A Characterization of Weakly Pairwise Nash Stable Networks

Valeska Groenert[*,†]

May 2010

Abstract

This paper defines a modified version of Myerson's linking game and shows that the set of weakly pairwise Nash stable networks is characterized by the set of networks arising from pure strategy Nash equilibria of this game. The game, called the linking game with player pairs, has a well-defined strategic form, and therefore the characterization allows to introduce mixed strategies to the network formation game.

In addition, it is shown that weakly pairwise Nash stable networks result from the subgame-perfect Nash equilibrium in undominated strategies of a two-stage network formation game. In the first stage of this game, players can form potential links with each other. A pair of players that has formed a potential link can coordinate to form a link in the second stage.

1 Introduction

Consider a dynamic process of network formation in which the network evolves according to successive modifications by myopic players. In each modification, either a single player severs a subset of his current links or a pair of players forms a link. A stable state of this process is a weakly pairwise Nash stable network: No player benefits from severing links and

[*]Contact: valeska.gronert@uab.cat

[†]I would like to thank Myrna Wooders and Sanjeev Goyal for their helpful comments. Support from the Spanish Ministry of Science and Innovation through grant "Consolidated Group-C" ECO2008-04756, and by the Generalitat de Catalunya, Departament d'Universitats, Recerca i Societat de la Informació through the Barcelona GSE Research Network is gratefully acknowledged.

for no pair both players benefit from forming a link.[1] While a weakly pairwise Nash stable network can be defined as the outcome of the dynamic process just described, it has not been defined in terms of a well-defined game in strategic form. In this paper I characterize the set of weakly pairwise Nash stable networks as the set of pure strategy Nash equilibrium networks of a network formation game in strategic form. It is then natural to consider the equilibrium of the mixed extension of this strategic form game. Because the game is finite, this equilibrium exists. Thus, my characterization result makes possible a prediction of the network formation process in cases when a weakly pairwise Nash stable network does not exist.

The network formation game I define is a variant of Myerson's linking game (Myerson, 1991). In Myerson's linking game all players simultaneously announce sets of players with whom they wish to form links. Links are formed if and only if two players announce each other. While in equilibrium no player wishes to cut links, equilibrium networks do permit situations in which two players would both benefit from forming a link.

To eliminate this coordination failure in Myerson's linking game, I introduce pairs of players as additional players to the game. The so modified game is called the linking game with player pairs. Each player pair announces whether it wishes to form a link. However, the strategy of a pair ij has an impact on the outcome of the game only if neither of the two players announces the other. In this case, if ij announces it wishes to form a link, the rules of the game are such that the link ij is added. The payoffs for player pairs are defined so that a pair benefits from a link if and only if both players benefit from it. Together with the rules of the game, this payoff specification ensures that the set of equilibrium outcomes is precisely the set of weakly pairwise Nash stable networks.

To provide intuition for the presence of pairs of players coordinating on the formation of their link, I define and analyze a game in which network formation occurs in two stages. In the first stage, players play Myerson's linking game. In the second stage, they play a constrained version of the linking game with player pairs in which only pairs that have formed a link in the first stage are added as players to the game. The outcome of the game is the network formed in the second stage. Thus, the only purpose of a first stage link is

[1] This paper uses the concept of weak pairwise Nash stability instead of the more familiar concept of pairwise Nash stability for technical reasons, which will be explained in due course.

that it allows players to coordinate in the second stage. A pair that has formed a link in the first stage ensures its (actual) link is formed in the second stage whenever both players benefit from doing so.

The two-stage linking game provides a natural model of link formation. Consider friendship formation as an example. There, the first stage might correspond to something like joining a club or getting to know the friends of friends and the second stage corresponds to actual friendship formation. The interpretation of stage one links eliminating coordination failure in stage two is that once two people get to know each other they will become friends provided they both like each other. Similarly, two people might never become friends, simply because they do not know (much about) each other. As a consequence, real-world networks might be stable even though they do not satisfy the definition of pairwise stability. For example, there are likely many people whose friendship we would enjoy if we only knew them. In this vein, Proposition 2 shows that the set of subgame perfect equilibrium networks of the two-stage linking game is larger than the set of pairwise Nash stable networks. However, as Proposition 3 shows, refining the equilibria of the two-stage linking game (by requiring undominated strategies) leads to weakly pairwise Nash stable equilibrium networks.

This paper relates to the literature of network formation that goes back to Myerson (1991) and the seminal work by Jackson and Wolinsky (1996). Here, I provide a noncooperative grounding of weakly pairwise Nash stable networks and thereby address the issue of their existence. The prior literature on network formation that uses pairwise stability or closely related concepts and addresses existence issues has either focused on establishing conditions that guarantee existence or shown existence in special settings (see e.g., Belleflamme and Bloch, 2004; Calvó-Armengol, 2004; Goyal and Joshi, 2006; Jackson and Watts, 2001). A novelty in this paper is that it presents a way to incorporate mixed strategies into a network formation game with coordinated moves. The paper thereby addresses the existence of pairwise Nash stable equilibrium.

The next section defines the linking game with player pairs and shows that its equilibrium outcomes are equivalent to the set of weakly Nash stable networks. At the end of the section, I show that the mixed strategy equilibrium might put weight on networks that are not part of an absorbing state of a dynamic process of network formation. Section 3 defines the two-stage linking game and shows that networks supported by its undominated subgame perfect equilibria are weakly pairwise Nash stable. Section 4 concludes.

2 The linking game with player pairs

Let $N = \{1, 2, ..., n\}$ be a finite set of players, let $g \subseteq \{ij : i, j \in N, i \neq j\}$ be a social network, and let G be the set of all networks. The payoff for individual $i \in N$ is given by the function $u_i : G \rightarrow R$. A network that is derived from network g by adding (deleting) link ij is denoted by $g+ij$ $(g-ij)$. The set of player i's neighbors in g is $N_i(g) = \{j \in N : ij \in g\}$.

