q}_s - q_s)]\\ &\geq \sum_{s=1}^{t}|\tilde{J}_s| + \tilde{q}_{t+1} - q_{t+1}\\ &\geq \sum_{s=1}^{t}|\tilde{J}_s|,\end{aligned}$$

where the first inequality follows from consistency and the second inequality follows from monotonicity. Hence, $|Ch(J)| = \sum_{s=1}^{T}|J_s| \geq \sum_{s=1}^{T}|\tilde{J}_s| = |Ch(\tilde{J})|$, which proves the statement.

□

The last result shows that a monotonic and consistent choice protocol induces a well defined college admissions problem. In particular, letting $f^I(R|(\theta^c, q^c, \succ^c)_{c\in C})$ denote the student optimal stable matching mechanism for the associated college admissions problem (abbreviated by SOSM henceforth) we obtain the following corollary.

Corollary 1. *Suppose $(\theta_c, \succ_c, q_c)_{c\in C}$ is monotonic and consistent.*

(i) For any profile of applicant preferences R, $f^I(R|(\theta_c, \succ_c, q_c)_{c\in C})$ is the student optimal stable matching in the associated college admissions problem.

(ii) $f^I(\cdot|(\theta_c, \succ_c, q_c)_{c\in C})$ is group strategy-proof for students.

Note that in each round of the SOSM students apply to their most preferred school/college among those that have not rejected her yet and each school/college then applies its admission process to decide which of the students applying to it are temporarily accepted. Since types are allowed to be multi-dimensional it is possible that a student i is temporarily accepted by c in some round t of the SOSM via the type specific quota for θ and temporarily accepted in some later round t' of the SOSM via the quota for some $\theta' \neq \theta$. Furthermore, note that the choice protocol in application 1 is consistent and monotonic. This shows in particular that stable admission procedures with desirable incentive properties can exist even when multi-dimensional affirmative action constraints are imposed on a school choice problem.

Before procceding it is instructive, not only for our application to the German university system, to develop an alternative representation of stability in the associated college admissions problem which uses the structure of the affirmative action with floating quotas problem. For notational simplicity I assume from now on that all schools consider student types in the same ordering, that is, $\theta_{(c,t)} \equiv \theta_t$ for all c and t. We begin with the following definition of matchings adapted to the choice protocol, which describes *who receives which place at which college.*

Definition 3. *A* ***matching adapted to the choice protocol*** *is a T-tuple of matchings $\mu = (\mu_1, \dots, \mu_T)$, such that*

- *(i) for all t, $\mu_t(i) \in C \cup \{i\}$,*
- *(ii) for all $i \in I$, $|(\cup_t \mu_t(i)) \cap C| \leq 1$,*
- *(iii) for all t and c, $|\mu_t(c)| \leq q^t(|\mu_1(c)|, \dots, |\mu_{t-1}(c)|)$, and*
- *(iv) for all t, $i \in \mu_t(c)$ if and only if $c = \mu_t(i)$.*

Note that if c has an upper bound $\overline{q}_c$ on the number of students it can admit and its choice protocol respects capacity constraints, then (iii) implies $| \cup_t \mu_t(c)| \leq q_c$. Next, I introduce an alternative definition of stability which takes into account that students might be evaluated according to different criteria in different quotas. For this definition, I view the capacity for round t of c's admission procedure as being determined by the numbers of students matched in earlier rounds $|\mu_1(c)|, \dots, |\mu_{t-1}(c)|$.[26]

Definition 4. *A matching sequence adapated to the choice protocol $\mu = (\mu_1, \dots, \mu_T)$ is* ***procedurally stable****, if*

- *(i) $\mu(i) R_i i$,*
- *(ii) $i \succ_{(c,\theta_t)} u$ for all $i \in \mu_t(c)$,*
- *(iii) if $i \in \mu_t(c)$, there is no $s < t$ such that $i \succ_{c,\theta_s} c$ and either $|\mu_s(c)| < q_{(c,s)}(|\mu_1(c)|, \dots, |\mu_{s-1}(c)|)$, or $i \succ_{c,\theta_s} j$ for some $j \in \mu_s(c)$.*
- *(iv) if $c P_i \mu(i)$ then there is no t such that $i \succ_{c,\theta_t} c$ and either $|\mu_t(c)| < q_{(c,t)}(|\mu_1(c)|, \dots, |\mu_{t-1}(c)|)$ and $i \succ_{(c,\theta_t)} j$ for some $j \in \mu_t(c)$*

