

Network formation with heterogeneous agents and absolute friction*

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Abstract

We present a model of endogenous network formation with absolute friction and heterogeneous agents. The individual payoffs from a given network are determined by the difference of an agent specific utility function that depends on the number of his/her direct links and the sum of his/her link-costs. These link-costs decompose in a symmetric function that represents the social and geographical distance between the two agents and an agent specific function representing the partner's wealth and status. From a theoretical point of view, we define a new stability concept that is situated between the notions of pairwise stability (see Jackson and Wolinsky (1996)) and strong stability (see Dutta and Mutuswami (1997)). We show that our model has a unique stable network and we demonstrate that it is also strongly stable. As such, we provide uniqueness and existence for a whole range of stability concepts situated between our new stability concept and strong stability. From a practical point of view, we provide an algorithm that reproduces this stable network from information on the individual payoff structure. We illustrate the use of this algorithm by applying it to an informal insurance data set from the village of Nyakatoke in rural Tanzania.

Keywords: network formation, heterogeneity, absolute friction

JEL-classification: C62; C78; C79

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1 Introduction and motivation

We present a model of endogenous network formation with absolute friction and heterogeneity among agents. The specifics of our model are based on the empirical regularities observed in networks of informal risk-sharing (see below). We show that our model leads to a unique network that is stable for a whole range of stability concepts. We provide an algorithm to compute this network and we illustrate its use by applying it to a data set from the small village of Nyakatoke.

Overview Social networks are among the most valuable contributors to apprehend modern human life. They determine, among others, communication, friendship, trust, marriage, trade, risk-sharing and wealth. In order to understand these phenomena it is crucial to study how these networks are formed and how they develop and change over time. This insight has led to an enormous growth of scientific research analyzing the development, stability and empirical regularities of social network formation.

In this research we present a model of endogenous network formation. Our model has three important features. First, the model assumes that link formation is only possible under mutual consent. This means that in order to establish a link, two agents have to agree to form this link, while every agent can unilaterally decide to cut a pre-existing link (see Jackson and Wolinsky (1996)). This feature distinguishes our research from models of unilateral link formation.¹In models of unilateral link formation, any agent can unilaterally decide to link with another agent. In this respect, it is also useful to mention that our model does not allow for transfers between agents, where it is possible that one agent compensates another agent for the formation a link between them.² Second, our model allows for a wide range of heterogeneity in the determination of the link-costs. The link-cost is the cost that each individual incurs when adding a certain link. Third, we restrict ourselves to networks with absolute friction. This means that the benefits of a certain link only attribute to the two agents that form this link. In this section, we discuss each of these three features in more detail. This will allow us to position and distinguish our research from other research in the network formation literature.

Bilateral link formation We assume that links can only be formed by mutual consent. In particular, no agent can form a link with another agent without the agreement of this agent but any agent can unilaterally decide to cut an existing link (see Myerson, 1977, 1991; Jackson and Wolinsky, 1996). We say that a network is stable if there is no *coalition* of agents that can *alter* the network in such a way that at least one agent in the coalition

¹See, among others, Bala and Goyal (1997); Dutta and Jackson (2000); Bala and Goyal (2000a,b); Haller and Sarangi (2005); Galeotti, Goyal, and Kamphorst (2006); Feri (2007); Feri and Meléndez-Jiménez (2009); Hojman and Szeidl (2008).

²See, among others, Dutta and Mutuswami (1997); Dutta, van den Nouweland, and Tijs (1998); Slikker and van den Nouweland (2000, 2001); Jackson (2005); Bloch and Jackson (2007).

benefits from the change without any agent in the coalition losing from the change.³ This definition hinges on two further specifications. First, we need to specify which coalitions are allowed to alter the existing network, and second, we need to specify in which manner these coalitions may alter the current network. Depending on these specifications, different stability concepts can be established. At the one extreme we may allow for coalitions of arbitrary size that can modify a given network by creating any link between two agents within the coalition and cut any number of links that involves at least one agent from the coalition. This corresponds to the notion of strong stability as introduced by Dutta and Mutuswami (1997) and Jackson and van den Nouweland (2005). At the other end of the spectrum we may consider the pairwise stability concept where deviating coalitions have size at most two and these coalitions can change the network by either creating the link between the two agents in the coalition or either cut a single link involving an agent in this coalition (see, Jackson and Wolinsky (1996); Jackson and Watts (2001)).

We introduce the notion of weak stability. Akin to the concept of pairwise stability, we only allow for deviating coalitions of maximal size two, but we broaden the range of potential deviating actions. In particular we allow a coalition to deviate by creating the link between the agents in the coalition and simultaneously cut at most a single link for each agent within this coalition. This stability concept strengthens the pairwise stability concept of Jackson and Wolinsky (1996) as it allows for the simultaneous creating and cutting of links. On the other hand, it is clearly weaker than the strong stability concept of Dutta and Mutuswami (1997). We show that, for our model, there exists a unique weakly stable network, and that this network is also strongly stable. In this sense, our equilibrium network will be stable for every stability concept that is stronger than our weak stability concept and weaker than strong stability. This class includes, for example, the notion of strong pairwise stability introduced by Belleflamme and Bloch (2004).

Heterogeneity in the payoff structure Within this model of network formation under mutual consent, we define the payoff structure for the various agents. The payoff from being part of a given network consists of a benefit part and a link-cost part, reflecting the idea that agents have to incur costs to establish a link. We assume that the link-cost can be split into two parts. A first part measures the social or geographical distance between the two agents, which reflects the idea that before establishing a link, agents have to invest time and effort to build a relationship of trust and that the better the ex ante relationship is, the less time and effort needs to be invested. In the literature on informal risk-sharing networks for example, De Weerd (2002) finds that kinship, geographical proximity, clan membership and religious affiliation are strong determinants in the formation of risk-sharing networks. This result is corroborated by the results of Fafchamps and Gubert (2007) who find that geographic proximity, kith and kin relationships are strong determinants of link formation

³This definition abstains from various forms of farsightedness (see for example Watts (2002); Deroan (2003); Watts (2002); Dutta, Ghosal, and Ray (2005); Page, Wooders, and Kamat (2005); Page and Wooders (2007); Herings, Mauleon, and Vannetelbosch (2009) or evolutionary dynamics (see, for instance, Watts (2001); Jackson and Watts (2002); Dutta et al. (2005); Tercieux and Vannetelbosch (2006); Carayol and Roux (2006); Feri (2007); Page and Wooders (2009)).

within an informal insurance network in the rural Philippines. Hence, the notion of social and geographical distance is highly relevant for the formation of a risk-sharing network. The second part measures the quality of the agent to whom a link is established. The implicit assumption is made that increased quality of the linking partner reduces link-costs.⁴ Again, empirical studies on risk-sharing networks reveal that some agents are more desirable linking partners than others because they have more wealth or more status. For example, Comola (2008) and De Weerd (2002), find that the probability of being linked to a household rises with the wealth of this household measured by the amount of livestock this household has, that the rich have a denser network than the poor and that a dyad with a rich person in it has more chance of being linked.

Formally, we assume that the total cost of i , for linking with j , $c(i, j)$, can be decomposed in a measure of social and geographical distance between i and j , $d(i, j)$, and an individual specific quality measure, represented by $\theta(j)$. We assume that the distance factor is specific to the pair of agents under consideration and it is symmetric, in the sense that the geographical and social distance from an agent i to an agent j is the same as the distance from j to i .