In the *linking game*, each player $i \in N$ has the strategy set $S_i = P(N\backslash\{i\})$ (the power set of $N\backslash\{i\}$) with typical element s_i. Given a strategy profile $s = (s_1, s_2, ..., s_n)$, the outcome of the game is the network $g(s)$ defined by $ij \in g(s)$ if and only if $i \in s_j$ and $j \in s_i$. If $g = g(s)$, the profile s *supports* g. A network is *Nash stable* if it is supported by a Nash equilibrium of the linking game. In other words, a network is Nash stable if there exists a strategy profile s for the linking game such that

$$u_i(g(s)) \geq u_i(g(s_i', s_{-i})) \text{ for all } s_i' \in S_i, \text{for all } i \in N.$$

As has been pointed out in the literature, Nash stability is an unsatisfactory concept in the context of network formation. Its predictive power is limited as the set of Nash stable networks tends to be large. In particular, the empty network, a network where no link is formed, is always Nash stable. This is due to the fact that a network in which two players are not linked but both would benefit from the addition of the link can be Nash stable if neither of the two players indicates the other player. To overcome this coordination failure, Jackson and Wolinsky (1996) introduced the concept of pairwise stability. A network is *pairwise stable* if

$$\begin{aligned} u_i(g) &\geq u_i(g - ij) \text{ for all } i, \text{ for all } ij \in g, \text{ and} \\ u_i(g + ij) &> u_i(g) \Rightarrow u_j(g + ij) < u_j(g), \text{ for all } ij \notin g. \end{aligned}$$

In a pairwise stable network no single player wishes to sever a single link and no pair of players wishes to add a link.

A network is *pairwise Nash stable* if it is pairwise stable and Nash stable. Pairwise Nash stability captures the idea that adding a link requires mutual consent but cutting a link is at the discretion of either of the linked players. Note that one could decompose the concept into a noncooperative component, Nash stability, and a cooperative component, the second condition of pairwise stability. In this paper I consider a slightly weaker version of pairwise stability: A network is *weakly pairwise stable* if

$$\begin{aligned} u_i(g) &\geq u_i(g-ij) \text{ for all } i, \text{ for all } ij \in g, \text{ and} \\ u_i(g+ij) &> u_i(g) \Rightarrow u_j(g+ij) \leq u_j(g), \text{ for all } ij \notin g. \end{aligned}$$

Here, the idea is that a coordinated deviation by two players requires that both players benefit from that deviation. A network is *weakly pairwise Nash stable* if it is Nash stable and weakly pairwise stable. Jackson and Wolinsky (1996) informally discuss this notion in the last section of their paper. They point out that most of their results are not sensitive to which notion of pairwise stability is used.

To see that a weak pairwise Nash stable network might not exist it is useful to consider the following dynamic network formation process.[2] The process starts out with an arbitrary network. If the network is pairwise Nash stable, the process ends. If not, there exists at least one pair of players who wishes to be linked or a at least one single player who wishes to sever at least one of his links. One of the pairs who wish to add links (if any) or one of the players who wish to sever links (if any) is selected and the network is modified accordingly. If the modified network is pairwise Nash stable, the process ends; if not, the procedure is repeated. This process itself can be depicted as a (directed) network, called a supernetwork.[3] The set of nodes of the supernetwork is the set of networks G. A *supernetwork* is a directed

[2]Such processes are examined by, e.g., Bala and Goyal (2000), Goyal and Vega-Redondo (2005), Jackson and Watts (2002), and Watts (2001).

[3]Supernetworks were introduced by Page, Wooders, and Kamat, 2005.

network $\mathbf{G} \subseteq G \times G$ such that $(g, g') \in \mathbf{G}$ if and only if

$$\begin{aligned} (i)\ g' &= g - \{ik\}_{k \in B} \text{ for some } B \subseteq N_i(g) \text{ and } u_i(g') > u_i(g), \text{ or} \\ (ii)\ g' &= g + ij \text{ and } u_i(g') > u_i(g) \text{ and } u_j(g') > u_j(g). \end{aligned}$$

An arc (g, g') in the supernetwork precisely means that network g is not weakly pairwise Nash stable because either (i) or (ii) holds. The commonly used terminology is that network g' *defeats* network g. Given supernetwork $\mathbf{G}$, it is straightforward to identify weakly pairwise Nash stable networks. One simply searches for networks without outgoing arcs. Such a network is not defeated by any other network and must be weakly pairwise Nash stable. If a weakly pairwise Nash stable network does not exist, by the finiteness of G, the process must eventually end up in a cycle which has no outgoing arc (see Lemma 1 in Jackson and Watts, 2002). Figure 1 shows how a supernetwork could look like. Here, g^8 is weakly pairwise Nash stable. In addition, the networks g^0, g^2, g^3, and g^1 form a cycle.[4]

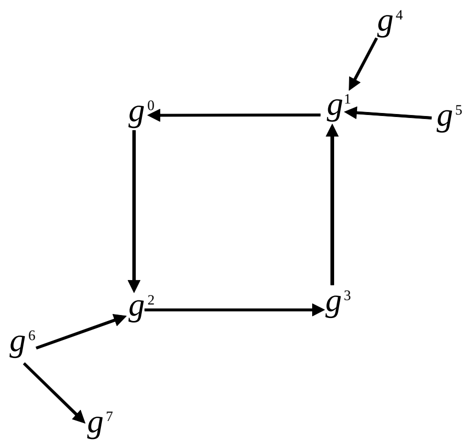

Figure 1

The possibility of a cycle in the supernetwork is demonstrated in more detail in Panel (a) of Figure 2.[5] The example is due to Jackson and Watts (2002), who also provide more details. Deviations by the four individual players and the pairs of players lead to a cycle.

[4] One could just as well define a supernetwork where a network is defeated by another network if and only if it is not pairwise Nash stable. In that case a network without outgoing arcs would be pairwise Nash stable.