Procedural stability requires that no student strictly prefers to be matched to some school at which she could have been admitted given its choice protocol and the set of competing applicants. Furthermore, condition (iii) requires that students who do receive a place at school c are admitted in the earliest possible step of c's admission procedure for $\mu(c)$. The intuition for this requirement is best understood in case c has a fixed upper bound $\overline{q}_c$ and an intended distribution of student types (as given by $(q_{(c,\theta)})_\theta$). In this case, (iii) can be understood as as the desire to stay as close as possible to this intended distribution. To see this note that if in

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[26] This is of course equivalent to defining the capacity sequence dependent on the number of vacant seats in earlier rounds.

each round of the choice protocol all available places are allocated then the school reaches its target distribution. The more capacity redistribution, the further the actual diverges from the desired distribution. In this sense, a procedurally stable matching strikes a balance between affirmative action and stability constraints. I now show that this notion of stability coincides with stability in the associated college admissions problem. First, we need to define how to transform matchings between the affirmative action with floating quotas problem and the associated college admissions problem since the latter does not specify which type of place an admitted applicant receives. For some matching $\mu = (\mu_1, \ldots, \mu_T)$ adapted to a given choice protocol, set $\nu^\mu(i) = c$ if there is some t such that $\mu_t(i) = c$, and $\nu(i) = i$ if $\mu_t(i) = i$ for all t. Similarly, set $\nu^\mu(c) = \cup_t \mu_t(c)$. This defines a matching for the associated college admissions problem. Given some matching ν for the associated college admissions problem, set $\mu_t^\nu(c) = J_t(\nu(c))$ and $\mu_t^\nu(i) = c$ if $i \in J_t(\nu(c))$, and $\mu_t^\nu(i) = i$ otherwise, where $J_t(\nu(c))$ is the set of students assigned a place at c in round t of c's admission process given the set of applicants $\nu(c)$.

Proposition 3. *(i) If μ is procedurally stable, then ν^μ is stable for the associated college admissions problem.*

(ii) If ν is stable for the associated college admissions problem, then $\mu^\nu = (\mu_1^\nu, \ldots, \mu_T^\nu)$ is procedurally stable.

Proof :

(i) We show first that $Ch_c(\nu^\mu(c)) = \nu^\mu(c)$. By (iii) of Definition 3, $\mu_1(c)$ contains the $q_{(c,1)}$ highest ranking acceptable students in $\nu^\mu(c)$ with respect to $\succ_{(c,\theta_1)}$. This implies $J_1(\nu^\mu(c)) = \mu_1(c)$ and $r_1(\nu^\mu(c)) = q_{(c,\theta_1)} - |\mu_1(c)|$. Proceeding inductively, suppose $J_s(\nu^\mu(c)) = \mu_s(c)$ for all $s \leq t$. This implies $q_{(c,t+1)}(r_1(\nu^\mu(c)), \ldots, r_t(\nu^\mu(c))) = q_{(c,t+1)}(|\mu_1(c)|, \ldots, |\mu_t(c)|)$. Again by (iii), $\mu_{t+1}(c)$ contains the $q_{(c,t+1)}(|\mu_1(c)|, \ldots, |\mu_t(c)|)$ highest ranking acceptable students in $\nu^\mu(c) \setminus (\mu_1(c) \cup \ldots \cup \mu_t(c))$ with respect to $\succ_{(c,\theta_{t+1})}$, implying $J_{t+1}(\nu^\mu(c)) = \mu_{t+1}(c)$ and $r_{t+1}(\nu^\mu(c)) = q_{(c,t+1)}(r_1(\nu^\mu(c)), \ldots, r_t(\nu^\mu(c))) - |\mu_{t+1}(c)|$.