All other studies of network models with bilateral link formation use more restrictive forms for the cost topology. Among others, Jackson and Wolinsky (1996); Goyal and Joshi (2006); Calvó-Armengol (2004) analyze models with homogeneous costs, i.e. the setting where linking costs are constant for all pairs of individuals. Johnson and Gilles (2000) assume that costs are described by a line cost topology. In this model, agents are situated on a line and the linking-cost between two individuals is determined by the Euclidean distance between them. Carayol and Roux (2006) uses a circle-cost topology, where agents are situated on a circle and the linking-cost between two agents is determined by their shortest distance. Finally, Jackson and Rogers (2005) impose an island cost topology. In this setting, agents are allocated to a finite number of islands and the linking-cost between two agents may take on one of two values conditional on the requirement that the link-cost between two agents on the same island is less costly than the link-cost between agents on distinct islands.

More general models of heterogeneity were investigated for models with non-bilateral link formation. In particular, Galeotti, Goyal, and Kamphorst (2006) investigate a model of one sided-link formation where costs can vary freely. Finally, Brueckner (2006) and Haller and Sarangi (2005) use the same heterogeneity concepts as we do — distance and quality — in a network formation model with probabilistic graphs. In probabilistic graph models, link formation may fail with a certain probability (Bala and Goyal, 2000b). More specifically, Brueckner (2006) assumes that this probability depends on the effort — costs — made by the agents involved. Hence, a major contribution of our model is that it is the first model which combines bilateral link formation with such general forms of heterogeneity.

⁴This impact may be indirect in that a higher quality may be reflected in an extra benefit, which increases the willingness to link, effectively reducing link-cost.

Absolute friction In addition, and contrary to previously mentioned studies of bilateral link formation, our model assumes absolute friction. Absolute friction states that the benefits from the network to a certain agent only depend on the agents that are directly linked to this agent. A model with less than absolute friction assumes that the benefits are decreasing in the length of the path while a model with no friction assumes that the length of the path does not matter in defining the benefits. The assumption of absolute friction is necessary to keep our model tractable. Indeed, the restrictive nature of the cost heterogeneity used in the literature with models of bilateral link formation (see above) strongly indicates that models with moderate levels of friction are only tractable by imposing more stringent restrictions on the cost topology.

The assumption of absolute friction is defensible for networks of informal insurance, trust or trade. Fafchamps and Lund (2003) show that risk sharing is not frictionless due to transaction costs, imperfect commitment, asymmetric information or other processes that limit exchange. This finding is corroborated by De Weerd and Dercon (2006) who indicate that direct network partners may be most relevant for understanding risk sharing. In addition, we could also assume that agents are shortsighted because they do not know the complete structure of the network and only take into account the benefits from their direct partners, while assuming other benefits to be non-existent. In this case, the network that emerges looks like the network that would emerge in the case of absolute friction, so that the absolute friction assumption is valid in this setting.

When we do not assume shortsightedness the assumption of absolute friction may be more difficult to maintain. In certain settings, like communications networks, an agent with many connections may be appealing to link to because she may pass on to you the information she got from her connections. Likewise, in a collaboration network an agent with many connections may be less attractive since his time is taken by many agents, leaving less time to work together.⁵ In these cases, the indirect network matters. Thus, we conclude that the validity of our assumption of absolute friction depends on the context and the additional assumptions made.

Main results This paper contributes to the present literature by providing a tractable model of two-sided network formation with few restrictions on linking costs, while considering a wide range of stability concepts. We show that for our model, there exists a unique stable network which is both weakly and strongly stable. In this sense, our equilibrium network will be stable for every stability concept that is stronger than our weak stability concept and weaker than the strong stability concept.

Besides the existence and uniqueness result, we provide an algorithm that reproduces this stable network. In this algorithm, starting from the complete network, we let a sequence of agents cut the links they do not want to maintain, given the cutting actions of the agents before them in the sequence. The stability of the outcome of the algorithm hinges

⁵See, among others, the connections and co-author models in Jackson and Wolinsky (1996), the job contact network in Calvó-Armengol (2004) and the local spillover game in Goyal and Joshi (2006).

on the order of the agents in the sequence. We can show that for the resulting network, in every iteration of the algorithm there will always be at least one agent whose set of links she would like to maintain in that network will be equal to her set of links in the unique stable network. Hence, the sequence of agents is such that in every iteration exactly those agents are selected. Naturally, an agent cannot be selected twice and is deleted from the selection pool once selected. This algorithm is one of the key innovations in this paper. Once we know the structure of the cost functions, and the form for the benefit functions, we can compute the particular sequence of agents and the emerging stable network, observe its general features and relate these features to these structures.

We illustrate the use of our algorithm by applying it to a data set on an informal insurance network drawn from the village of Nyakatoke in rural Tanzania⁶. Although our approach is rather crude in the measurement of the cost structure, implying that our exercise can be considered as an illustration at best, we find that our algorithm outperforms a random network formation model where probabilities were fitted from a logit regression which uses the same data.

Section 2 introduces the model and provides the definition of a stable network. Section 3 contains and discusses the main result relating to the uniqueness and existence of the stable network. Section 4 provides the empirical illustration and section 5 concludes.

2 The model

In this section, we present the various concepts used in this research. We begin by laying the foundations of network formation. Then we move on to discuss the various stability concepts and finally we introduce our specific model.

Models of network formation Consider a set of agents $N = \{1, \dots, n\}$. An *undirected network* g is determined by a collection of two element sets $\{i, j\}$ where $i, j \in N$ and $i \neq j$. For ease of notation, we will write ij instead of $\{i, j\}$. If $ij \in g$, we say that agents i and j are linked in the network g . We denote the *complete network*—the network where every agent is linked to every other agent—by g_c , i.e. $g_c = \{ij | i, j \in N, i \neq j\}$. We denote by \mathcal{G}_n the collection of all networks on a set of n agents. To each agent $i \in N$, we endow a *payoff function* π_i from the set \mathcal{G}_n to the set \mathbb{R} defining the payoff, $\pi_i(g)$ that agent i receives when the network g is established. A *payoff structure* $\{\pi_i\}_{i \leq n}$ consists of a finite number, n , of agents and payoff functions π_i for every agent $i \leq n$. We denote by \mathcal{S} the set of all payoff structures.

Stability concepts A *stability concept*, Π , is a correspondence from the set of payoff structures, \mathcal{S} , to the set of all networks $\bigcup_n \mathcal{G}_n$ such that for all payoff structures, $\{\pi_i\}_{i \leq n} \in \mathcal{S}$,

$$\Pi(\{\pi_i\}_{i \leq n}) \subseteq \mathcal{G}_n.$$

⁶This data set is also used in the research of Comola (2008); De Weerd (2002); De Weerd and Dercon (2006); Comola and Fachamps (2008).

The set $\Pi(\{\pi_i\}_{i \leq n})$ determines the set of stable networks corresponding to the payoff structure $\{\pi_i\}_{i \leq n}$ and stability concept Π .

We begin by defining three particular stability concepts of which one is new. In order to model these concepts, we borrow the idea of a *linking game* as introduced by Myerson (1991). For the moment let us fix a payoff structure $\{\pi_i\}_{i \leq n}$ and the set of agents $N = \{1, \dots, n\}$. For an agent, $i \in N$, his/her strategy set S_i consists of all subsets of $N - \{i\}$. This definition includes the empty set, \emptyset . The interpretation is that $j \in s_i$ if i proposes to j to form a link and $j \notin s_i$ if i is not willing to link with j . For a given strategy profile, (s_1, \dots, s_n) , a link between i and j will be formed if $j \in s_i$ and $i \in s_j$. In other words, i and j will be linked if i proposes to form a link to j and j proposes to form a link to i . We denote by $g(s_1, \dots, s_n)$ the network that is formed when individual $i \in N$ chooses strategy $s_i \in S_i$. In particular, we will have that $ij \in g(s_1, \dots, s_n)$ if and only if $j \in s_i$ and $i \in s_j$. Observe that, in general, there are several strategy profiles that may lead to the same network, i.e. there will exist distinct profiles (s_1, \dots, s_n) and (s'_1, \dots, s'_n) such that $g(s_1, \dots, s_n) = g(s'_1, \dots, s'_n)$. One notable exception is the complete network, g_c , where $g(s_1, \dots, s_n) = g_c$ if and only if $s_i = N - \{i\}$ for all $i \in N$. For every coalition $C \subseteq N$ and strategy profile $s \in S$, we also write $s = (s_C, s_{N-C})$, where s_C is the strategy profile s restricted to individuals within the coalition C . We call s_C a coalition strategy profile.