[5] The significance of Panel (b) will be explained later.

At g^0, players 2 and 3 benefit from forming a link. So g^0 is defeated by g^1. There, player 3 has an incentive to sever his link to player 4, so g^2 defeats g^1. Network g^3 defeats g^2 because player 2 has an incentive to sever her link to player 3. Finally, at g^3 players 3 and 4 benefit from forming a link. If they do so, the process is back to g^0, closing the cycle.

Such cycles are symptomatic not only for the nonexistence of pairwise Nash stable networks. For example, when a pure strategy equilibrium of a finite noncooperative game does not exist, there will be a cycle of outcomes such that each outcome in the cycle can be reached from the previous outcome through a unilateral deviation that is beneficial for the deviating player. The standard example to illustrate this point is the game of Matching Pennies. Similarly, if the core of a finite cooperative game is empty, there will be a cycle of outcomes that are linked through coalitional deviations that are beneficial for the deviating coalition.

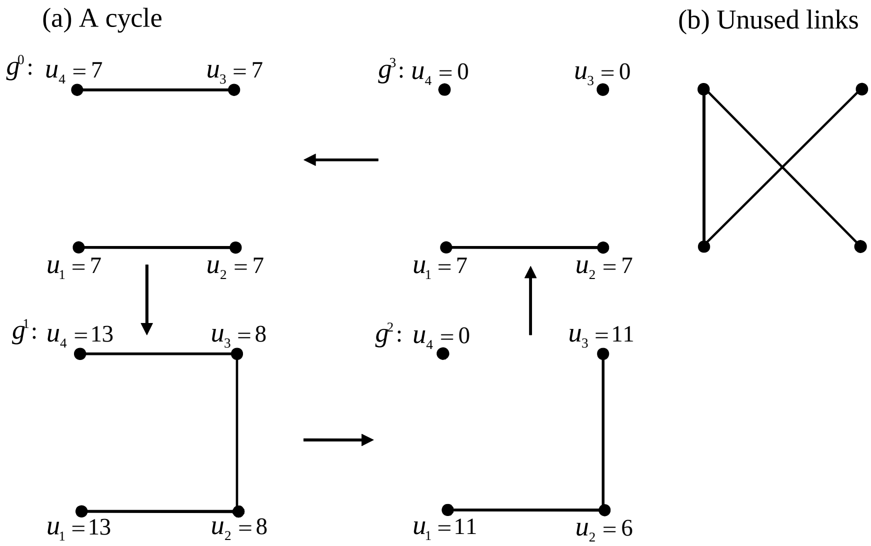

Figure 2

In noncooperative games the problem of existence is resolved by reverting to mixed strategies. Mixed strategies are probability distributions over sets of pure strategies. However, since no well-defined strategic form network formation game underlies the concept of weak pairwise Nash stability, it is not clear what a mixed strategy means in this context.[6]

[6] Calvó-Armengol and Ilkiliç (2009) relate pairwise Nash stable networks to the noncooperative concept

In the following, I define such a strategic form network formation game. With that game at hand, a natural way to introduce mixed strategies is via the mixed extension of that network formation game.

Formally, the linking game with player pairs is a noncooperative game with players acting in their own interest. In fact though, the game incorporates cooperation. This is done by treating pairs of players as players of the game, with strategies that are independent from the individual players' strategies. Each pair indicates whether it wishes to form a link. The game is designed so that a pair's indication of wanting to form a link has an effect on the outcome only when the corresponding individual players do not indicate each other. In addition, a pair's payoff is specified so that a pair wishes to add their link if and only if both players benefit from it. This design eliminates the possibility of two players failing to form a link even though both would benefit from it.

The linking game with player-pairs

The set of players is $N \cup \{ij : i, j \in N,\, i \neq j\}$.[7] The set of pure strategies available to each $i \in N$ is the same as in the linking game, $S_i = P(N\backslash\{i\})$. A pair ij has only two pure strategies, $S_{ij} = \{Y, N\}$, with the interpretation that $s_{ij} = Y$ indicates the pair ij wants to form the link ij and $s_{ij} = N$ that it does not want to form the link. A strategy profile is denoted by (s^s, s^p) where s^s is the subprofile for individual players and s^p is the subprofile for pairs. Whether or not the link ij is formed for a given strategy profile depends on s_i, s_j, and s_{ij}. Table 1 lists the possible combinations of s_i, s_j, and s_{ij}, showing that a pair's strategy only affects the outcome if neither of the individual players indicates the other player. Note also that, at profiles for which the link is formed, individual players always have the power to cut the link. In Cases 1 and 2, the individual player can switch to not indicating the other player. In Case 7, he could switch to *indicating* the other player. This last feature might seem somewhat paradoxical, but is necessary to avoid an equilibrium in which a player would like to sever several of his connections.

proper equilibrium. They find conditons on payoffs so that pairwise Nash stable networks and the proper equilibrium networks of Myerson's linking game coincide.

Bloch and Jackson (2006) define a pairwise Nash equilibrium as the Nash equilibrium profile s of the linking game that satisfies $u_i(g(s) + ij) > u_i(g(s)) \Rightarrow u_j(g(s) + ij) < u_j(g(s))$. This, however, does not permit the introduction of mixed strategies either.

[7] Compared to the linking game, there are $n(n-1)/2$ additional players, one for each pair of players.