Next, we show that if $cP_i\nu^\mu(i)$ for some student $i \in I$, then $i \notin Ch_c(\nu^\mu(c) \cup \{i\})$. Note that if $i \succ_{(c,\theta_1)} c$ and $i \in J_1(\nu^\mu(c) \cup \{i\})$, we must have either $|\mu_1(c)| < q_{(c,1)}$ or $i \succ_{(c,\theta_1)} j$ for some $j \in \mu_1(c)$ so that (iv) of Definition 3 must be violated with respect to $t = 1$. This and the individual rationality of ν^μ in the associated college admissions problem imply $J_1(\nu^\mu(c) \cup \{i\}) = \mu_1(c)$. Proceeding inductively, suppose $J_s(\nu^\mu(c) \cup \{i\}) = \mu_s(c)$ for all $s \leq t$. This assumption implies $q_{(c,t+1)}(r_1(\nu^\mu(c)), \ldots, r_t(\nu^\mu(c))) = q_{(c,t+1)}(|\mu_1(c)|, \ldots, |\mu_t(c)|)$.

Hence, if $i \succ_{(c,\theta_{t+1})} c$ and $i \in J_{t+1}(\nu^{\mu}(c) \cup \{i\})$, we must obtain a contradiction to (iv) of Definition 3 with respect to $t+1$ as above. This inductive argument shows $i \notin J_1(\nu^{\mu}(c) \cup \{i\}) \cup \ldots \cup J_T(\nu^{\mu}(c)) = Ch_c(\nu^{\mu}(c) \cup \{i\})$ and completes the proof of (i).

(ii) We show first that if $Ch_c(\nu(c)) = \nu(c)$, then μ^{ν} satisfies (iii) of Definition 2. To see this note that $\mu^{\nu}_t(c) = J_t(\nu(c))$ consists of the $q_{(c,t)}(r_1(\nu(c)), \ldots, r_{t-1}(\nu(c))) = q_{(c,t)}(|\mu^{\nu}_1(c)|, \ldots, |\mu^{\nu}_{t-1}(c)|)$ highest ranking acceptable applicants in $\nu(c) \setminus (J_1(\nu(c)) \cup \ldots \cup J_{t-1}(\nu(c))) = (\mu^{\nu}_t(c) \cup \ldots \cup \mu^{\nu}_T(c)) \setminus (\mu^{\nu}_1(c) \cup \ldots \cup \mu^{\nu}_{t-1}(c))$ with respect to $\succ_{(c,\theta_t)}$ by the admission process of school c.

Next, suppose that (iv) is violated for some t so that $cP_i\nu(i)$, $i \succ_{(c,\theta_t)} c$, and either $|\mu^{\nu}_t(c)| < q_{(c,t)}(|\mu^{\nu}_1(c)|, \ldots, |\mu^{\nu}_{t-1}(c)|)$ or $i \succ_{(c,\theta_t)} j$ for some $j \in \mu^{\nu}_t(c)$. Note that in both cases we would obtain $i \in \cup_{s=1}^{t} J_s(\nu(c) \cup \{i\})$ so that $i \in Ch_c(\nu(c) \cup \{i\})$, which contradicts stability in the associated college admissions problem.

□

This together with Corollary 1 implies the following.

Corollary 2. *Suppose $(\theta_c, \succ_c, q_c)_{c \in C}$ is monotonic and consistent. Then for all profiles of student preferences R, $f^I(R|(\theta_c, \succ_c, q_c)_{c \in C})$ is the unanimously most preferred procedurally stable matching for students.*

5.1.1 Comparison to related literature

Konishi and Ünver (2006) introduce the class of categorywise responsive preferences for two-sided matching problems. A school c's preferences belong to this class if the set of students can be partitioned into a finite number of categories such that preferences are (i) separable across categories (in the sense that the ranking of students within some category does not depend on which applicants from other categories are available), and (ii) responsive restricted to any single category of students. If student types are uni-dimensional and capacity distribution is not allowed, affirmative action problems induce a college admissions problem with categorywise responsive preferences, where categories are described by student types. But in this case seats in the different type-specific quotas could be allocated in separate assignment procedures anyway. If types can be multi-dimensional and capacity redistribution is allowed, preferences may be neither separable across categories/types nor responsive within a type-category. Hence, affirmative action with floating quotas problems cannot be solved adequately by relying on two-sided matching problems with categorywise responsive preferences.