Consider an individual i and a strategy $s_i \in S_i$. The *close neighborhood* of strategy s_i , denoted by $\mathcal{N}_c(s_i)$ is defined as the set of strategies s'_i such that the cardinality (number of elements) of the set,

$$\{j \in N - \{i\} | j \in s_i \text{ and } j \notin s'_i\} \cup \{j \in N - \{i\} | j \notin s_i \text{ and } j \in s'_i\}$$

is at most unity. This concept is symmetric in the sense that $s_i \in \mathcal{N}_c(s'_i)$ if and only if $s'_i \in \mathcal{N}_c(s_i)$. Any strategy $s'_i \in \mathcal{N}_c(s_i)$, not equal to s_i itself, contains either one element more than s_i or one element less than s_i . The interpretation, for agent i , for changing his strategy from s_i to $s'_i \in \mathcal{N}_c(s_i)$ is that he chooses maximally 1 other agent to whom he proposes to form a new link or for which he decides that he does not want to form a link with.

We can now define the stability concept of pairwise stability originally introduced by Jackson and Wolinsky (1996),⁷

Definition 1 (Pairwise Stability). *A network g is pairwise stable if there exist a strategy profile (s_1^*, \dots, s_n^*) such that $g = g(s_1^*, \dots, s_n^*)$ and for all nonempty coalitions C of size less than or equal to 2, there does not exist a coalition strategy profile s_C such that for all $i \in C$, $s_i \in \mathcal{N}_c(s_i^*)$ and*

$$\pi_i(g(s_C, s_{N-C}^*)) \geq \pi_i(g(s_C^*, s_{N-C}^*)),$$

with at least one inequality being strict.

⁷The pairwise stability concept of Jackson and Wolinsky (1996) is usually defined in a different way. However, it is quite easy to show that our definition is equivalent to theirs. We use this formulation in order to compare the pairwise stability concept with the other stability concepts in this section.

Given the payoff structure $\{\pi_i\}_{i \leq n}$, we denote by $\Pi_{ps}(\{\pi_i\}_{i \leq n})$ the (possibly empty) collection of pairwise stable networks.

We now define a new stability concept that is slightly stronger than pairwise stability. We call it weak stability, to contrast with the definition of strong stability, which will be introduced later on. Similar to the concept of close neighborhood, we introduce the concept of a *neighborhood*, $\mathcal{N}(s_i)$ of a strategy $s_i \in S_i$. The neighborhood of s_i consists of all strategies $s'_i \in S_i$ such that the cardinalities of both sets,

$$\{j \in N - \{i\} | j \in s_i \text{ and } j \notin s'_i\} \text{ and } \{j \in N - \{i\} | j \notin s_i \text{ and } j \in s'_i\}.$$

are at most unity. The interpretation, for agent i , for changing his strategy from s_i to $s'_i \in \mathcal{N}(s_i)$ is that he chooses maximally 1 other agent with who he decides not to form a link and, at the same time, chooses maximally 1 other agent to whom he proposes to form a new link. Restricting individuals to choose from the set $\mathcal{N}(s_i)$ clearly restricts the degree to which i can deviate from a previous chosen strategy. Our weak stability concept proposes robustness against this kind of deviations for coalitions of maximal size two.

Definition 2 (Weak Stability). *A network g is weakly stable if there exists a strategy profile (s_1^*, \dots, s_n^*) such that $g = g(s_1^*, \dots, s_n^*)$ and for all nonempty coalitions C of size less than or equal to 2, there does not exist a coalition strategy profile s_C such that for all $i \in C$, $s_i \in \mathcal{N}(s_i^*)$ and*

$$\pi_i(g(s_C, s_{N-C}^*)) \geq \pi_i(g(s_C^*, s_{N-C}^*)),$$

with at least one inequality being strict.

Given the payoff structure $\{\pi_i\}_{i \leq n}$, we denote by $\Pi_{ws}(\{\pi_i\}_{i \leq n})$ the (possibly empty) collection of weakly stable networks. Clearly, every weakly stable network is also pairwise stable. On the other hand, the weak stability concept is weaker than the strong stability concept developed in Dutta and Mutuswami (1997) and Jackson and van den Nouweland (2005). A network is strongly stable if for all possible coalitions $C \subseteq N$ of individuals there does not exist a rearrangement of links within the coalition and cutting of links that involves at least one individual within the coalition such that at least one individual within the coalition strictly gains from the rearrangement, and no agent within the coalition loses from this rearrangement. Formally,⁸

Definition 3 (Strong Stability). *A network g is strongly stable if there exists a strategy profile (s_1^*, \dots, s_n^*) such that $g = g(s_1^*, \dots, s_n^*)$ and for all nonempty coalitions C , there does not exist a coalition strategy profile s_C such that for all $i \in C$, $s_i \in S_i$ and*

$$\pi_i(g(s_C, s_{N-C}^*)) \geq \pi_i(g(s_C^*, s_{N-C}^*)),$$

with at least one inequality being strict.

⁸We follow the definition as stated in Jackson and van den Nouweland (2005), since this definition is compatible with pairwise stability.

For a given payoff structure $\{\pi_i\}_{i \leq n}$, we denote by $\Pi_{ss}(\{\pi_i\}_{i \leq n})$ the (possibly empty) collection of strongly stable networks. The strong stability concept strengthens the weak stability concept in two ways. First it allows deviating agents to use any possible strategy in S_i . In contrast, the weak stability concept restricts deviations to be contained in the neighborhood set $\mathcal{N}(s_i^*)$. Second, the strong stability concept allows for deviations of coalitions of arbitrary size while weak stability only allows for deviating coalitions of maximal size two. Of course, we could imagine stability concepts where the choice of deviating strategy and the size of the deviating coalition is between these two extremes. In this sense, we can define a collection of stability concepts that are between the weak stability concept and the strong stability concept.

Definition 4 (Intermediate Stability concepts). *A stability concept Π is intermediately stable if for all payoff structures $\{\pi_i\}_{i \leq n}$,*

$$\Pi_{ss}(\{\pi_i\}_{i \leq N}) \subseteq \Pi(\{\pi_i\}_{i \leq n}) \subseteq \Pi_{ws}(\{\pi_i\}_{i \in N}).$$

In words, every strongly stable network is also stable with respect to Π and every network that is stable with respect to Π is also weakly stable. The collection of intermediate stable correspondences contains, for example, the stability concept of strongly pairwise stability as introduced by Belleflamme and Bloch (2004).

Definition 5 (Strong Pairwise Stability). *A network g is strongly pairwise stable if there exists a strategy profile (s_1^*, \dots, s_n^*) such that $g = g(s_1^*, \dots, s_n^*)$ and for all nonempty coalitions C of size less than or equal to 2, there does not exist a coalition strategy profile s_C such that for all $i \in C$, $s_i \in S_i$ and*

$$\pi_i(g(s_C, s_{N-C}^*)) \geq \pi_i(g(s_C^*, s_{N-C}^*)),$$

with at least one inequality being strict.

Like the weak stability concept, the strong pairwise stability concept allows deviation for coalitions of size 2 but it allows the deviating strategy to be any strategy in the set S_i .

In the following section we will show that our model allows for a unique weakly stable network that is also strongly stable. Given above definition, this network will be stable for all intermediate stability concepts. As such, for our specific model, the whole range of intermediate stability concepts coincides. Given the uniqueness of this stable network, this makes it a particular attractive equilibrium network.