Case	s_i	s_j	s_{ij}	Outcome
1)	$j \in s_i$	$i \in s_j$	Y	$ij \in g$
2)	$j \in s_i$	$i \in s_j$	N	$ij \in g$
3)	$j \notin s_i$	$i \in s_j$	Y	$ij \notin g$
4)	$j \notin s_i$	$i \in s_j$	N	$ij \notin g$
5)	$j \in s_i$	$i \notin s_j$	Y	$ij \notin g$
6)	$j \in s_i$	$i \notin s_j$	N	$ij \notin g$
7)	$j \notin s_i$	$i \notin s_j$	Y	$ij \in g$
8)	$j \notin s_i$	$i \notin s_j$	N	$ij \notin g$

Table 1

The individual players' payoffs are the same as in the original game. The payoff for a pair ij is

$$u_{ij}(g) = \begin{cases} 0 & \text{if } ij \notin g \\ \min\{u_i(g) - u_i(g - ij), u_j(g) - u_j(g - ij)\} & \text{if } ij \in g. \end{cases}$$

Thus, if the current network is $g - ij$ and both players benefit from the addition of ij, then $u_{ij}(g) > u_{ij}(g - ij) = 0$. Whenever possible, the pair would then switch to a strategy that induces the link in the network.[8]

If g is supported by a Nash equilibrium of the game with player pairs, I will say that g is *Nash stable for the linking game with player pairs*.

Proposition 1. A network is Nash stable for the linking game with player pairs if and only if it is weakly pairwise Nash stable.

Proof.

(1) Let g be a Nash stable network for the game with player pairs with (s^s, s^p) being the supporting Nash equilibrium strategy profile. The proof of this part proceeds by first showing that g must be Nash stable, and then that g must be weakly pairwise stable.

(i) Nash stability

[8] Note that, to induce the deviation, the payoff from $u_{ij}(g)$ has to be strictly positive, which requires that each of the individual players benefit from adding the link. This feature is what makes it necessary to use the weak notion of pairwise stability to obtain the result in Proposition 1.

Let s' be a strategy profile for the linking game in which, for all $i, j \in N$, $j \in s'_i$ if and only if $ij \in g$, that is s' supports g. No player can gain by indicating additional players because this would not change the outcome. Moreover, since in the linking game with player pairs, every individual player can sever any number of his links in $g(s^s, s^p)$ by changing his strategy, and since (s^s, s^p) is a Nash equilibrium, severing links cannot be beneficial.[9] Thus g is Nash stable.

(ii) Weak pairwise stability

To see that g must be weakly pairwise stable, note that the first requirement of weak pairwise stability, $u_i(g) \geq u_i(g - ij)$ for all $ij \in g$, is implied by the already established fact that g is Nash stable. Next, consider some $ij \notin g$ (if any), and suppose that $u_i(g+ij) > u_i(g)$ and that $u_j(g + ij) > u_j(g)$, contrary to the requirement of weak pairwise stability. In this case, $u_{ij}(g + ij) = \min \{u_i(g + ij) - u_i(g), u_j(g + ij) - u_j(g)\} > 0$. There are two kinds of cases, in which $ij \notin g(s^s, s^p)$. Each case leads to a contradiction:

1. $j \notin s_i$, $i \notin s_j$, and $s_{ij} = N$, a contradiction because ij can deviate to $s_{ij} = Y$ and increase the pair's payoff from 0 to $\min \{u_i(g + ij) - u_i(g), u_j(g + ij) - u_j(g)\} > 0$.

2. $j \notin s_i$, $i \in s_j$, and $s_{ij} = N$ or $s_{ij} = Y$, a contradiction because i can deviate to $j \in s_i$ and induce the network $g + ij$ (the same holds true for the cases $j \in s_i$, $i \notin s_j$, and $s_{ij} = N$ or $s_{ij} = Y$).[10]

Thus, g is Nash stable and weakly pairwise stable, implying that g is weakly pairwise Nash stable.

(2) Conversely, suppose that g is weakly pairwise Nash stable. Let (s^s, s^p) be a strategy profile for the linking game with player pairs, in which $j \in s_i$, $i \in s_j$, and $s_{ij} = Y$ for all i, j such that $ij \in g$, and $j \notin s_i$, $i \notin s_j$, and $s_{ij} = N$ for all i, j such that $ij \notin g$. The profile (s^s, s^p) supports g. By deviating, an individual player i can only sever links, which is not beneficial for him because g is Nash stable. The deviation of a pair of players ij can only lead to the addition of a link, which is not beneficial both players in the pair because g is

[9] If player i wishes to sever a link to player j, he can, if $j \in s_i$ and $i \in s_j$, simply change to a strategy s'_i such that $j \notin s_i$, and if $j \notin s_i$, $i \notin s_j$, and $s_{ij} = Y$, to s'_i such that $j \in s'_i$.

[10] The case $j \in s_i$, $i \notin s_j$, and $s_{ij} = N$ is the one that requires a weakening of pairwise stability. Here, j has to deviate in order to have the link added to the network. However, if one allows j to be indifferent he would not necessarily want to deviate, not yielding the desired contradiction.

weakly pairwise stable. Thus, the profile (s^s, s^p) is a Nash equilibrium, showing that g is Nash stable for the linking game with player pairs. □

Proposition 1 shows that a weakly pairwise Nash stable network can be viewed as the outcome of a "noncooperative" game. Because this game is finite, a Nash equilibrium of its mixed extension exists. The following example illustrates such a mixed strategy equilibrium. The example is based on the one depicted in Figure 2.

Example

To simplify the analysis, let the payoffs for networks containing any of the links, $14, 13$, or 24 (the "unused" links in Panel (b) of Figure 2) be as follows. For all $i \in N$, if g contains all three of these links, $u_i(g) = -3$, if g contains exactly two of these links, $u_i(g) = -2$, and if g contains exactly one such link, $u_i(g) = -1$. All other networks, except for the ones depicted in Panel (a), yield a payoff of zero to the individual players. For this specification of payoffs, no network is pairwise stable.