Next, it is illustrative to compare regular choice protocols with the endowed assignment valuations introduced by Hatfield and Milgrom (2005). Preferences in this class can be described as follows: Each hospital/college has a fixed number of (potentially heterogenous) *jobs* to fill and is endowed with some existing set of doctors/students; new doctors/students are evaluated according to their productivity in the various jobs and from each set of applicants a hospital chooses the subset of doctors which, if doctors are optimally distributed among jobs, maximizes productivity. As they note, it is an *open question whether all substitutes valuations are endowed assignment valuations* (p.927). While the class of endowed assignment valuations is flexible enough to handle type specific quotas and capacity constraints (as shown by Hatfield and Milgrom (2005)), it cannot accommodate floating quotas. To see this note that the set of *jobs* would have to depend on the set of applicants in a non-trivial way if we allow capacity to be redistributed between type-specific quotas. Hence, the class of endowed assignment valuations does not exhaust the class of substitutable preferences.

Finally, I compare affirmative action with floating quotas problems to the affirmative action problems in Abdulkadiroglu and Sönmez (2003) and Abdulkadiroglu (2005). In this model, schools have type specific upper bounds and each school's ranking is the same for all student types. Abdulkadiroglu (2005) shows that if such affirmative action constraints are multi-dimensional, e.g. place upper bounds on ethnicities and sexes, schools' induced preferences over groups of students may fail to be substitutable. The maximal domain result in Hatfield and Milgrom (2005) then implies that there may not be a stable matching for the associated college admissions problem.[27] However, as shown above we obtain a well defined associated college admissions problem if we start from an affirmative action with floating quotas problem in which schools initially divide their capacity according to their intended type distribution and then accommodate to actual demands by redistributing capacities. Hence, affirmative action with floating quotas problems seem to provide a more positive perspective on the possibility of achieving stable outcomes in light of multi-dimensional affirmative action constraints.

5.2 Application to the German System

I now describe how the theory developed in the last subsection can be applied to develop a redesign for the German admission system. Let $(U, A, \succ^1, \succ^2, \succ^3, q)$ be an extended university admissions environment as defined in section 4.1 and R be an arbitrary profile of applicant

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[27] Their result is correct if attention is restricted to matching problems in which there is only a single contract between each pair of agents as in the school choice problem.

preferences, which I will identify with a university admissions problem in the following as everything else is fixed. To a given university admissions problem we can associate an affirmative action with floating quotas problem in which

(i) the set of students is identified with the set of applicants and the set of colleges is identified with the set of universities,

(ii) $\Theta = \{G, W, D\}$, where $G \in \tau(i)$ means that applicant/student i is a *top-grade applicant*, $W \in \tau(i)$ mains that i is a *wait-time applicant*, and D is some default type such that $D \in \tau(i)$ for all i,

(iii) $q_{u,G} = q_{u,W} = \frac{1}{5}q_u$ and $q_{u,D} = \frac{3}{5}q_u$ for all u, and

(iv) the choice protocol is given by $(\theta_1, \theta_2, \theta_3) = (G, W, D)$ and $q^3(r^1, r^2)_{u,D} = \frac{3}{5}q_c + r^1 + r^2$, $q^3(r^1, r^2)_{u,G} = q^3(r^1, r^2)_{u,W} = 0$ for all u and all r^1, r^2.

The main idea for a redesign will be to use the SDA for the associated college admissions problem, denoted by f^A in the following, to solve the university admissions problem. Before discussing the advantages of this procedure, the next example illustrates the above construction for a simple example and calculates f^A.