Model specification We are now ready to determine the specific functional structure of the payoff functions π_i . We assume that the payoff for i of a network g , $\pi_i(g)$, can be decomposed in two parts. A first part provides the benefit that i obtains from the network g and a second part provides the costs that i incurs from the network g .

For the benefit part, we impose the assumption of absolute friction, meaning that individuals do not have any advantage or disadvantage of the presence of indirectly connected individuals. In particular, each individual $i \in N$ is endowed with an individual specific

utility function $v_i : \mathbb{N} \rightarrow \mathbb{R}$ representing the utility that i derives from her number of direct links $n_i(g)$,

$$n_i(g) = |L_i(g)| \equiv |\{j \in N - \{i\} | ij \in g\}|.$$

By making the assumption that only the number of direct links matter, we impose homogeneity on the benefit side, i.e. all potential partners are deemed to be equal in terms of linking gains. Furthermore, we assume that the change in utility is strictly decreasing in the number of links and that the additional benefit of adding an extra link becomes arbitrarily small or negative if the number of links is large enough. Formally, for all $n \in \mathbb{N}$:

- a) $v_i(n+2) - v_i(n+1) < v_i(n+1) - v_i(n)$ and,
- b) for all $\varepsilon > 0$, there exists a number $n(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n(\varepsilon)$, $v_i(n+1) - v_i(n) \leq \varepsilon$.

Establishing and maintaining a link is costly. We introduce the notion of a link-cost function to model these linking costs. As mentioned in the introduction, we assume that this link-cost function depends on two factors: a distance factor and a quality factor. If the social or geographical distance between individuals is large, they will have to incur more costs to overcome this distance and establish a link. For every pair of agents i and j , we define a number $d(i, j)$ representing the distance between i and j . We assume that this function is symmetric. Formally,

- c) for all $i, j \in N$: $d(i, j) = d(j, i)$.

Besides this distance function, we assume that the cost of link formation is also determined by the quality of the different individuals. For each agent we consider a number $\theta(i)$ that represents his quality. The larger $\theta(i)$ the higher the quality of agent i . We assume that the cost of i from linking with j , $c(i, j)$, can be written as a difference between these two parts:

$$c(i, j) = d(i, j) - \theta(j).$$

At the end of section 3, we show that our model can be adapted to include alternative forms for this cost function.

Finally, in order to obtain our uniqueness result, we impose two additional assumptions:

- d) for all i, j and $k \in N$: $c(i, j) \neq c(i, k)$ and,
- e) for all $n \in \mathbb{N}$ and all i and $j \in N$, $v(n+1) - v(n) \neq c(i, j)$.

The first assumption states that every agent can perfectly order the other agents in terms of linking costs, without there being any indifference between potential linking partners. The second assumption implies that no agent can ever be indifferent between adding (or cutting) a certain link. These requirements are generic in the sense that any violation of them will be undone for a small perturbation in the distance or quality functions.

Bringing together the cost and benefit sides, we get that the payoff of i derived from the network g is given by,

$$\begin{aligned}\pi_i(g) &= v_i(n_i(g)) - \sum_{j \in L_i(g)} c(i, j) \\ &= v_i(n_i(g)) - \sum_{j \in L_i(g)} (d(i, j) - \theta(j)).\end{aligned}$$

3 Stable networks

Having outlined the various concepts of stability and the specific payoff structure of our model, we proceed by showing existence and uniqueness of the (weakly) stable network. We also give some additional insights into the configuration of this stable network. In particular, we provide an algorithm that allows the computation of the stable network from information on the individual benefit functions $v_i(\cdot)$ and cost structure $c(i, \cdot)$. In this section, we restrict ourselves to providing the necessary definitions and concepts to understand our main results and the main idea behind the algorithm. We illustrate this algorithm by an example. The formal proof of our result can be found in the appendix.

Consider four individuals, $n = 4$. The distance, quality and cost functions are given below in table 1 together with the benefit function v_i which is assumed to be identical for all $i = 1, \dots, 4$. Observe that the cost function, $c(i, j)$, is constructed using the formula, $c(i, j) = d(i, j) - \theta(j)$.

$d(i, j)$					$\theta(i)$		$c(i, j)$					$v(n)$	
$i \setminus j$	1	2	3	4	i		$i \setminus j$	1	2	3	4	n	
1	/	49	76	23	1	10	1	/	44	56	8	0	0
2	49	/	58	16	2	5	2	39	/	38	1	1	100
3	76	58	/	62	3	20	3	66	58	/	47	2	150
4	23	16	62	/	4	15	4	13	11	42	/	3	180

Table 1: Overview

The algorithm, given in figure 1, selects for each iteration $t = 0, 1, 2, \dots$ a set of agents $A_t \in N_t$ — with N_t a subpopulation from N — that deletes the links in network $g_t \in \mathcal{G}_n$ they do not want to maintain, which results in the network g_{t+1} . We start with the complete network $g_0 = g_c$ and the grand population $N_0 = N$. The algorithm guarantees that for each iteration $t = 1, 2, \dots$, $g_t \subseteq g_{t-1}$, $N_t \subseteq N_{t-1}$ and $N_t \neq N_{t-1}$. We end the algorithm when $N_t = \emptyset$, lets say at iteration T , which is guaranteed to happen in finite time.

In every round of the algorithm t , the set $A_t \subseteq N_t$ is constructed containing those agents $i \in N_t$ whose set of links they would like to maintain in g_t is equal to their set of links in the weakly stable network. The set A_t can be shown to be never empty. The

network g_{t+1} is then constructed from g_t by letting all members of A_t cut all the links they do not want to keep. The set N_{t+1} is constructed by deleting A_t from N_t and we move on to iteration $t + 1$. Since the remaining links are part of the weakly stable network, when $N_T = \emptyset$ we end up with the stable network.

We now discuss each step in the algorithm, for a certain iteration t , and we relate it to our numerical example above.

Assume that we arrive at step t of the algorithm with network g_t and coalition $N_t \subseteq N$. In order to determine A_t we take the following steps.

In a first step, we compute for every $i \in N_t$ and $j \in N$ for which $ij \in g_t$, the number of individuals, k , in N such that $ik \in g_t$ and $c(i, k) \leq c(i, j)$. We denote this number by $s(i, j, g_t)$.

Definition 6. For all $i, j \in N$ and $g \in \mathcal{G}_n$,

$$s(i, j, g) = |\{k \in N | ik \in g \text{ and } c(i, k) \leq c(i, j)\}|.$$

Considering our numerical example, we have that, for example, $s(1, 3, g_0) = 3$: there are three individuals, namely 2, 3 and 4, who are linked to 1 in g_0 and have linking-costs less than or equal to $c(1, 3)$.

In a second step, we compute for each $i \in N_t$ and $j \in N$ with $ij \in g_t$ the maximal number $k(i, j)$ such that $v_i(k(i, j)) - v_i(k(i, j) - 1) \geq c(i, j)$ if it exists. If it does not exist (i.e. $v_i(n) - v_i(n - 1) < c(i, j)$ for all $n = 1, 2, \dots$), we set $k(i, j) = 0$.

Definition 7. For all $i, j \in N$,

$$k(i, j) = \max \{ \{n \in \mathbb{N} | v_i(n) - v_i(n - 1) \geq c(i, j)\} \cup \{0\} \}.$$

The existence of $k(i, j)$ is guaranteed by assumption b . For our example, we have $k(1, 3) = 1$.

Given the first and second step above, we can form, for each pair of individuals (i, j) ($i \in N_t, j \in N$) the pair $(s(i, j, g_t), k(i, j))$. These pairs are shown in the first four columns of table 2 below.