Let us further simplify the analysis by eliminating weakly dominated strategies. It is clear that for any player i, if s_i indicates the willingness to form any of the links $14, 13$, or 24, then s_i is is weakly dominated by $s_i' = s_i \setminus \{14, 13, 24\}$. Similarly, for every player pair $ij = 14, 13, 24$, the strategy $s_{ij} = N$ weakly dominates $s_{ij} = Y$. Furthermore, note that for player 3, a strategy s_3 involving $2 \notin s_3$ is weakly dominated by a strategy $s_3' = s_3 \cup \{2\}$. Similarly, a strategy for player 4 with $3 \notin s_4$ is weakly dominated by $s_4' = s_4 \cup \{3\}$, and a strategy for player 1 with $2 \notin s_1$ is weakly dominated by $s_1' = s_1 \cup \{2\}$. Lastly, for player 2, the strategy s_2 such that $3 \in s_2$ but $1 \notin s_2$ is weakly dominated by a strategy s_2' with $3 \in s_2'$ and $1 \in s_2'$, and for player 3 the strategy s_3 such that $4 \in s_3$ but $2 \notin s_3$ is weakly dominated by a strategy s_3' with $4 \in s_3'$ and $2 \in s_3'$. Hence, in a Nash equilibrium in undominated strategies, the following strategies are played with probability one: $s_1 = \{2\}$, $s_4 = \{3\}$, and $s_{13} = s_{14} = s_{24} = N$. In addition, for players 2 and 3, the only strategies that are not weakly dominated are $s_2 = \{1\}$, $s_2' = \{1, 3\}$, $s_3 = \{2\}$, and $s_3' = \{2, 4\}$.

Note that every combination of the individual players' undominated pure strategies is such that the strategies of the pairs $12, 23$, or 34 will not affect the outcome (recall that a pair ij only affects the outcome if $i \notin s_j$ and $j \notin s_i$). Therefore, if ρ_{ij} denotes the probability for $s_{ij} = Y$, then any $\rho_{ij} \in [0, 1]$, for $ij = 12, 23$, or 34, supports the equilibrium. Let

α denote the probability that player 2 chooses s_2 and let β denote the probability that player 3 attaches to s_3. In a mixed strategy equilibrium, α and β must solve the following optimization problems for players 2 and 3.

Player 2:

$$\begin{aligned} & \max_{\alpha} \alpha\left(\beta 7+(1-\beta) 7\right)+(1-\alpha)(\beta 6+(1-\beta) 8) \\ & Foc \text{ (interior solution)} \\ \beta 7+(1-\beta) 7 &= \beta 6+(1-\beta) 8 \Leftrightarrow \beta^*=\frac{1}{2}. \end{aligned}$$

Player 3:

$$\begin{aligned} & \max_{\beta} \beta\left(\alpha 0+(1-\alpha) 11\right)+(1-\beta)(\alpha 7+(1-\alpha) 8) \\ & Foc \text{ (interior solution)} \\ (1-\alpha) 11 &= \alpha 7+(1-\alpha) 8 \Leftrightarrow \alpha^*=\frac{3}{10} \end{aligned}$$

In summary, the set of equilibrium strategy profiles in undominated strategies for the linking game with player pairs is as follows:

$$\begin{aligned} s_1 &= \{2\}; \\ s_2 &= \{1\} \text{ and } s_2'=\{1,3\} \text{ with probability } \frac{1}{2} \text{ each;} \\ s_3 &= \{2\} \text{ with probability } \frac{3}{10},\ s_3'=\{2,4\} \text{ with probability } \frac{7}{10}; \\ s_4 &= \{3\}; \\ s_{ij} &= N, \text{ for } ij=13,14, \text{ and } 24; \\ s_{ij} &= Y \text{ with probability } \rho_{ij}\in[0,1], \text{ for } ij=12,23, \text{ and } 34. \end{aligned}$$

Only the four networks in Panel (a) occur with positive probability. Under the induced distribution, g^0, g^1, g^2, and g^3 occur with probabilities $\frac{7}{20}$, $\frac{7}{20}$, $\frac{3}{20}$, and $\frac{3}{20}$, respectively.

It is worth noting that there are other specifications for the player pairs' payoffs that yield the result in Proposition 1. In particular, any specification such that $u_{ij}(g+ij)>u_{ij}(g)$ if and only if $u_i(g+ij)>u_i(g)$ and $u_j(g+ij)>u_j(g)$ yields the same set of pure strategy equilibria of the linking game with player pairs. This fact might lead to the concern that other payoff specifications result yield different mixed strategy equilibrium outcomes. For affine transformations of the payoff function used here, a standard result from expected utility theory guarantees that the set of mixed strategy equilibria remains the same. All other kinds of transformations (that preserve the above condition), might lead to different mixed strategy equilibria. However, I believe that the specification chosen here is the most natural one. Moreover, it can be applied to any set of preferences, while other specifications will likely work only for a subset of preferences.

As already discussed, in the supernetwork $\mathbf{G}$, the nonexistence of a weakly pairwise Nash stable network implies at least one cycle with no outgoing arc to a network outside the cycle. Such cycles of networks have been labeled basins of attraction (Page and Wooders, 2008).[11] Formally, a set of networks $A \subseteq G$ is a *basin of attraction* if

$$\begin{aligned} (i)\ g &\in A,\ g' \notin A \text{ implies that } \left(g, g'\right) \notin \mathbf{G}, \text{ and} \\ (ii)\ g, g' &\in A,\ g \neq g', \text{ implies that } \exists\ g_0, g_{1,}..., g_L \in G \text{ such that} \\ (g_l g_{l+1}) &\in \mathbf{G} \text{ for } l = 0, 1, ..., L-1, \text{ where } g_0 = g \text{ and } g_L = g'. \end{aligned}$$

Basins of attractions are "absorbing states." Once a basin is reached, the process remains within the basin. Weakly pairwise Nash stable networks are precisely the networks that belong to a singleton basin of attraction. Let $\mathcal{A}$ be the collection of basins of attraction for the supernetwork $\mathbf{G}$. A natural conjecture is that networks that are supported by a mixed strategy equilibrium of the linking game with player pairs must belong to a basin of attraction, or, more formally: If g occurs with positive probability under some mixed strategy Nash equilibrium of the linking game with player pairs, then $g \in A$ for some $A \in \mathcal{A}$.