Example 4. *Consider again the university admissions problem of Example 1. For the convenience of the reader, I briefly summarize priorities and preferences. Remember that each university was assumed to have one place to allocate for each step of the ZVS procedure. Preferences of students are given by*

R	R_{a_1}	R_{a_2}	R_{a_3}	R_{a_4}	R_{a_5}	R_{a_6}	R_{a_7}	R_{a_8}	R_{a_9}
	u_1	u_1	u_3	u_2	u_2	u_3	u_2	u_2	u_1
	u_2	u_3	u_2	u_1	u_3	u_2	u_1	u_1	u_2
	u_3	u_2	u_1	u_3	u_1	u_1	u_3	u_3	u_3

.

Applicants are indexed in increasing order of average grades so that a_1, a_2, a_3 are top-grade applicants. We assumed that applicants a_7, a_8, a_9 were wait-time applicants and that the priority structure for step 2 was (we only list wait-time applicants here)

$$\succ^2_{u_1}: \; a_8, \; a_7, \; a_9$$
$$\succ^2_{u_2}: \; a_9, \; a_7, \; a_8$$
$$\succ^2_{u_3}: \; a_7, \; a_8, \; a_9$$

The preferences of universities for step 3 were given by the following

$\succ^3_{u_1}$: a_1, a_4, a_5, a_2, a_3, a_6, a_7, a_8, a_9
$\succ^3_{u_2}$: a_1, a_6, a_2, a_3, a_5, a_4, a_7, a_8, a_9
$\succ^3_{u_3}$: a_1, a_4, a_5, a_2, a_3, a_6, a_7, a_8, a_9

For this university admissions problem we have $Ch_{u_2}(\{a_4, a_5, a_7, a_8\}) = \{a_4, a_5, a_7\}$ since (i) no applicant in $\{a_4, a_5, a_7, a_8\}$ is a top-grade applicant, (ii) a_7, a_8 are wait-time applicants and $a_7 \succ^2_{u_2} a_8$, and (iii) $\{a_4, a_5\} \succ^3_{u_2} a_8$ so that a_4 and a_5 admitted in step U. Similarly, one can establish that $Ch_{u_1}(\{a_1, a_2, a_8, a_9\}) = \{a_1, a_2, a_8\}$ and $Ch_{u_2}(\{a_4, a_5, a_7, a_9\}) = \{a_4, a_5, a_9\}$. We now calculate the outcome of the SDA.

In the first round, a_1, a_2, a_9 apply to u_1, a_4, a_5, a_7, a_8 apply to u_2, and a_3, a_6 apply to u_3. Given that $Ch_{u_2}(\{a_4, a_5, a_7, a_8\}) = \{a_4, a_5, a_7\}$, a_8 applies to her second choice university u_1 in the second round of the SDA. Since $Ch_{u_1}(\{a_1, a_2, a_8, a_9\}) = \{a_1, a_2, a_8\}$ this causes a_9 to apply to u_2 in the third round. This in turn leads to the rejection of a_7 since $Ch_{u_2}(\{a_4, a_5, a_7, a_9\}) = \{a_4, a_5, a_9\}$. In the fourth round a_7 applies to u_1, where she is immediately rejected since $Ch_{u_1}(\{a_1, a_2, a_7, a_8\}) = \{a_1, a_2, a_8\}$. In the fifth, and final, round she applies to a_3. Since she is accepted and no other applicant is unmatched the procedure ends and the resulting matching is thus

$$f^A(R) = \begin{array}{ccc} u_1 & u_2 & u_3 \\ \{a_1, a_2, a_8\} & \{a_4, a_5, a_9\} & \{a_3, a_6, a_7\} \end{array}$$

Note that this coincides with the Pareto dominant equilibrium outcome μ_1 in Example 1. We will see shortly that this is not peculiar to the above example.

We now discuss the benefits of using f^A to solve the university admissions problem. Note first that the choice protocol defined above satisfies all the requirements introduced in Definition 3. By Corollary 1 this implies that (i) for each university admissions problem R, $f^A(R)$ is the student optimal stable matching for the associated college admissions problem, (ii) f^A is group-strategyproof for applicants. Furthermore, note that given the above construction procedural stability coincides with the stability concept introduced in Definition 2. But then Corollary 2 immediately yield the following result.