	$(s(i, j, g_0), k(i, j))$				$\Lambda_i(g_0)$	$\rho_i(g_0)$	$\rho_i(g_0) - \theta(i)$
	1	2	3	4			
1	/	(2,2)	(3,1)	(1,3)	{4, 2}	44	34
2	(3,2)	/	(2,2)	(1,4)	{4, 3}	38	33
3	(3,1)	(2,1)	/	(1,2)	{4}	47	27
4	(2,3)	(1,3)	(3,2)	/	{2, 1}	13	-1

Table 2: Summary table for $t=0$

In a third step, we look for all pairs of individuals (i, j) with $i \in N_t$ and $j \in N$ with $ij \in g_t$ for which $s(i, j, g_t) \leq k(i, j)$. These are indicated in bold in the first four columns

of table 2. The set of all these individuals $j \in N$ is called the set of acceptable agents for i in network g_t , denoted $\Lambda_i(g_t)$.

Definition 8. For all $i, j \in N$ and all $g \in \mathcal{G}_n$, the set of acceptable agents for i in network g is given by,

$$\Lambda_i(g) = \{j \in N \mid ij \in g \text{ and } s(i, j, g_t) \leq k(i, j)\}.$$

In particular, both agents 2 and 4 are acceptable for 1 in network g_0 . Hence, $\Lambda_1(g_0) = \{4, 2\}$. Observe that it is possible that $\Lambda_i(g)$ is empty for some i and g . For our example, the set of acceptable agents is given in the table above (second column) for round 0.

Consider a network g , and an individual $i \in N$. Assume that individual i is granted the opportunity to alter the network g unilaterally. In other words, i is allowed to cut any number of his links. It can be shown that i will keep all links ij for which $j \in \Lambda_i(g)$ and cut all links with agents $j \notin \Lambda_i(g)$. The idea of the algorithm is to select a certain coalition, A_t , of N_t and allow for all $i \in A_t$ to perform this action: we delete all links for which $i \in A_t$, $j \in N$ and $j \notin \Lambda_i(g)$. The set A_t is determined in the following step of our algorithm.

In the fourth step, we compute for each $i \in N_t$ and each $j \in \Lambda_i(g_t)$, the cost $c(i, j)$ and we retain the highest value. We call this the critical cost for i in g_t and we denote it by $\rho_i(g_t)$. If i has no critical agent in network g_t , then we set $\rho_i(g_t) = 0$. For our example, we have that $c(1, 2) > c(1, 4)$, hence $c(1, 2) = 44$ is the critical cost for 1 in g_0 . From the discussion above, we see that $i \in N$, in network g , would like to cut all links with individuals j for which $ij \in g$ and $c(i, j) \leq \rho_i(g)$.

Definition 9. For all $i, j \in N$ and all $g \in \mathcal{G}_n$, the critical cost for i in g is given by,

$$\rho_i(g) = \max \{ \{0\} \cup \{c(i, j) \mid j \in \Lambda_i(g)\} \}.$$

Now, we compute for all agents $i \in N_t$ the value $\rho_i(g_t) - \theta(i)$ and we identify the coalition of agents $i \in N_t$ for which $\rho_i(g_t) - \theta(i)$ is smallest. This coalition determines A_t . For our numerical example, we obtain $A_0 = \{4\}$.

The intuition behind the particular choice of A_t is the following. We know that any agent $i \in N_t$ would like to cut all links $ij \in g_t$ for which $c(i, j) > \rho_i(g_t)$ (or equivalently, $j \notin \Lambda_i(g_t)$) and keep all links with $c(i, j) \leq \rho_i(g_t)$ (or equivalently $j \in \Lambda_i(g_t)$). Now consider an individual $i \in A_t$ and let $c(i, j) \leq \rho_i(g_t)$. Then,

$$\begin{aligned} c(j, i) &= d(i, j) - \theta(i) \\ &= c(i, j) - \theta(i) + \theta(j) \\ &\leq \rho_i(g_t) - \theta(i) + \theta(j) \\ &\leq \rho_j(g_t). \end{aligned}$$

The first and second equality follows from the definition of $c(j, i)$ and $c(i, j)$. The first inequality follows from the fact that $c(i, j) \leq \rho_i(g_t)$ while the last inequality follows from the definition of A_t . As such, we see that $c(j, i) \leq \rho_j(g_t)$ or equivalently $i \in \Lambda_j(g_t)$, i.e. i is

an acceptable agent for j in g_t . As such, j will not have an incentive to cut his link with i in network g_t .

We are now ready to update the algorithm to the next step. To construct g_{t+1} from g_t we take the collection of individuals obtained from the previous step, A_t , and we delete all links ij for which $ij \in g_t$, $i \in A_t$ and $j \notin \Lambda_i(g_t)$ (or equivalently $c(i, j) > \rho_i(g_t)$). Considering our particular example, we have that $A_0 = \{4\}$ and $\Lambda_4(g_0) = \{2, 1\}$, hence, $g_1 = g_0/\{34\}$. Next, we construct the set $N_{t+1} = N_t - A_t$, giving $N_1 = \{1, 2, 3\}$. Finally, we go back to step 1 replacing g_t with g_{t+1} and N_t by N_{t+1} .

For our example, we see that in the next iteration, $A_1 = \{2, 3\}$ and that g_2 is formed by deleting the link between 1 and 2 and 1 and 3 (see table 3). Note that since 4 cut his link with 3, agent 2 became a member of the set of acceptable agents of agent 3, $\Lambda_3(g_1)$.

	$(s(i, j, g_1), k(i, j))$				$\Lambda_i(g_1)$	$\rho_i(g_0)$	$\rho_i(g_0) - \theta(i)$
	1	2	3	4			
1	/	(2,2)	(3,1)	(1,3)	{4, 2}	44	34
2	(3,2)	/	(2,2)	(1,4)	{4, 3}	38	33
3	(2,1)	(1,1)	/	/	{2}	53	33

Table 3: Summary table for t=1

	$(s(i, j, g_2), k(i, j))$				$\Lambda_i(g_2)$	$\rho_i(g_0)$	$\rho_i(g_0) - \theta(i)$
	1	2	3	4			
1	/	/	/	(1,3)	{4}	8	-2

Table 4: Summary table for t=2

Finally, we end with $N_2 = \{1\}$ in round 2, see table 4. No link is deleted in round 2. This is a general feature of our algorithm: any link that is not deleted in a certain iteration t will not be deleted further in the algorithm (see lemma 2 in the appendix). As such, we may end the algorithm as soon as N_t is a singleton. We see that g_3 is the final network in the algorithm because $N_3 = \emptyset$. We obtain $g_3 = \{\{1, 4\}, \{2, 3\}, \{2, 4\}\}$. The full algorithm is summarized in figure 1.

The following theorem shows that a weakly stable network exists, in the form of the output of figure 1, and that it is unique. The proof can be found in the appendix.

Theorem 1. *There exists a unique weakly stable network g_{ws} and it is reproduced by the output of the algorithm in figure 1.*