However, the following example refutes this conjecture. Let $N = \{1, 2, 3, 4\}$ and consider

[11] The term basin of attraction is used for similar concepts in mathematics and the sciences. Page and Wooders (2008) have introduced it to the context of strategic network formation.

the following four networks, depicted in Figure 3:

$$\begin{aligned} g^0 &= \{14, 23\} \\ g^1 &= \{14, 23, 12\} \\ g^2 &= \{14, 23, 34\} \\ g^3 &= \{14, 23, 12, 34\}. \end{aligned}$$

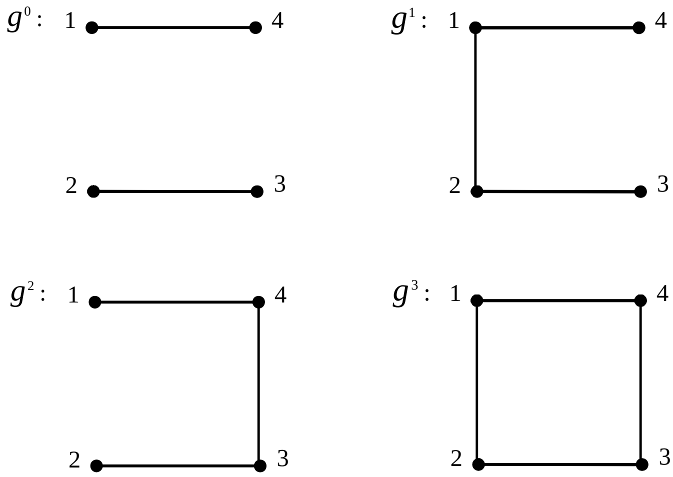

Figure 3

Suppose that the individual players' preferences over networks are represented by the following utility functions

$$\begin{aligned} u_1\left(g^0\right) &= u_1\left(g^1\right) = 1 \text{ and } u_1(g) = -1 \text{ for } g \in G \setminus \{g^0, g^1\}; \\ u_2\left(g^0\right) &= u_2\left(g^1\right) = 1 \text{ and } u_2(g) = -1 \text{ for } g \in G \setminus \{g^0, g^1\}; \\ u_3\left(g^0\right) &= 0,\ u_3\left(g^2\right) = u_3\left(g^3\right) = 1,\ u_3\left(g^1\right) = 2, \text{ and} \\ u_3(g) &= -1 \text{ for } g \in G \setminus \{g^0, g^1, g^2, g^3\}; \\ u_4\left(g^0\right) &= 0,\ u_4\left(g^2\right) = u_4\left(g^3\right) = 1,\ u_4\left(g^1\right) = 2, \text{ and} \\ u_4(g) &= -1 \text{ for } g \in G \setminus \{g^0, g^1, g^2, g^3\}, \end{aligned}$$

implying utilities for player pairs:

$$\begin{aligned}
u_{14}\left(g^0\right) &= 1,\ u_{14}\left(g^1\right) = 2, \text{ and } u_{14}\left(g\right) = 0 \text{ for } g \in G \setminus \{g^0, g^1\}; \\
u_{23}\left(g^0\right) &= 1,\ u_{23}\left(g^1\right) = 2, \text{ and } u_{23}\left(g\right) = 0 \text{ for } g \in G \setminus \{g^0, g^1\}; \\
u_{12}\left(g\right) &= 0 \text{ for } g \in G; \\
u_{34}\left(g^2\right) &= 1,\ u_{34}(g^3) = -1, \text{ and } u_{34}\left(g\right) = 0 \text{ for } g \in G \setminus \{g^2, g^3\}; \\
u_{13}(g) &\leq 0 \text{ for } g \in G; \\
u_{24}(g) &\leq 0 \text{ for } g \in G.
\end{aligned}$$

The supernetwork $\mathbf{G}$ restricted to the networks in $\{g^0, g^1, g^2, g^3\}$ contains only two arcs: $\left(g^0, g^2\right)$ and $\left(g^3, g^1\right)$. Moreover, for any $g \in \{g^0, g^1, g^2, g^3\}$ and $g' \in G \setminus \{g^0, g^1, g^2, g^3\}$, we have $(g, g') \notin \mathbf{G}$, as illustrated in Figure 4. Therefore, networks g^1 and g^2 each constitute a singleton basin of attraction while g^3 does not belong to any basin of attraction because it is defeated by g^1.

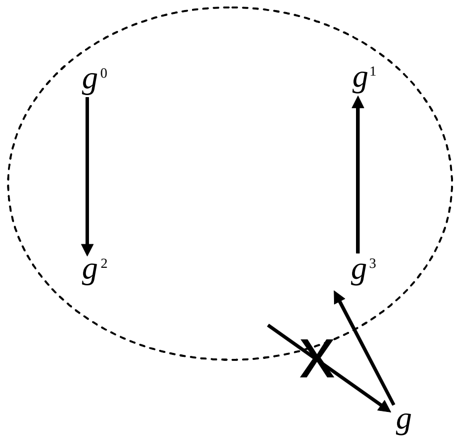

Figure 4

It is straightforward (though somewhat tedious) to verify that a strategy profile satisfying the following conditions constitutes a mixed strategy Nash equilibrium of the linking game with player pairs:

$$\begin{aligned}
s_1 &= \{4\}; \\
s_2 &= \{3\}; \\
s_3 &= \{2,4\}; \\
s_4 &= \{1,3\}; \\
s_{12} &= Y \text{ and } s_{12} = N \text{ with probability } \frac{1}{2} \text{ each;} \\
s_{13} &= N; \\
s_{24} &= N.^{12}
\end{aligned}$$

No player (individual or pair) has an incentive to deviate. Under such a profile networks g^2 and g^3 occur with equal probability $\frac{1}{2}$ each. Since g^3 does not belong to any basin of attraction, the example shows that the above conjecture is false.