Corollary 3. *Let R be a university admissions problem and let $NE^Z(R)$ denote the set of Nash equilibrium outcomes of the game induced by the ZVS mechanism. Then for all $Q \in NE^Z(R)$, $f^A_a(R)\tilde{R}_a f^Z_a(Q)$ for all $a \in A$.*

Hence, interpreting a university admissions problem as a college admissions problem with floating quotas and using the SDA of the associated college admissions problem yields an assignment procedure which *Pareto dominates all* complete information equilibria supported by the current assignment procedure. Thus applicants are *unambiguously* better off under this alternative procedure. Furthermore, the procedure satisfies the goals set out in the beginning of this section as it produces a stable assignment that is as favorable as possible to applicants and provides (groups of) applicants with dominant strategy incentives to reveal their true ranking of available universities.

Note that the above construction mimics the current ZVS procedure in the sense that we first check whether an applicant can be admitted as a top-grade applicant, then check whether she can be admitted as a wait-time applicant, and, finally, consider the applicant's chances of being admitted on basis of the criteria chosen by universities. However, the floating quota system is in sharp contrast to the current ZVS procedure, where a top-grade student who does not claim a place early in the procedure loses her top-grade priority since places are irreversibly converted to places that are allocated on basis of universities' preferences. Also note that the number of places a university allocates according to its own criteria decreases through the course of the algorithm: If at some point of the algorithm a top-grade applicant takes one of the places reserved for her, she will keep this place unless another top-grade applicant with higher priority applies in one of the later rounds. Finally, note that the algorithm works equally well with other quota systems or different quotas across universities (as long as the associated choice protocols satisfy monoticity and consistency). This is particularly important if the above is also taken to be a proposal for reform of the decentralized system that runs parallel to the ZVS procedure and in which the division of capacities can be quite different.

6 Conclusion and Discussion

This paper analyzed the assignment procedure that is used to allocate places at public universities for medicine and related subjects in Germany. The procedure uses two algorithms, the Boston algorithm and the college proposing deferred acceptance algorithm, that have been studied extensively in the matching literature. The major difference to previous studies is that the German system combines these two algorithms into a complicated sequential assignment procedure. Assuming universities to be non-strategic, we derived a full characterization of complete information equilibria of the revelation game between applicants induced by the admissions procedure. It was established that applicants' strategic incentives lead to outcomes that satisfy a notion of stability that takes the specifics of the German university admissions system, a floating quota system with different admission criteria, into account. Outside of the complete information setup, two examples demonstrated the problems created by the sequential allocation of seats and the use of ranking constraints by universities.

We introduced the college admissions problem with floating quotas and showed how it gives rise to a well-defined associated college admissions problem as long as choice protocols satisfied two simple conditions of monotonicity and consistency. It was shown how this approach to affirmative action constraints strikes a reasonable balance between colleges' desires to control the distribution of types in entering classes and stability constraints. We then showed how the German university admissions problem could be understood as an affirmative action problem with floating quotas. Our results show that, provided students follow their dominant strategy incentives to submit their true ranking of universities, the alternative mechanism developed in this paper Pareto dominates any (complete information) equilibrium of the current assignment procedure.

My hope is that these results direct public attention towards the actual problems of the university admissions system in Germany and how these might be overcome. The widespread refusal of a centralized procedure seems to be based on a wrong assessment of the benefits and disadvantages of a decentralized system. The analysis is particularly important given that there have recently been significant problems for those subjects that are not part of the centralized procedure.[28] A big problem of these decentralized procedures is that the market often fails to clear since some applicants hold multiple offers and this leads to congestion. In light of these problems, Germany's federal states agreed to establish a centralized system to coordinate the admission system which is to be used for the first time in the winter term 2010/2011. While the move towards a partially centralized system can be expected to alleviate some of the problems of a decentralized admission system described above, it is important to use an algorithm which ensures that applicants do not have to participate in a complicated revelation game. Affirmative action with floating quotas problems provide universities with ample flexibility to express their preferences over entering classes through monotonic and consistent choice protocols. As shown above, the associated SDA procedure provides students with straightforward incentives and achieves a normatively appealing assignment of applicants to public universities.

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[28] See e.g. http://www.spiegel.de/unispiegel/studium/0,1518,610971,00.html.