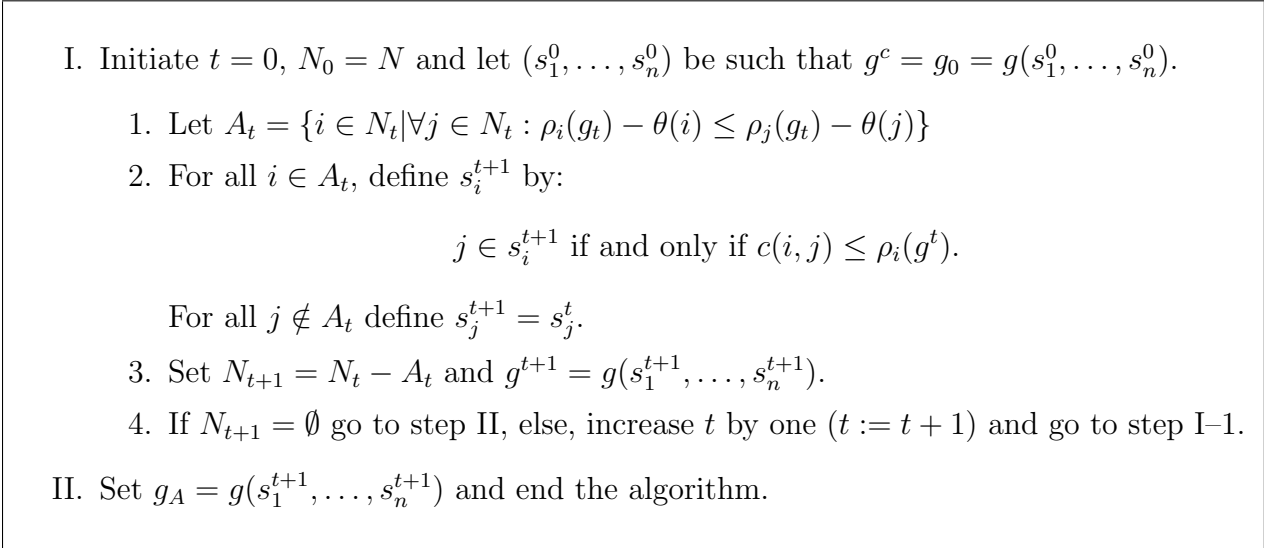


Figure 1: Stable network algorithm.

Allowing for the simultaneous formation and cutting of a link, as presented in the definition of a weakly stable network, is essential for uniqueness of the stable network. Indeed, it is possible that there are multiple pairwise stable networks (see definition 1). To see this, consider an example with three individuals i, j and k and assume that both j and k are willing to link with i but not with each other. Further assume that i would benefit from linking to either j and k , but not to both. This happens if, for example,

$$\begin{aligned}
c(j, i) < v_j(1) - v_j(0) < c(j, k), & \quad c(k, i) < v_k(1) - v_k(0) < c(k, j), \\
v_i(1) - v_i(0) > c(i, k), c(i, j) & \quad \text{and} \quad v_i(2) - v_i(1) < c(i, k), c(i, j).
\end{aligned}$$

Assume that $c(i, j) < c(i, k)$ and let $g = \{ik\}$. Since i and k have no other links, neither of them will have an incentive to cut the link ik . Agent j may propose to form a link with i . In the pairwise stability framework, where i cannot cut and form a link at the same time, i will decline j 's proposal although i would like to cut his link with k and link with j because this would give him a positive net benefit of $c(i, k) - c(i, j)$. In this setting, the concept of pairwise stability does not lead to a unique stable network as both networks with $g = \{ij\}$ and $g = \{ik\}$ are pairwise stable. On the other hand, if we impose the weak stability concept, we see that, in this case, i will cut his link with k and accept the proposal to link with j at the same time leaving only $g = \{ij\}$ to be weakly stable.

As for the strongly stable network, we show in the following proposition that the unique weakly stable network is also strongly stable. This implies that if we would impose no restrictions on the actions available to the individuals and the size of the deviating coalition we would still obtain the same stable network. Observe that this also implies that the output of our algorithm is also stable with respect to every intermediate stability concept, in particular the strongly pairwise stability concept from Belleflamme and Bloch (2004)

(see definition 5).

Proposition 1. *The unique weakly stable network g_{ws} is strongly stable.*

We finalize this section by giving an alternative cost structure for which above existence and uniqueness result remains to hold. In the basic model, we assumed that $c(i, j)$ could be written as the difference between a symmetric distance function $d(i, j)$ and an agent specific quality function $\theta(j)$. On the other hand, we could also specify $c(i, j)$ as the ratio of a symmetric distance function $d(i, j)$ and an agent specific (non-zero) quality function $\theta(j) : N \rightarrow \mathbb{R}_{++}$:

$$c(i, j) = \frac{d(i, j)}{\theta(j)}.$$

All the results of this section are valid for this cost function with a minor adjustment in the definition of the set, A_t , which is used in step I-1 in the algorithm of figure 1, which changes to,

$$A_t \in \left\{ i \in N_t \mid \forall j \in N_t : \frac{\rho_i(g_t)}{\theta(i)} \leq \frac{\rho_j(g_t)}{\theta(j)} \right\}.$$

4 Empirical Illustration

Using data from a risk sharing network in Nyakatoke, Tanzania, we verify whether our model is able to predict the existence of such risk sharing links and whether it can capture the features of risk sharing networks observed in real life. We keep this illustration as simple as possible, using the geographical distance between two households as a proxy for our distance measure $d(i, j)$, while the value of livestock owned by a household serves as a proxy for our quality measure $\theta(i)$. Various empirical studies⁹ on risk sharing networks identify geographical proximity as one of the important determinants of risk sharing links. For the case of livestock, Comola (2008) argues that livestock is a real wealth dimension, more so than land ownership.

As a measure of predictive power we look at the fraction of existing links in the Nyakatoke network that also exist in the stable network produced by the algorithm, as well as the fraction of non-existing links in the Nyakatoke network that do not exist in the stable network. In order to get a sense of how accurate the predictions based on the model are, we compare our results with predictions based on using fitted link probabilities from a logit regression model using geographical distance and/or livestock values. Since both predictions rely on the same set of information — geographical distance and livestock value — we are able to filter out the contribution of our network formation model by comparing the two prediction fractions. We fully realize we use very crude measures of distance and quality and in this respect this exercise should not be viewed as an empirical application but rather as a way of showing that the network formation process presented in this paper might have some validity when it comes to predicting the existence of links.

⁹See Fafchamps and Lund (2003), Fafchamps and Gubert (2007), De Weerd (2002) and Comola (2008).

At the time of the survey in 2000, Nyakatoke consisted of 119 households, 116 for which data are fully available. All households were asked the following question: *Can you give a list of people [...], who you can personally rely on for help and/or that can rely on you for help in cash, kind or labour?* The 116 households mentioned a total of 960 intra-village network partners, which amounts to 480 risk sharing links. However, the degree (i.e. the number of network partners) is unevenly distributed, ranging from 1 to 32 partners. Furthermore, the Nyakatoke risk sharing network has features corresponding to other empirical regularities of large social networks, as noted by Comola (2008): there is only one component which has an average path length of 2.5 steps, clustering is higher than would be expected if the formation process would be totally random and clustering tends to be negatively correlated with degree.¹⁰ The key characteristics of the Nyakatoke network are shown in the first column of table 6 below.

First, we focus on distances only, simply inputting geographical distances between any pair while assuming $\theta(i) = 0$ for all $i \in N$. The payoff function is of the form

$$\pi_i(g) = \alpha * n_i(g)^{2/3} - \sum_{j \in L_i(g)} d(i, j),$$

with the scale parameter α chosen such that the total number of links in the network is as close to 480 (the total number of links in the real-life network) as possible. For this exercise, $\alpha = 0,454$, with a total number of 479 links. The prediction results are given in the first row of table 5. Keeping in mind that our handling of the data is rather raw, the results are mixed. Somewhat less than one third of all existing links is predicted in our model. Earlier simulations — not shown here — show that if we randomly select 480 pairs to be linked, we get a prediction rate for linked pairs between 3 and 13% and between 92,5 and 93,2% for unlinked pairs. Hence, we get prediction rates well above what we would expect from a non-informative model. In order to filter out the contribution of our model to predicting the Nyakatoke network, we compare the results from the first row of table 5 with the prediction results from the fitted probability model as shown in the second row. In this model, we ran the following logit regression with standard errors in parentheses:

$$\ln \left(\frac{Pr(ij \in g)}{1 - Pr(ij \in g)} \right) = \underset{(0,09)}{-1,3} - \underset{(0,22)}{2,9} * \text{distance}_{ij}$$

Then, networks are simulated where the probability that a link exists between a pair is equal to the fitted probability from the logit regression. The numbers shown in the table are the [5-95] percentiles, obtained over 100 000 simulations. The prediction rates for our model are above these intervals, indicating that the network formation process as described in this paper is relevant to explain the existence or absence of certain links in the Nyakatoke risk sharing network.