3 The two-stage linking game

This section models explicitly how coordination between two players might arise in a noncooperative two-stage game. In the first stage, players play the linking game. A stage-one link constitutes a coordination possibility in stage two. Given a network of coordination possibilities formed in the first stage, players play a constrained version of the linking game with player pairs in the second stage. In the constrained version, only pairs that have formed a link in the first stage can coordinate on whether they form a link. Thus, if the network of coordination possibilities formed in the first stage is given by $h \subseteq \{ij : i, j \in N, i \neq j\}$, then the set of players in the constrained linking game with player pairs is $N \cup h$. The corresponding constrained game is denoted by $\hat{\Gamma}(h)$.

Intuitively, the two stages capture the idea that coordination as assumed by pairwise stability can arise between two individuals only if they have some sort of connection to each other; therefore the terminology of a coordination possibility. For example, to form friendships, first people become acquainted with each other (through work, by joining a club, because they have common friends, etc.). Then they might form a closer friendship, which, arguably, will happen if and only if both sides wish to establish that friendship.

I show that any network that can be supported by a Nash equilibrium of the linking game can also be supported by a subgame perfect Nash equilibrium of the two-stage linking game. However, if this network is not weakly pairwise stable, then the subgame perfect Nash equilibrium that supports it will have at least one player playing a weakly dominated strategy. Thus, a network supported by an undominated subgame perfect Nash equilibrium of the two-stage linking game, must be weakly pairwise stable.

As in the linking game with player pairs, the set of players in the two-stage linking game is $N \cup \{ij : i, j \in N, \ i \neq j\}$. A pure strategy for player i is denoted by t_i and a pure strategy for a player pair ij is denoted by q_{ij}. A strategy t_i for player i consists of a set $r_i \in R_i = P(N \backslash \{i\})$ and a function $f_i : G \rightarrow R_i$. The set r_i indicates with whom player i would like to form a coordination possibility. The function f_i indicates, for each first stage "network" formed in the first stage, the set of players player i wishes to form a link with. Let $G_{ij} = \{g : ij \in g\}$, that is G_{ij} is the set of networks in which players i and j are directly linked to each other. A strategy q_{ij} for player pair ij is a function $f_{ij} : G_{ij} \rightarrow \{Y, N\}$. The corresponding mixed strategies for a player (individual or pair) are probability distributions over the player's set of pure strategies. For ease of notation, mixed strategies are not formally introduced. A profile of strategies and sets of strategies are denoted in the usual way.

Proposition 2. *A network $\hat{g}$ is Nash stable if and only if it is supported by a subgame perfect Nash equilibrium of the two-stage linking game.*

Proof.

(1) Let s be a Nash equilibrium of the linking game that supports $\hat{g}$. The proof proceeds by constructing a profile $(t, q) = ((r_i, f_i)_{i \in N}, q)$ that is subgame perfect and supports $\hat{g}$. Let (t, q) be as follows.

Stage 1:

For all $i, j \in N$, let $j \in r_i$ if and only if $ij \in \hat{g}$.

Stage 2:

(a) Subgame $\hat{g}$: For all $i \in N$, let $f_i(\hat{g}) = r_i$. For all $ij \in \hat{g}$, let $f_{ij}(\hat{g}) = N$.

Let $B \subseteq \{ik : ik \in \hat{g}\}$.

(b) Subgames $g = \hat{g} \setminus B$: For all $i \in N$, let $f_i(g) = r_i$. For all $ij \in g$, let $f_{ij}(g) = N$.

(c) All other subgames g: Pick a Nash equilibrium of the constrained linking game with player pairs $\hat{\Gamma}(g)$. (Note that this might be a Nash equilibrium in mixed strategies).

The outcome of this profile is $\hat{g}$. It remains to verify that the profile is a subgame perfect equilibrium. Because $\hat{g}$ is Nash stable, the play induced on $\hat{\Gamma}(\hat{g})$ is a Nash equilibrium: No individual player wants to deviate, and since player pairs only have the power to add links but not to delete them and there is no player pair in $\hat{\Gamma}(\hat{g})$ whose link is not in $\hat{g}$, no pair can deviate either. The same logic applies to the play induced on subgames in (b). Lastly, the profile induces a Nash equilibrium on subgames in (c). Next, we need to verify that the specified profile also induces a Nash equilibrium on the entire game. It was already shown that no player can benefit from a deviation in any of the subgames. A unilateral deviation by a single player in stage 1 can only lead to a subgame of type (b). However, the Nash equilibrium outcome (t, q) induces on these subgames is the same as the outcome of subgame $\hat{g}$. Thus, the profile is a Nash equilibrium for the entire game.

(2) Conversely, let $\hat{g}$ be supported by a subgame perfect Nash equilibrium of the two-stage linking game. Let g' be the set of potential links formed in stage 1. Because $\hat{g}$ is the outcome of a subgame perfect equilibrium, it is supported by a Nash equilibrium of the constrained linking game with player pairs $\Gamma(g')$. Therefore no individual player wishes to delete any subset of his links in $\hat{g}$, showing that $\hat{g}$ is Nash stable. $\square$

Proposition 2 shows that, even when individuals can form links to foster cooperation, the result can be a complete coordination failure. For example, as in the linking game, the empty network is trivially supported by a subgame perfect Nash equilibrium of the two-stage linking game. However, Proposition 3 shows that if a subgame perfect equilibrium of the two-stage linking game supports a network that is not weakly pairwise Nash stable, at least one player plays a weakly dominated strategy. In other words, any undominated subgame perfect equilibrium of the two-stage linking game will only support weakly pairwise Nash stable outcomes. This is not true for the linking game. Jackson (2008, p. 374) provides an example of a network which is supported by a Nash equilibrium in undominated strategies of the linking game but is not weakly pairwise Nash stable.