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A Appendix

A.1 An example showing that Assumption A1 is Restrictive

The following example shows that the No Empty Lists assumption A1 from section 4 is restrictive. There are seven applicants $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ indexed in order of increasing average grades. For simplicity, we assume that there are only two universities u and u' who have one place to allocate in each of the three steps of the ZVS procedure. Preferences of applicants are as follows:

R	R_{a_1}	R_{a_2}	R_{a_3}	R_{a_4}	R_{a_5}	R_{a_6}	R_{a_7}
	u	u	u'	u'	u'	u	u'
			u				

Assuming that a_5 and a_6 are the applicants with the longest waiting time, $a_3 \succ_u^3 a_2$ as well as $a_5 \succ_{u'}^3 a_4 \succ_{u'}^3 a_3$, and that all applicants submit their true ranking of universities for each step of the procedure, the outcome of the ZVS procedure is

$$f^{Z1}(R) = \begin{array}{cc} u & u' \\ a_1 & \emptyset \end{array}, \qquad f^{Z2}(R) = \begin{array}{cc} u & u' \\ a_6 & a_7 \end{array}, \qquad f^{Z3}(R, \succ^3) = \begin{array}{cc} u & u' \\ a_3 & \{a_4, a_5\} \end{array}.$$

Note that if we keep the profile of reports by everyone but a_2 fixed, a_2 cannot obtain a place at u if she applies for a place in step 1. Suppose then that she decides to apply only for step 3 and submits $Q_{a_2}^3 = u$. If everyone else submits the same preferences as before, a_3 would be a top-grade applicant and could obtain a place at her most preferred university u' in step 1 of the ZVS procedure. But then a_2 would receive a place at u in step 3 of the procedure since no one else will apply to u in that step. Thus, she benefits from not applying for a place in step 1.

□

A.2 Evaluation Procedures in the current ZVS procedure

In this Appendix we provide some further details on the different evaluation procedures used by universities in step 3. All of the below concerns the ZVS procedure for the winter term 2009/2010. The evaluation process takes place after assignments in steps 1 and 2 have been determined and only those applicants who did not receive a place in these steps are considered. In principle, the ZVS informs each university about all remaining applicants who have listed the university in their ranking for step 3.

A university may, however, limit the set of applicants it will consider for step 3 in advance on basis of its rank in the preference lists submitted for step 3, average grades, or a combination of the two criteria. For example, a university with, say, a hundred seats to be allocated in step 3 may consider only the 300 applicants with the best average grades among those who ranked it first. This practice is called *pre-selection* and the ZVS informs each university only about those applicants who “survived” its pre-selection process.

In case an applicant is not rejected in the pre-selection process of a university, the ZVS provides the university with detailed information including its rank in the submitted preference list, average grade, waiting-time, and so on. Universities can then use average grades, interviews, statements of purpose, completion of on-the-job training in a relevant field, prizes in scientific competitions, and so on, to evaluate remaining applicants. Furthermore, universities are allowed to split their capacity into several parts and to apply different admission criteria across these parts. For example, a university may decide to allocate 50 percent of places according to average grade and 50 percent on basis of performance in an interview. In this case, the university has to specify which place an applicant receives if she could be admitted in more than one of these quotas. The official information brochure of the ZVS states that *in the determination of an applicant’s rank average grade has to be a decisive factor*,[29] although the exact requirement that needs to be fulfilled is not specified.

A university uses a *mechanical evaluation procedure* if the marginal cost of evaluating an additional candidate according to its criteria is negligible. Under this label I summarize all universities who do not use “subjective” criteria such as performance in interviews or the evaluation of statements of purpose. While such universities may still require to elicit additional information from applicants, computing the rank of an applicant is completely mechanical. In this category, I also include universities who use standardized tests in their evaluation procedure. For example, several universities offering medical subjects use the outcome of a standardized medical subject test in their evaluation. Applicants have to pay a fixed fee in order to take the test and tests are graded by an independent company. If, for example, a university uses some weighted average of average grade and performance in the standardized test to rank applicants, its cost of considering one additional applicant amounts to the (computational) cost of calculating the weighted average of two numbers.