The simulation with the quality measure is a bit more complex. The quality measure $\theta(i)$ is the value of livestock of i . First, since the livestock value is of a different magnitude

¹⁰Another empirical regularity is a positive correlation in degree of connected households for which we could not find evidence in the Nyakatoke risk sharing network.

Fraction of Correct predictions	Linked	Unlinked
Model (only distance)	0,313	0,947
Fitted Probability (only distance)	[0,088 - 0,133]	[0,926 - 0,936]
Model (distance and quality)	0,333	0,948
Fitted Probability (distance and quality)	[0,098 - 0,146]	[0,927 - 0,937]

Table 5: Prediction results

as the geographical distance measure,¹¹ we scale it down using the ratio of coefficients from the following logit regression:

$$\ln \left(\frac{Pr(ij \in g)}{1 - Pr(ij \in g)} \right) = -1,4 - \frac{3}{(0,09)} * distance_{ij} + \frac{1 * 10^{-6}}{(1,3 * 10^{-7})} * \sum_{k=i,j} livestock_k$$

which turns out to be $\frac{3}{1000000}$. Second, adding quality adds a weighting parameter β to the net benefit function:

$$\pi_i(g) = \alpha * n_i(g)^{2/3} - \sum_{j \in L_i(g)} ((\beta * d(i, j) - (1 - \beta) * \theta(j)).$$

For different values of β — between 0 and 1 with steps of 0,02 — we calibrate α such that the total number of links is approximately 480. The third row of table 5 gives the prediction results for the $\{\beta, \alpha\}$ -combination which yielded the best prediction results, $\{\beta = 0,48, \alpha = 2,49\}$. Again, comparing these results with the results from the fitted probability model with distance and quality as shown in the fourth row indicates that our model can help explain the existence of links in the Nyakatoke network, although prediction fractions do not improve markedly when adding quality.

As far as the features of the simulated networks, shown in table 6, are concerned, we can say that they match at some points the features of networks observed in real life — high degree of clustering, negative correlation between clustering and degree and positive correlation in degree of connected agents. At other points — the uneven distribution of degree as well as the low characteristic path length — the model fails to replicate, which can be attributed to the usage of the raw distance and quality measures. For example, geographical distance is a measure which is a metric, satisfying triangle inequality¹², hence producing high characteristic path lengths. If we would introduce more accurate social distance measures, which would not necessarily satisfy triangle inequality, characteristic path length may be lower. The inaccuracy of the data may also explain the fact that we did not obtain an uneven degree distribution. Additionally, we assumed identical benefit

¹¹Distance is expressed in kilometers with values ranging from close to 0 to 1,8, while livestock can take on values to over 1,5 million.

¹² $d(i, j) + d(j, k) \geq d(i, k)$

	Nyakatoke	Model (distance)	Model (distance and quality)
[Min Median Max]	[1 7 32]	[1 8 16]	[3 8 10]
Char. Path length	2,54	4,23	3,64
Clustering	0,19	0,62	0,52
Corr. Clust. - Degree	-0,21	-0,043	-0,39
Corr. Degree linked agents	0,04	0,70	0,49

Table 6: Key statistics of networks: The clustering coefficient used is the overall clustering coefficient (see Jackson (2008)).

function for every household. Introducing some heterogeneity on the benefit side may lead to a more uneven degree distribution.

5 Concluding remarks

In an endogenous network formation setting where only the direct network matters for utility, this paper introduced very general forms of heterogeneity on the cost side. The costs are characterized by the distance between individuals and individual quality. The only restrictions we impose is that the distance is symmetric, that each individual has a perfect ordering of linking costs without indifference and that individuals are never indifferent between forming and cutting a link. Using an algorithm we were able to identify the unique weakly (strongly) stable network. Individuals will try to link up to individuals with the lowest linking costs so as to maximize the net gains from the direct network. Not everyone will succeed in doing this however, as some desired network partners may refuse to form a link with them.

The algorithm provides an elegant way to determine the emerging unique stable network. Different structures will give rise to different configurations and hence, using the framework, it might become possible to relate differences in features of networks to differences in underlying cost structure.

The uniqueness result of the stable network hinges on the assumptions made regarding the cost function — perfect preference ordering and symmetric distance function — as well as on the actions available to the individuals — the fact that individuals can simultaneously cut and form a link, i.e. the fact that weak stability strengthens the notion of pairwise stability. Relaxing these assumptions would invalidate the uniqueness result by including some history dependence in the model. However, the basic configuration of the emerging stable networks will be little affected by this relaxation, so that we can consider this kind of randomness as irrelevant to the model. A non-trivial type of history dependence would result from relaxing the assumptions with respect to the utility function — such as imposing more intermediate forms of friction or relaxing the concavity assumption — and hence, may be an important direction for future research.

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A Proofs

A.1 Preliminary results

Before proving theorem 1, we introduce some lemmata.

Lemma 1. *If $i \in A_t$, $ij \in g_{t+1}$ and $s \geq t$, then $ij \in g_{s+1}$.*

Proof. Assume that $i \in A_t$ and $ij \in g_{t+1}$. Further, let $j \in A_p$. If $p < t$, we have, by $g_{t+1} \subseteq g_{p+1}$, that $ij \in g_{p+1}$. As there is no iteration between t and s where ij could be cut, we conclude that $ij \in g_{s+1}$

Therefore assume that $p \geq t$. Observe that it suffices to show that $c(j, i) \leq \rho_j(g_p)$ because this guarantees that j will not cut the link ij in iteration p of our algorithm. From $i \in A_t$ and $ij \in g_{t+1}$, we conclude that,

$$c(j, i) = c(i, j) - \theta(i) + \theta(j) \leq \rho_i(g_t) - \theta(i) + \theta(j) \leq \rho_j(g_t).$$

This implies,

$$s(j, i, g_t) \leq k(j, i),$$

and from $g_p \subseteq g_t$, we derive,

$$s(j, i, g_p) \leq s(j, i, g_t) \leq k(j, i),$$

Hence, $c(j, i) \leq \rho_j(g_p)$. □

An important corollary of 1 is that for all iterations t of the algorithm and $i \in A_t$, if $ij \in g_{t+1}$, then $ij \in g_A$ (here g_A is the output of the algorithm).

Lemma 2. *If the network g is weakly stable and $g \subseteq g'$, then for all $ij \in g'$, if $c(i, j) \leq \rho_i(g)$ and $c(j, i) \leq \rho_j(g)$, then $ij \in g$.*

Proof. Assume that $c(i, j) \leq \rho_i(g')$ and $c(j, i) \leq \rho_j(g')$. If $ij \notin g$, then by weak stability of g it follows that (using assumption e),

$$v_i(n_i(g) + 1) - v_i(n_i(g)) < c(i, j) \quad \text{or} \quad v_j(n_j(g) + 1) - v_j(n_j(g)) < c(j, i).$$

Otherwise we would have that both i and j would benefit from creating the link ij . This shows that either $k(i, j) < n_i(g) + 1$ or $k(j, i) < n_j(g) + 1$. Assume without loss of generality that the first of the two holds. From $c(i, j) \leq \rho_i(g')$, it follows that

$$s(i, j, g') \leq k(i, j) \leq n_i(g).$$

As $g \subseteq g'$, $ij \in g'$ and $ij \notin g$, it follows that there must be an agent, k , which is linked to i in g and for which $c(i, k) > c(i, j)$. This shows that i would benefit by cutting the link with k and creating the link with j . g is weakly stable, hence, it must be that this is not beneficial for j , i.e.

$$v_j(n_j(g) + 1) - v_j(n_j(g)) < c(j, i).$$

This implies $k(j, i) < n_j(g) + 1$. Analogously to the previous case, we conclude that there must be an individual h such that $jh \in g$ and $c(j, h) > c(j, i)$.