Proposition 3. *If a network is supported by a subgame perfect equilibrium in undominated* *strategies of the two-stage linking game then it is weakly pairwise Nash stable.*

Proof. I prove the contrapositive. Let $\hat{g}$ be a network that is not weakly pairwise Nash stable. If $\hat{g}$ is not Nash stable, Proposition 2 implies that it cannot be supported by a subgame perfect equilibrium of the two-stage linking game. If $\hat{g}$ is not weakly pairwise stable, pick a pair $\{i, j\}$ such that $ij \notin \hat{g}$ but $u_i(\hat{g}+ij) > u_i(\hat{g})$ and $u_j(\hat{g}+ij) > u_j(\hat{g})$. Let (t, q) be a subgame perfect equilibrium profile of the two-stage linking game that supports $\hat{g}$, and let h be the first stage links formed under the profile (t, q). Since (t, q) supports $\hat{g}$, it must hold that $ij \notin h$ for otherwise $\hat{g}$ could not be supported by a Nash equilibrium of the constrained game $\hat{\Gamma}(h)$ (a deviation by ij would make sure that the link ij was formed). So either $j \notin r_i$ or $i \notin r_j$. Without loss of generality, suppose that $j \notin r_i$. Now, consider the following strategy $t_i' = (r_i', f_i')$. Let t_i' be a strategy for player i that coincides with t_i except that $j \in r_i'$ and that $f_i'(\hat{g}+ij) = f_i(\hat{g})$. To show that t_i' weakly dominates t_i, we need to show that there exists at least one play of all other players, say (t_{-i}', q') to which t_i' is a better response than t_i, and so that t_i' yields at least as much as t_i to every other play by the remaining players.

Let (t_{-i}', q') coincide with (t_{-i}, q) except that $i \in r_j'$, $f_j'(\hat{g}+ij) = f_j(\hat{g})$, and $f_{ij}'(\hat{g}+ij) = Y$. The outcome of the profile (t_i, t_{-i}', q') is $\hat{g}$, while the outcome of the profile (t_i', t_{-i}', q') is $\hat{g}+ij$. Thus t_i' is a better response to (t_{-i}', q') than t_i.

Now, let (t_{-i}', q') be an arbitrary play. If t_j' is such that $i \notin r_j'$, then the same subgame is reached under (t_i, t_{-i}', q') and (t_i', t_{-i}', q'), and it is not the subgame $\hat{g}+ij$. On all these subgames the play induced by the two strategy profiles coincides, and therefore i obtains the same utility from both profiles. If t_j' is such that $i \in r_j'$, then $\hat{g}+ij$ is reached under (t_i', t_{-i}', q') and $\hat{g}$ is reached under (t_i, t_{-i}', q'). In this case, again, the two profiles lead to the same outcome because $f_i'(\hat{g}+ij) = f_i(\hat{g})$.

Thus t_i' weakly dominates t_i, which is what we wanted to show. □

4 Conclusion

This paper demonstrates that weakly pairwise Nash stable networks are the equilibrium networks a modified version of Myerson's linking game, in which pairs of players are treated as additional players. Two features of the game bring about the type of coordination required by weak pairwise stability. First, pairs have the discretion to add links whenever individual players fail to coordinate on the formation of a link. Second, a pair's payoff is designed so that the pair benefits from the addition of the link if and only if both players benefit from it. My characterization allows to introduce mixed strategies to coordinated moves.

The paper also defines a two-stage game of network formation which provides a natural model of how coordination can arise in a noncooperative setting. The equilibrium of the two-stage game provides a rationale for networks in which players are not linked even though both would benefit from being linked. In the game, such networks can arise in equilibrium if players fail to form coordination links in the first stage. In a real-world network, such networks might be stable simply because two players do not know enough about each other (or do not know each other at all).

The nonexistence of solutions to games with cooperative elements has been studied extensively in the literature with a focus on finding conditions when such a solution exists. In all these games, reverting to mixed strategies is infeasible because they have no well-defined strategic form. To resolve this, it might be feasible to define equivalent strategic form noncooperative games, using methods similar to the ones used in this paper.

References

[1] Bala, V. and S. Goyal (2000), "A Noncooperative Model of Network Formation", Econometrica, 1181 -1229.

[2] Belleflamme, P. and F. Bloch (2004), "Market Sharing Agreements and Collusive Networks", International Economic Review, 45 (2), 387-411.

[3] Bloch, F. and M. O. Jackson (2006), "Definitions of equilibrium in network formation games", International Journal of Game Theory, 34, 305-318.

[4] Calvó-Armengol, T. (2004), "Job contact networks", Journal of Economic Theory, 115, 191-206.

[5] Calvó-Armengol, T. and R. Ilkiliç (2009), "Pairwise-Stability and Nash Equilibria in Network Formation", International Journal of Game Theory, 38, 51-79.

[6] Goyal, S. and S. Joshi (2006), "Unequal connections", International Journal of Game Theory, 34, 319-349.

[7] Goyal, S. and F. Vega-Redondo (2005), "Network formation and social coordination", Games and Economic Behavior, 50, 178-207.

[8] Jackson, M. O. (2005), "A Survey of Network Formation Models: Stability and Efficiency" in Group Formation in Economics: Networks, Clubs, and Coalitions by Demange, G. and M. H. Wooders (eds), Cambridge University Press.

[9] Jackson, M. O. and A. Watts (2001), "The existence of pairwise stable networks", Seoul Journal of Economics 14, 3, 299-321.

[10] Jackson, M. O. and A. Watts (2002), "The Evolution of Social and Economic Networks", Journal of Economic Theory 106, 265-295.

[11] Jackson, M. O. and A. Wolinsky (1996), "A Strategic Model of Social and Economic Networks", Journal of Economic Theory, 71, 44-74.

[12] Myerson, R. B. (1991), "Game Theory", Harvard University Press, Cambridge, Massachusetts.

[13] Page, F.H., Wooders, M. and S. Kamat (2005), "Networks and farsighted stability", Journal of Economic Theory, 120 (2), 257-269.

[14] Page, F.H. and M. Wooders, (2008), "Strategic basins of attraction, the path dominance core, and network formation games", Games and Economic Behavior, forthcoming.