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[29] *Merkblatt M09: Auswahlverfahren der Hochschulen*, available at www.zvs.de. Translation by the author.

Subject	# U	#Pre 1	#Pre 2	#Pre 3	#Pre 4	# M1	# M2	# M+Pre
Vet. Med.	5	3	0	0	0	2	0	0
Pharmacy	22	0	3	4	0	21	0	6
Dentistry	29	6	3	6	1	21	5	13
Medicine	34	8	5	6	1	24	4	16

Table 1: Preselection and Mechanical Evaluation

Subject	$\#INT$	$\#INT_{>0.5}$
Vet. Med.	3	3
Pharmacy	2	2
Dentistry	7	2
Medicine	10	8

Table 2: Interviews in the Evaluation Process

In the first table, $\#U$ lists the number of universities, $\#Pre_k$ lists the number of universities that consider only applicants who ranked them at least kth ($k = 1, \ldots, 4$), $\#M1$ lists the number of universities who use a completely mechanical evaluation procedure, $\#M2$ lists the number of universities who allocate at least half of the available capacity using a mechanical evaluation criteria,[30] and $\#M + Pre$ lists the number of universities who have a partly mechanical evaluation procedure but only consider applicants who rank them sufficiently high.[31]

In the second table, $\#INT$ is the number of universities that use interviews to allocate at least part of their capacity and $\#INT_{>0.5}$ is the number of universities who assign more than half of their seats on basis of interviews. I only report number for medicine, dentistry, pharmacy, and veterinary medicine. The ZVS also allocates places for biology at five universities, but for the majority of universities places in biology are not allocated through the centralized procedure.

A.3 Omitted details of the ZVS Procedure

This appendix lists some of the more substantial simplifications made in the main body of the text. Readers interested in all details of the current ZVS procedure may still want to consult

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[30] As mentioned above, a university may decide to allocate some of the places available for step 3 according to subjective criteria and remaining places according to objective criteria. In this case, a university has to specify which place an applicant is supposed to receive if she can be admitted according to both criteria.

[31] Here, all universities are included who allocate at least part of their capacity according to objective criteria but require all applicants to rank them sufficiently high.

the *Vergabeverordnung ZVS [Stand: WS 2009/2020]* (available at www.zvs.de).

Capacities: The total number of places at each university is determined by the application of federal laws. For each state there is a so called *Kapazit"atsverodnung* (KaPVO) which prescribes a formula for calculating the number of applicants a university can admit on basis of the number of professors, available teaching facilities, and so on.[32]

Special Quotas: Up to approximately fifteen percent of total available places are allocated in advance among foreign applicants, applicants pursuing a second university degree, and so on. These applicants are not allowed to participate in the regular assignment procedure.

Step 1: The education system in Germany is federalized and the general opinion is that average grades are not directly comparable across federal states. For this reason, there are actually sixteen separate assignment procedures in step 1, one for each federal state. This is achieved by splitting the 20 percent of (remaining) capacity available in step 1 into sixteen parts. In the assignment procedure of a given federal state only those applicants are considered who have received their high school diploma in this state.

Step 3:

- Once assignments are determined by the ZVS procedure described in section 2, successful applicants have to enroll at their assigned university. If some applicants fail to do so, their places are allocated according to the rules of step 3. Here, only those applicants are considered who did not receive a place in previous rounds of the assignment procedure. Again, students have to enroll at their assigned university (if any) and if they fail to do so, another round of step 3 is used to allocate remaining places (again only students who were not previously assigned a place are considered). Any places that remain after all of this are allocated via lottery by universities.
- In order to prevent multiple rounds of the assignment procedure in step 3, a university can demand the ZVS to overbook its capacities. Thus, a university with, say 100 places, may ask the ZVS to be assigned 150 applicants since it expects some students not to accept their assigned places.

Lotteries: If a university does not fill its capacity in the ZVS procedure, remaining places are allocated on basis of lottery. Each university conducts its own lottery and applicants have to apply to universities directly in order to participate.

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[32] There has been some discussion about the KaPVO in recent years, see e.g. *Die fiese Formel* in Die Zeit, Nr. 39(2007).