Conclude that both i and j can benefit from linking with each other and cutting their link with respectively k and h . Therefore, g cannot be weakly stable. □

A.2 Proof of theorem 1

We only show uniqueness. The property that the output of the algorithm g_A is weakly stable follows from the proof of proposition 1 as every strongly stable network is also weakly stable.

Therefore, assume that there exists a weakly stable network g_{ws} . We need to show that $g_A = g_{ws}$. We work by induction on the iterations in the algorithm. In particular we show that for all $t = 0, 1, \dots, T - 1$,

1. if $i \in A_t$ and $ij \in g_{t+1}$, then $ij \in g_{ws}$,
2. $g_{ws} \subseteq g_{t+1}$.

If these two conditions hold, then from lemma 1 it follows that $g_{ws} = g_T = g_A$.

First of all, observe that $g_{ws} \subseteq g_0 = g_c$. Consider the case $t = 0$. Now, let $i \in A_0$ and $ij \in g_1$. It follows that $c(i, j) \leq \rho_i(g_0)$. As such,

$$c(j, i) = c(i, j) - \theta(i) + \theta(j) \leq \rho_i(g_0) - \theta(i) + \theta(j) \leq \rho_j(g_0).$$

It follows, from lemma 2 that $ij \in g_{ws}$. In addition, consider $i \in A_0$ and $ih \notin g_1$. It follows that $c(i, h) > \rho_i(g_0)$ or equivalently

$$v_i(n_i(g_1)) - v_i(n_i(g_1) - 1) < c(i, h).$$

Assume that $ih \in g_{ws}$. Since all $ij \in g_{t+1}$ are part of g_{ws} , it must be that $n_i(g_1) < n_i(g_{ws})$. Then,

$$v_i(n_i(g_{ws}) - v_i(n_i(g_{ws}) - 1) < c(i, h).$$

As such, i will benefit from cutting his link with h in g_{ws} . Hence, g_{ws} cannot be a stable network. We have that $ih \notin g_{ws}$ and $g_{ws} \subseteq g_1$.

Assume that the induction holds for all iterations up to $t - 1$ and consider iteration t . Let $i \in A_t$ and $ij \in g_{t+1}$, then

$$c(j, i) = c(i, j) - \theta(i) + \theta(j) \leq \rho_i(g_t) - \theta(i) + \theta(j) \leq \rho_j(g_t).$$

Again, using lemma 2, it follows that $ij \in g_{ws}$. This shows the first requirement. We still need to show the second requirement. Towards this end, assume on the contrary that for some $i \in A_t$, $n_i(g_t) \geq n_i(g_{ws}) > n_i(g_{t+1})$. From lemma 1 one can easily deduce that for all $i \in A_t$, $\rho_i(g_t) = \rho_i(g_{t+1})$. Hence for $ij \in g_{ws}, g_t$ and $ij \notin g_{t+1}$,

$$c(i, j) > \rho_i(g_{t+1}) \quad \text{or equivalently} \quad v_i(n_i(g_{t+1})) - v_i(n_i(g_{t+1}) - 1) < c(i, j).$$

From $n_i(g_{ws}) > n_i(g_{t+1})$, we deduce that,

$$v_i(n_i(g_{ws}) - v_i(n_i(g_{ws}) - 1) < c(i, j).$$

As such, i will benefit from cutting his link with j in g_{ws} . As such, g_{ws} cannot be a stable network. Now, we have that $g_{ws} \subseteq g_t$, $n_i(g_{ws}) \leq n_i(g_{t+1})$ for all $i \in A_t$ and $ij \in (g_t - g_{t+1})$ only if i or $j \in A_t$. Conclude that $g_{ws} \subseteq g_{t+1}$.

A.3 Proof of proposition 1

Take any $C \subseteq N$ and let g_A be the output of the algorithm. We prove that there will always be at least one individual $i \in C$ who loses by any rearrangement of the linking pattern among the members of C , under the condition that links are formed under mutual consent. Consider an agent, i , defined by:

$$i \in \{j \in C | j \in A_t \text{ and } \forall k \in C \text{ if } k \in A_p, \text{ then } t \leq p\},$$

Agent i is the member of C who, in our algorithm, cuts links first.

Let g_A be the output of the algorithm and let g' be a network where only links within C are rearranged and perhaps links with at least one agent in the coalition are cut. Further, assume on the contrary, that all individuals within C benefit from changing from g_A to g' . Let $A = L_i(g_A) \setminus L_i(g')$ and $B = L_i(g') \setminus L_i(g_A)$. A is the set of agents who are linked to i in network g_A but not in g' and B is the set of agents who are linked to i in g' but not in g_A . We assume that at least one of the sets A or B is nonempty. Otherwise, i is indifferent between network g_A and g' . In that case, we may exclude i from the set C and consider the smaller set $C - \{i\}$. Now, for all $j \in N$, if $ij \in g_{t+1}$, then it follows (from lemma 1) that $ij \in g_A$. This implies that whenever $j \in A$ then $c(i, j) \leq \rho_i(g_t)$. On the other hand, because i is the first individual to choose in the algorithm, it must be that for all $j \in B$, $c(i, j) > \rho_i(g_t)$. Conclude that for all $j \in A$ and $k \in B$: $c(i, j) < c(i, k)$. We have that:

$$\pi_i(g_A) - \pi_i(g') = v_i(n_i(g_A)) - v_i(n_i(g')) - \sum_{j \in A} c(i, j) + \sum_{j \in B} c(i, j).$$

If $|A| = |B|$, the proof follows from the facts that $c(i, j) < c(i, k)$ for all $j \in A$ and $k \in B$ and $n_i(g_A) = n_i(g')$. We distinguish two other cases.

(1) If $|B| > |A|$, we can write:

$$\pi_i(g_A) - \pi_i(g') = \sum_{k=0}^{|B|-|A|-1} (v_i(n_i(g_A) + k) - v_i(n_i(g_A) + k + 1)) - \sum_{j \in A} c(i, j) + \sum_{j \in B} c(i, j).$$

Now, select $|A|$ individuals from B and call the remaining set B' , then:

$$\pi_i(g_A) - \pi_i(g') > \sum_{k=0}^{|B|-|A|-1} (v_i(n_i(g_A) + k) - v_i(n_i(g_A) + k + 1)) + \sum_{j \in B'} c(i, j).$$

For all $k \geq 0$ and $j \in B$, it is also the case that:

$$v_i(n_i(g_A) + k + 1) - v_i(n_i(g_A) + k) < c(i, j).$$

Hence,

$$\pi_i(g_A) - \pi_i(g') > - \sum_{j \in B'} c(i, j) + \sum_{j \in B'} c(i, j) = 0.$$

(2) Now, consider the case where $|A| > |B|$. Then we can write:

$$\pi_i(g_A) - \pi_i(g') = \sum_{k=0}^{|A|-|B|+1} (v_i(n_i(g_A) - k) - v_i(n_i(g_A) - k - 1)) - \sum_{j \in A} c(i, j) + \sum_{j \in B} c(i, j).$$

Select $|B|$ individuals from A and call the remaining set A' . This gives:

$$\pi_i(g_A) - \pi_i(g') > \sum_{k=0}^{|A|-|B|-1} (v_i(n_i(g_A) - k) - v_i(n_i(g_A) - k - 1)) - \sum_{j \in A'} c(i, j).$$

For all $k \geq 0$ and $j \in A$, it is also the case that:

$$v_i(n_i(g_A) - k) - v_i(n_i(g_A) - k - 1) > c(i, j).$$

Hence,

$$\pi_i(g_A) - \pi_i(g') > \sum_{j \in A'} c(i, j) - \sum_{j \in A'} c(i, j) = 0.$$

Conclude that i has higher payoffs in g_A than in g' contradicting the assumption that g_A is not strongly stable.